

## On $\tilde{\beta}_g(\theta)$ Coconvergence and Adherence in Topological Spaces via Grill

<sup>1</sup>K. Kannan and <sup>2</sup>N. Nagaveni

<sup>1</sup>Department of Mathematics, Hindusthan College of Engineering and Technology,  
 32 Coimbatore, India

<sup>2</sup>Department of Mathematics, Coimbatore Institute of Technology,  
 14 Coimbatore, India

**Abstract:** A new class of sets called  $\tilde{\beta}$ -generalized closed sets and  $\tilde{\beta}$ -generalized open sets in topological spaces and its properties are studied. A subset  $A$  of a topological spaces  $(X, \tau)$  is called  $\tilde{\beta}$ -generalized closed sets (briefly  $\tilde{\beta}_g$ -closed) if  $\text{cl}(\text{int}(\text{cl}(A)))$  contains  $U$  whenever  $A$  contains  $U$  and  $U$  is open in  $X$ . A new class of  $\tilde{\beta}_g$ -continuous maps and  $\tilde{\beta}_g$ -irresolute maps in topological spaces and study some of its basic properties. In this study, we introduce the notion of  $\tilde{\beta}_g(\theta)$  convergence and  $\tilde{\beta}_g(\theta)$ -adherence in grill topological spaces and study some of its basic properties and relations among them.

**Key words:**  $\tilde{\alpha}\tilde{\delta}^{\theta\#}$ -convergence,  $\tilde{\alpha}\tilde{\delta}^{\theta\#}$ -adherence, grill topological, irresolute maps, properties and relations

### INTRODUCTION

Generalized open sets play a very important role in general topology and they are now the research topics of many topologists worldwide. Indeed a significant the main general topology and real analysis concerns the variously modified forms of continuity, separation axioms etc., by utilizing generalized open sets and the idea of grills on a topological space was first introduced by Choquet. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds.

Throughout the present study, spaces  $X$  and  $Y$  always mean topological spaces. Let  $X$  be a topological space and  $A$  a subset of  $X$ . For a subset  $A$  of a topological space  $(X, \tau)$ ,  $\text{cl}(A)$  and  $\text{int}(A)$  denote the closure of  $A$  and the interior of  $A$  respectively. In this study, we investigate some more properties of this type of closed spaces. Before entering to our research, we recall the following definitions which are useful in the sequel.

**Definition 1.1:** A subset  $A$  of a topological space  $(X, \tau)$  is:

- A pre open set (Sheik, 2000) if  $A \subseteq \text{int}(\text{cl}(A))$  and a preclosed set if  $\text{cl}(\text{int}(A)) \subseteq A$

- A semi open set (Levine, 1963) if  $A \subseteq \text{cl}(\text{int}(A))$  and a semi closed set if  $\text{int}(\text{cl}(A)) \subseteq A$
- A  $\alpha$ -open set (Njastad, 1965) if  $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$  and a  $\alpha$ -closed set if  $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$
- A semi-preopen set (Andrijevic, 1986) if  $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$  and a semi-preclosed set if  $\text{int}(\text{cl}(\text{int}(A))) \subseteq A$
- A regular open set (Stone, 1937) if  $A = \text{int}(\text{cl}(A))$  and a regular closed set if  $A = \text{cl}(\text{int}(A))$
- A  $\tilde{\beta}_g$ -closed set (Kannan and Nagaveni, 2012) ( $\tilde{\beta}$ -generalized closed set) if  $\text{cl}(\text{int}(\text{cl}(A))) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open in  $X$
- Researcher define the  $\tilde{\beta}$ -closure of  $A$  (Kannan and Nagaveni, 2015) as follows:

$$\text{Cl}_{\tilde{\beta}_g}(A) = \bigcap \{F : F \text{ is } \tilde{\beta}_g\text{-closed in } X, A \subseteq F\}$$

