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Numerical Solution of Hallen's Integral Equation by the Chebyshev Pseudospectral Method

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Abstract: This study present a numerical method for solving Hallen's integral equation based on Chebyshev pseudospectral method. The method consists of representing the solution of the Hallen's integral equation by Nth degree interpolating polynomial, using Chebyshev nodes and then discritizing the problem using a cell-averaging technique. Properties of Chebyshev pseudospectral method are presented, then utilize to reduce the computation of Hallen's integral equation to some algebraic equation. The method computationally attractive and applications are demonstrate through an illustrative example.

Key words: Hallen's integral equation, pseudospectral method, Chebyshev nodes, cell-averaging technique

INTRODUCTION

The class of solution methods based on orthogonal polynomials have become known as spectral methods. Spectral methods are implemented in various ways. For example, the tau, Galerkin and collocation methods have all been proposed as implementation strategies (Gottlieb and Orszag, 1989; Foruberg, 1996). The collocation method (also known as the pseudospectral method) has established itself as the one that permits the most convenient computer implementation. However, in pseudospectral methods the nodes must correspond to the zeros of the derivatives of classical orthogonal polynomials on the interval [-1,1], including the end points. These points are generally based on the Legendre or Chebyshev polynomials.

Hallen (1965) wrote his famous integral equation to give an exact treatment of antenna current wave reflection at the end of the tube shaped cylindrical antenna in 1956, but his first research on this subject (Hallen, 1938) probably goes back to 1938. This equation enabled him to show that on thin wire the current distribution is approximately sinusoidal and propagates with nearly the speed of the light. The Hallen integral equation is a Fredholm integral equation of the first kind.

This equation for the thin-wire cylindrical antenna of length l and radius a with a « l is given by

$$\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} K(x',y')I(y')dy' = \frac{j}{2\zeta_0}V\sin(\beta|x'|) + A\cos(\beta|x'|) \tag{1}$$

There are two choices of $K(x', y')$. The 2 kernels are usually referred to as the exact and the approximate or reduced kernel. With the approximate kernel, the integral equation has no solution.

Nevertheless, the same numerical method is often applied to both forms of the integral equation (Fikioris and Wu, 2001). The approximate kernel, which we used in this study, is given by

$$K(x',y') = \frac{1}{4\pi}\frac{e^{-j\beta\sqrt{(x'-y')^2+a^2}}}{\sqrt{(x'-y')^2+a^2}} \tag{2}$$

In Eq. 1 and 2 and $\zeta_0 = 120\pi$ and $\beta = 2\pi/\lambda$ is the free wavenumber where λ is wavelenght, $I(y')$ is the current, V is the driven voltage and A is a constant to be determined from the conditions $I(-\ell/2) = I(\ell/2) = 0$.

CHEBYSHEV PSEUDOSPECTRAL METHOD

Proposed method: The Chebyshev pseudospectral method is one special case of more general class of spectral methods. The basic formulation of these methods involves two essential steps: one is to choose a finite-dimensional space (usually a polynomial space) from which on approximation to the solution of differential equation is made. The other step is to choose a projection operator, which imposes the differential equation in the finite-dimensional space. One important feature of spectral methods, is that the underlying polynomial space is spanned be orthogonal polynomials that are infinitely differentiable global function. Among of these orthogonal polynomial are Legendre and Chebyshev polynomials which are orthogonal on the interval [-1,1], whith respect to an appropriate weight function ($w(x) = 1$ for Legendre polynomial and $w(x) = (1-x^2)^{-½}$ for Chebyshev

polynomial). Let P_N denote the space of algebraic polynomial of degree «N and let $T_m(x)$, m» 0, $-1 \le x \le 1$, denote the orthogonal family of Chebyshev polynomial of the first kind in this space, with respect to the weight function $w(x) = (1-x^2)^{-1/2}$.
We choose the grid points (interpolation) to be

$$x_j = \cos\left(\frac{j\pi}{N}\right), \qquad j = 0, 1, \cdots, N \tag{3}$$

of the Nth order Chebyshev polynomials $T_N(x)$, $x \in [-1, 1]$. These points are $x_N = -1 < x_N - 1 < \ldots < x_1 < x_0 = 1$, also views as the zeros of $(1-x^2)T_N(x)$, where $T_N(x) = dT_{N(x)}/dx$.