### $\tilde{\beta}_g(\theta)$ -convergence and $\tilde{\beta}_g(\theta)$ -adherence

**Definition 2.1:** A grill  $g$  on a topological space  $(X, \tau)$  is defined to be a collection of nonempty subsets of  $X$  such that (Kannan and Nagaveni, 2012):

$$A \in G \text{ and } A \subseteq B \subseteq X \Rightarrow B \in G$$

And:

$$A, B \subseteq X \text{ and } A \cup B \in G \Rightarrow A \in G \text{ or } B \in G$$

**Definition 2.2:** (Kannan and Nagaveni) If  $g$  is a grill (or a filter) on a space  $(X, \tau)$ , then the section of  $g$  denoted by  $\text{sec } g$  is given by:

$$\text{secg} = \{A \subseteq X : A \cap G \neq \emptyset, \text{ for all } G \in g\}$$

$$\Rightarrow x \in \widehat{\beta}g(\theta)\text{cl}(g), \text{ for all } G \in g$$

**Definition 2.3:** A grill  $g$  on a topological space  $(X, \tau)$  is said to be a:

$$\Rightarrow G \in g(\widehat{\beta}g(\theta), x), \text{ for all } G \in g$$

- $\widehat{\beta}g(\theta)$ -adhere (briefly  $\widehat{\beta}g(\theta)A$ ) at  $x \in X$  if for each  $U \in \widehat{\beta}gO(x)$  and each  $G \in gCl_{\widehat{\beta}g}(U)G \neq \emptyset$
- $\widehat{\beta}g(\theta)$ -converge (briefly  $\widehat{\beta}g(\theta)C$ ) to a point  $x \in X$  if for each  $U \in \widehat{\beta}gO(x)$ , there is some  $G \in g$  such that  $G \subseteq Cl_{\widehat{\beta}g}(U)$  (in this case we shall also say that  $g$  is  $\widehat{\beta}g(\theta)$ -convergent to  $x$ )

$$\Rightarrow g \subseteq g(\widehat{\beta}g(\theta), x)$$

Conversely, let  $g \subseteq g(\widehat{\beta}g(\theta), x)$ . Then for all  $G \in g$ ,  $x \in \widehat{\beta}g(\theta)\text{cl}(g)$ , so that for all  $U \in \widehat{\beta}gO(x)$  and for all  $G \in g$ ,  $Cl_{\widehat{\beta}g}(U) \cap G \neq \emptyset$ . Hence,  $g$  is  $\widehat{\beta}g(\theta)A$  at  $x$ .

**Remark 2.4:** A grill  $g$  is  $\widehat{\beta}g(\theta)C$  to a point  $x \in X$  if and only if  $g$  contains the collection  $\{Cl_{\widehat{\beta}g}(U) : U \in \widehat{\beta}gO(x)\}$ .

**Theorem 2.9:** A grill  $g$  on topological space  $(X, \tau)$  is  $\widehat{\beta}g(\theta)C$  to a point  $x$  of  $X$  if and only if  $g \subseteq \text{secg}(\widehat{\beta}g(\theta), x)$ .

**Definition 2.5:** A filter  $F$  on a space  $(X, \tau)$  is  $U \in \widehat{\beta}gO(x)$  said to  $\widehat{\beta}g(\theta)A$   $x \in X$  ( $\widehat{\beta}g(\theta)C$  to  $x \in X$ ) if for each  $F \in F$  and each  $U \in \widehat{\beta}gO(x)$ ,  $F \cap Cl_{\widehat{\beta}g}(U) \neq \emptyset$  (resp. to each there corresponds  $F \in F$  such that  $F \cap Cl_{\widehat{\beta}g}(U) \neq \emptyset$ ).

**Proof:** Let  $g$  be a grill on  $X$ ,  $\widehat{\beta}g(\theta)C$  to  $x \in X$ . Then for each  $U \in \widehat{\beta}gO(x)$ , there exists  $G \in g$  such that  $G \subseteq Cl_{\widehat{\beta}g}(U)$  and hence  $Cl_{\widehat{\beta}g}(U) \in g$  for each  $U \in \widehat{\beta}gO(x)$ . Now,  $B \in \text{secg}(\widehat{\beta}g(\theta), x)$ .

We note at this stage that unlike the case of filters, the notion of  $\widehat{\beta}g(\theta)A$  of a grill is strictly stronger than that of  $\widehat{\beta}g(\theta)C$ . In fact, we have.

$$\Rightarrow X - B \notin g(\widehat{\beta}g(\theta), x)$$

$$\Rightarrow x \notin \widehat{\beta}g(\theta)\text{cl}(X - B)$$

**Theorem 2.6:** If a grill  $g$  on a space  $(X, \tau)$   $\widehat{\beta}g(\theta)A$  at some point  $x \in X$ , then  $g$  is  $\widehat{\beta}g(\theta)C$  to  $x$ .

$$\Rightarrow \text{there exists } U \in \widehat{\beta}gO(x) \text{ such that}$$

$$Cl_{\widehat{\beta}g}(U) \cap (X - B) = \emptyset$$

**Proof:** Let a grill  $g$  on  $(X, \tau)$ ,  $\widehat{\beta}g(\theta)A$  at  $x \in X$ . Then for each  $U \in gO(x)$  and each  $G \in g$ ,  $Cl_{\widehat{\beta}g}(U) \cap G \neq \emptyset$  so that  $Cl_{\widehat{\beta}g}(U) \in \text{secg}$ , for each  $U \in gO(x)$  and hence  $X - Cl_{\widehat{\beta}g}(U) \notin g$ . Then  $Cl_{\widehat{\beta}g}(U) \in g$  (as  $g$  is a grill and  $X \in g$ ) for each  $U \in \widehat{\beta}gO(x)$ . Hence,  $g$   $\widehat{\beta}g(\theta)C$  must to  $x$ .

$$\Rightarrow Cl_{\widehat{\beta}g}(U) \subseteq B, \text{ where } U \in \widehat{\beta}gO(x)$$

$$\Rightarrow B \in G$$

**Remark 2.7:** Let  $X$  be a topological space. Then for any  $x \in X$ , we adopt the following notation:

Conversely, let if possible,  $g$  not a  $\widehat{\beta}g(\theta)C$  to  $x$ . Then for some  $U \in gO(x)$ ,  $Cl_{\widehat{\beta}g}(U) \notin g$  and hence  $Cl_{\widehat{\beta}g}(U) \notin \text{secg}(\widehat{\beta}g(\theta), x)$ . Thus for some  $A \in g(\widehat{\beta}g(\theta), x)$ ,  $A \cap Cl_{\widehat{\beta}g}(U) = \emptyset$ . But  $A \in g(\widehat{\beta}g(\theta), x)$

$$g(\widehat{\beta}g(\theta), x) = \{A \subseteq X : x \in \widehat{\beta}g(\theta)\text{cl}(A)\}$$

$$\Rightarrow x \in \widehat{\beta}g(\theta)\text{cl}(A)$$

$$\Rightarrow Cl_{\widehat{\beta}g}(U) \cap A \neq \emptyset$$

$$\text{secg}(\widehat{\beta}g(\theta), x) = \{A \subseteq X : A \cap G \neq \emptyset, \text{ for all } G \in g(\widehat{\beta}g(\theta), x)\}$$

which is a contradiction.

In the next two theorems, we characterize the  $\widehat{\beta}g(\theta)A$  and  $\widehat{\beta}g(\theta)C$  of grills in terms of the above notations.

**Theorem 2.10:** A grill  $g$  on a topological space  $(X, \tau)$ ,  $\widehat{\beta}g(\theta)C$  to a point  $x$  of  $(X, \tau)$ , if and only if  $\text{secg}(\widehat{\beta}g(\theta), x) \subseteq g$ .

**Theorem 2.8:** A grill  $g$  on a space  $(X, \tau)$ ,  $\widehat{\beta}g(\theta)A$  to a point  $x \in X$  if and only if  $g \subseteq g(\widehat{\beta}g(\theta), x)$ .