In order to construct the interpolation of a function f(x) at the point $x \in [-1, 1]$ and k = 0, 1, • • • , N, we define the following Lagrange polynomials

$$\phi_k(x) = \frac{(-1)^{k+1}\left(1-x^2\right)\dot{T}(x)}{c_k N^2\left(x - x_k\right)} = \frac{2}{c_k}\sum_{j=0}^{N}\frac{T_j(x_k)T_j(x)}{c_j} \tag{4}$$

Which $c_0 = c_N = 2$ and $c_j = 1$, j = 1, 2, • • • ,N - 1.
It is readily verified that

$$\phi_k\left(x_j\right) = \delta_{jk} \tag{5}$$

Associated with the N+1 Chebyshev nodes (grid points), is a unique Nth-degree interpolating polynomial (projection operator) $I_N f(x)$

$$I_N f(x) = \sum_{j=0}^{N}\phi_j(x) f\left(x_j\right) \tag{6}$$

Such that $I_N f(x_k) = f(x_k)$, k = 0, 1, • • • ,N. Alternatively, the interpolating polynomial $I_N f(x)$ can be expressed in terms of series expansion of the classical Chebyshev polynomials

$$I_N f(x) = \sum_{j=0}^{N} T_j(x)\hat{F}\left(x_j\right) \tag{7}$$

Where

$$\hat{F}\left(x_j\right) = \frac{2}{Nc_j}\sum_{r=0}^{N}\frac{T_r\left(x_j\right)f\left(x_r\right)}{c_r} \tag{8}$$

It is well known that the spectral projection operators, such as I_N, based on Chebyshev nodes x_j are well behaved compared to those based on equally grid points (Elnager and Kezemi, 1998). Clearly IN is a linear projection operator on C[-1, 1], the banach space of continuous, real-valued function on [-1, 1].

Discretization of integral: We shall use the cell-averaging Chebyshev integration rule (Elnager and Kezemi, 1998). This rule states that there exists an N × (N + 1) matrix R_{jk}, $1 \le j \le N$, $0 \le k \le N$ such that for all $f \in C^r[-1, 1]$, r > 0, we have

$$\int_{-1}^{1} f(x)dx = \sum_{j=1}^{N}\int_{x_j}^{x_{j-1}} f(x)dx = \sum_{j=1}^{N}\left(x_{j-1} - x_j\right)\hat{f}_{j-\frac{1}{2}} = \sum_{j=1}^{N}\left(x_{j-1} - x_j\right)\sum_{k=0}^{N} R_{jk} f\left(x_k\right) \tag{9}$$

Now let that

$$w_k = \sum_{j=1}^{N}\left(x_{j-1} - x_j\right)R_{jk} \tag{10}$$

then the Eq. 9 can be written as

$$\int_{-1}^{1} f(x)dx = \sum_{k=0}^{N} w_k f\left(x_k\right) \tag{11}$$

Where the cell-averages

$$\hat{f}_{\frac{1}{2}}, \hat{f}_{\frac{3}{2}}, \cdots, \hat{f}_{\frac{2N-1}{2}}$$

are related to $f(x_0)$, $f(x_1)$, • • • , $f(x_N)$ throught the matrix R_{jk}, $1 \le j \le N$, $0 \le k \le N$. The entries of the cellaveraging matrix R_{jk}, $1 \le j \le N$, $0 \le k \le N$ are given by

$$R_{jk} = g_k\left(x_{j-\frac{1}{2}}\right) \tag{12}$$

Where

$$x_{j-\frac{1}{2}} = \cos\left(\frac{\left(j-\frac{1}{2}\right)\pi}{N}\right) \tag{13}$$

and also we have that

$$g_k(x) = \frac{1}{Nc_k}\left[\begin{array}{l} 1 + \sigma_1 T_1\left(x_k\right)U_1(x) + \\ \sum_{r=2}^{N}\frac{T_r(x)\left[\sigma_r U_r(x) - \sigma_{r-2}U_{r-2}(x)\right]}{c_r}\end{array}\right] \tag{14}$$

Which

$$\text{Sigma}_r = \frac{\sin\left(\frac{r+1}{2N}\pi\right)}{(r+1)\sin\left(\frac{\pi}{2N}\right)}, \qquad U_r(x) = \frac{1}{r+1}\dot{T}_{r+1}(x) \tag{15}$$