**Proof:** Let  $g$  be a grill on a topological space  $(X, \tau)$ ,  $\widehat{\beta}g(\theta)C$  to a point  $x \in X$ . Then for each  $U \in \widehat{\beta}gO(x)$  there exists  $G \in g$  such that  $G \subseteq Cl_{\widehat{\beta}g}(U)$  and hence  $Cl_{\widehat{\beta}g}(U) \in G$  for each  $U$ . Now,  $B \in \text{secg}(\widehat{\beta}g(\theta), x) \Rightarrow X/B \notin g(\widehat{\beta}g(\theta), x) \Rightarrow x \notin Cl_{\widehat{\beta}g}(U) \Rightarrow$  there exists  $U \in \widehat{\beta}gO(x) \rightarrow B \in G$  such that  $Cl_{\widehat{\beta}g}(U) \cap (X/B) = \emptyset \Rightarrow Cl_{\widehat{\beta}g}(U) \subseteq B$ , where  $U \in \widehat{\beta}gO(x) \Rightarrow B \in G$ . Conversely, let if possible,  $g$  not to  $\widehat{\beta}g(\theta)A$  to  $x$ . Then for some  $U \in \widehat{\beta}gO(x)$ ,  $Cl_{\widehat{\beta}g}(U) \notin g$

**Proof:** A grill  $g$  on a space  $(X, \tau)$ ,  $\widehat{\beta}g(\theta)A$  at  $x \in X$

$$\Rightarrow Cl_{\widehat{\beta}g}(U) \cap G \neq \emptyset \text{ for all } U \in \widehat{\beta}gO(x) \text{ and all } G \in g$$

and hence  $Cl_{\beta_g}(U) \notin \text{sec } g(\beta_g(\theta), x)$ . Thus for some  $A \in g(\beta_g(\theta), x)$ ,  $A \cap Cl_{\beta_g}(U) = \emptyset$ . But  $A \in g(\beta_g(\theta), x) \Rightarrow x \in Cl_{\beta_g}(A) \Rightarrow Cl_{\beta_g}(U) \cap U \neq \emptyset$ .

**Definition 2.11:** A nonempty subset  $A$  of a topological space  $X$  is called  $\beta_g$  closed relative to  $X$  if for every cover  $u$  of  $A$  by  $\beta_g$ -open sets of  $X$  there exists a finite subset  $u_0$  of  $u$  such that  $A \subseteq \cup\{Cl_{\beta_g}(U) : U \in u_0\}$ . If in addition,  $A = X$  then  $X$  is called a  $\beta_g$ -closed space.

**Theorem 2.12:** For a topological space  $X$ , the following statements are equivalent:

- $X$  is  $\beta_g$ -closed
- Every maximal filterbase  $\beta_g(\theta)C$  to some point of  $X$
- Every filterbase  $\beta_g(\theta)$ -adhere to some point of  $X$
- For every family  $\{V_\alpha : \alpha \in I\}$  of  $\beta_g$ -closed sets that  $\cap\{V_\alpha : \alpha \in I\} = \emptyset$ , there exists a finite subset  $I_0$  of  $I$  such that  $\cap\{Int_{\beta_g}(V_\alpha) : \alpha \in I_0\} = \emptyset$ .

**Proof (a)  $\Rightarrow$  (b):** Let  $F$  be a maximal filterbase on  $X$ . Suppose that  $F$  does not  $\beta_g$ -converge to any point of  $x$ . Since,  $T$  is maximal,  $T$  does not  $\beta_g(\theta)$ -accumulate at any point of  $X$ . For each  $x \in X$ , there exist  $F_x \in F$  and  $V_x \in \beta_g O(X, x)$  such that  $Cl_{\beta_g}(V_x) \cap F_x = \emptyset$ . The family  $\{V_x : x \in X\}$  a cover of  $X$  by  $\beta_g$ -open sets of  $X$ . By (a), there exists a finite number of points  $x_1, x_2, x_3, \dots, x_n$  of  $X$  such that  $X = \cup\{Cl_{\beta_g}(V_{x_i}) : i=1, 2, \dots, n\}$ . Since,  $F$  is a filter base on  $X$ , there exists  $F_0 \in F$  such that  $F_0 \subseteq \cap\{F_{x_i} : i=1, 2, \dots, n\}$ . Therefore, we obtain  $F_0 = \emptyset$ . This is a contradiction.

**(b)  $\rightarrow$  (c):** Let  $F$  be any filterbase on  $X$ . Then, there exists a maximal filterbase  $F_0$  such that  $F \subseteq F_0$ . By (b),  $F_0$   $\beta_g(\theta)$ -converges to some point  $x \in X$ . For every  $F \in F$  and every  $V \in \beta_g O(X, x)$ , there exists  $F_0 \in F_0$  such that  $F_0 \subseteq Cl_{\beta_g(\theta)}(V)$ ; hence  $\emptyset \neq F \cap Cl_{\beta_g}(V) \subseteq F$ . This shows that  $F$   $\beta_g(\theta)$ -accumulates at  $x$ .

**(b)  $\rightarrow$  (c):** Let  $\{V_\alpha : \alpha \in I\}$  be any family of  $\beta_g$ -closed subsets of  $X$  such that  $\cap\{V_\alpha : \alpha \in I\} = \emptyset$ . Let  $\Gamma(I)$  do note the ideal of all finite  $\beta_g$ -closed subsets of  $A$ . Assume  $\cap\{Int_{\beta_g}(V_\alpha) : \alpha \in I\} = \emptyset$  that for every  $I \in \Gamma(I)$ . Then, the family  $F = \{\cap_{\alpha \in I} Int_{\beta_g}(V_\alpha) : I \in \Gamma(I)\}$  is a filterbase on  $X$ . By (c),  $F$   $\beta_g(\theta)$ -accumulates at some point  $x \in X$ . Since,  $\{X/V_\alpha : \alpha \in I\}$  is a cover of  $X$ ,  $x \in X/V_{\alpha_0}$  for some  $V_{\alpha_0} \in I$ . Therefore, we obtain  $x/V_{\alpha_0} \in \beta_g O(X, x)$ ,  $Int_{\beta_g}(V_{\alpha_0}) \in F$  and  $Cl_{\beta_g}(V_{\alpha_0}) \cap Int_{\beta_g}(V_{\alpha_0}) = \emptyset$  which is a contradiction.

**(d)  $\rightarrow$  (a):** Let  $\{V_\alpha : \alpha \in I\}$  be a cover of  $X$  by  $\beta_g$ -open sets. Then,  $\{X/V_\alpha : \alpha \in I\}$  is a family of  $\beta_g$ -closed subset, so  $fX$  such that  $\cap\{X/V_\alpha : \alpha \in I\} = \emptyset$ . By (d), there exists a finite subset  $I_0$  of  $I$  such that  $\cap\{Int_{\beta_g}(X/V_\alpha) : \alpha \in I_0\} = \emptyset$  hence  $X = \cup\{Cl_{\beta_g}(V_\alpha) : \alpha \in I_0\}$ . This shows that  $X$  is  $\beta_g$ -closed.

**Theorem 2.13:** A topological space  $X$  is  $\beta_g$ -closed if and only if every grill on  $X$  is  $\beta_g(\theta)$ -convergent in  $X$ .

**Proof:** Let  $g$  be any grill on a  $\beta_g$ -closed space  $X$ . Then by Theorem 2.6,  $\text{sec } g$  is a filter on  $X$ . Let  $B \in \text{sec } g$ , then  $X/B \neq \emptyset$  and hence  $B \in g$  (as  $g$  is a grill). Thus,  $\text{sec } g \subseteq g$ . Then by Theorem 2.6 (b), there exists an ultrafilter  $u$  on  $X$  such that  $\text{sec } g \subseteq u \subseteq g$ . Now as  $X$  is  $\beta_g$ -closed in view of Theorem 3.2, the ultrafilter  $u$  is  $\beta_g(\theta)$ -convergent to some point  $x \in X$ . Then for each  $U \in (\beta_g O(X, x))$ , there exists  $F \in u$  such that  $F \subseteq Cl_{\beta_g}(U)$ . Consequently,  $Cl_{\beta_g}(U) \in u \subseteq g$ , that is  $Cl_{\beta_g}(U) \in g$ , for each  $U \in \beta_g O(X, x)$ . Hence,  $g$  is  $\beta_g(\theta)$ -convergent to  $x$ . Conversely, let every grill on  $X$  be  $\beta_g(\theta)$ -convergent to some point of  $X$ . By virtue of Theorem 3.2 it is enough to show that every ultrafilter on  $X$  is  $\beta_g(\theta)$ -convergent in  $X$  which is immediate from the fact that an ultrafilter on  $X$  is also a grill on  $X$ .

**Theorem 2.14:** A topological space  $X$  is  $\beta_g$ -closed relative to  $X$  if and only if every grill  $g$  on  $X$  with  $A \in g$  to a point in  $A$ .

**Proof:** Let  $A$  be  $\beta_g$ -closed relative to  $X$  and  $g$  a grill on  $X$  satisfying  $A \in g$  such that  $g$  does not  $\beta_g(\theta)$ -convergent to any  $a \in A$ . Then to each  $a \in A$ , there corresponds some  $U_a \in \beta_g O(X, a)$  such that  $Cl_{\beta_g}(U_a) \notin g$ . Now  $\{U_a : a \in A\}$  is a cover of  $A$  by  $\beta_g$ -open sets of  $X$ . Then,  $A \subseteq \cup_{i=1}^n Cl_{\beta_g}(U_{a_i}) = U$  (say) for some positive integer  $n$ . Since  $g$  is a grill,  $U \in g$ ; hence  $A \in g$  which is a contradiction.

Conversely, let  $A$  be not  $\beta_g$ -closed relative to  $X$ . Then for some cover  $u = \{U_\alpha : \alpha \in A\}$  of  $A$  by  $\beta_g$ -open sets of  $X$ ,  $F = \{A/U_{\alpha \in I_0} Cl_{\beta_g}(U_\alpha) : I_0 \text{ is finite subset of } I\}$  is a filter base on  $X$ . Then, the family  $F$  can be extended to an ultra filter  $F^*$  on  $X$ . Then  $F^*$  is a grill on  $X$  with  $A \in F^*$  (as each  $F$  of  $F$  is a subset of  $A$ ). Now for each  $x \in A$ , there must exist  $\beta \in I$  such that  $x \in U_\beta$  as  $U$  is a cover of  $A$ . Then for any  $G \in F^*$ ,  $G \cap (A/Cl_{\beta_g}(U_\beta)) \neq \emptyset$ , so that  $G \supseteq Cl_{\beta_g}(U_\beta)$  for all  $G \in g$ . Hence,  $F^*$ , cannot  $\beta_g(\theta)$ -convergent to any point of  $A$ . The contradiction proves the desired result.

### CONCLUSION

By using generalized closed sets and topology, new class of sets in topological spaces namely  $\beta$ -generalized closed sets and  $\beta$ -generalized open sets have been introduced and some of their properties are investigated. Various functions, namely  $\beta$ -continuous functions, almost contra  $\beta$ -continuous functions in topological spaces and also introduces a new spaces like  $\beta_g(\theta)$ -convergence and  $\beta_g(\theta)$ -adherence in grill topological spaces have been defined and their characteristics are investigated.

**REFERENCES**

- Andrijevic, D., 1986. Semi-preopen sets. *Mat. Vesnik*, 38: 24-32.
- Choquet, G., 1947. Theory of sets on-the terms and filter grid. *Weekly Rep. Meetings Acade. Sci.*, 224: 171-173.
- Kannan, K. and N. Nagaveni, 2012. On-generalized closed sets and open sets in topological spaces. *Int. J. Math. Anal.*, 6: 2819-2828.
- Levine, N., 1963. Semi-open sets and semi-continuity in topological spaces. *Amer. Math. Monthly*, 70: 36-41.
- Njastad, O., 1965. On some classes of nearly open sets. *Pac. J. Math.*, 15: 961-970.
- Sheik, M.J., 2000. On w-closed sets in topology. *Acta India Sci.*, 4: 389-392.
- Stone, M.H., 1937. Applications of the theory of Boolean rings to general topology. *Trans. Am. Math. Soc.*, 41: 375-481.