DISCRETIZATION OF HALLEN'S INTEGRAL EQUATION

First, we introduce the transformations x′ = ℓ/2x and y′ = ℓ/2y. The Hallen's integral equation and the condition I (-ℓ/2) = I (ℓ/2) = 0 may be writing as follows

$$\frac{\ell}{2}\int_{-1}^{1} K\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right)I(y)\,dy = f(x), \quad -1 < x < 1 \qquad (16)$$

and

$$I(-1) = I(1) = 0 \qquad (17)$$

Where

$$f(x) = \frac{j}{2\zeta_0}V\sin\left(\beta\left|\frac{\ell}{2}x\right|\right) + A\cos\left(\beta\frac{\ell}{2}x\right), -1 < x < 1$$

When x = y, the kernel in Eq. 16 is sharply peaked, particulary for small value of a. Therefore, from the computational point of view, it would be advantageous to isolate and extract the singularity from kernel. This may be accomplish by writing K (ℓ/2x, ℓ/2y)as

$$K\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right) = K_n\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right) + K_s\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right) \qquad (18)$$

Where K_n (ℓ/2x, ℓ/2y) and K_s (ℓ/2x, ℓ/2y) denote the nonsingular and singular parts of kernel K, respectively and are given in Fornberg (1996) as

$$K_n\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right) = \frac{1}{4\pi}\frac{e^{-j\beta\sqrt{\left(\frac{\ell}{2}x-\frac{\ell}{2}y\right)^2+a^2}}}{\sqrt{\left(\frac{\ell}{2}x-\frac{\ell}{2}y\right)^2+a^2}} \qquad (19)$$

$$K_s\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right) = \frac{1}{4\pi}\frac{1}{\sqrt{\left(\frac{1}{2}x-\frac{1}{2}y\right)^2+a^2}} \qquad (20)$$

By using Eq. 18 we can express Eq. 16 as

$$\frac{\ell}{2}\int_{-1}^{1} K_s\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right)I(y)\,dy + \frac{\ell}{2}\int_{-1}^{1} K_n\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right)I(y)\,dy = f(x), \ -1 < x < 1 \qquad (21)$$

The integrand of the first integral in Eq. 21 is well behaved and as a consequence may be evaluated numerically. The integrand of the second integral in Eq. 21 cantain a singularity and will be evaluated as follows. Let

$$\frac{\ell}{2}\int_{-1}^{1} K_s\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right)I(y)\,dy = S_1(x) + S_2(x) \qquad (22)$$

Where

$$S_1(x) = \frac{\ell}{2}\int_{-1}^{1} K_s\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right)\left(I(y) - I(x)\right)dy \qquad (23)$$

and

$$S_2(x) = I(x)\int_{-1}^{1} K_s\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right)dy \qquad (24)$$

The integrand of the integral in Eq. 23 is well behaved and the integral in Eq. 24 can be evaluated as

$$H(x) = \int_{-1}^{1} K_s\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right)dy = \frac{1}{4\pi l}\ln\left(\frac{\sqrt{(lx-l)^2+4a^2}+lx-l}{\sqrt{(lx-l)^2+4a^2}-lx-l}\right) \qquad (25)$$

In view of Eq. 18-25, 13 is expressed by

$$\frac{\ell}{2}\int_{-1}^{1} K_n\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right)I(y)\,dy + \frac{\ell}{2}\int_{-1}^{1} K_s\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right)\left(I(y)-I(x)\right)dy + \frac{\ell}{2}I(x)H(x) = f(x), -1 < x < 1 \qquad (26)$$

Now we discretize Eq. 24 by using Chebyshev pseudospectral method. For this propose, we use Eq. 6 to approximate

$$I(y) = \sum_{j=0}^{N}\phi_j(x)I(y_j) \qquad (27)$$

RESULTS AND DISCUSSION

Where y_j and ϕ_j (y) for $0 \le j \le N$, respectively are given Eq. 3 and 4. A collocation scheme is defined by substituting Eq. 25 into 24 and evaluating the result at the points x_k for $0 \le k \le N$ given in (3). This give we

$$\frac{\ell}{2}\sum_{j=0}^{N}\int_{-1}^{1} K_n\left(\frac{\ell}{2}x_k,\frac{\ell}{2}y\right)\phi_j(y)I(y_j)\,dy + \frac{\ell}{2}\sum_{j=0}^{N}\int_{-1}^{1} K_s\left(\frac{\ell}{2}x_k,\frac{\ell}{2}y\right)\left(\phi_j(y)I(y_j) - I(x_k)\right)dy + \frac{\ell}{2}I(x_k)H(x_k) = f(x_k), k = 0,1,\cdots,N \qquad (28)$$

Where we used $I(x_N) = 0$ $I(x_j) = I_j$ for $0 \le j \le N$. Furthermore, since the integrals in Eq. 28 are well

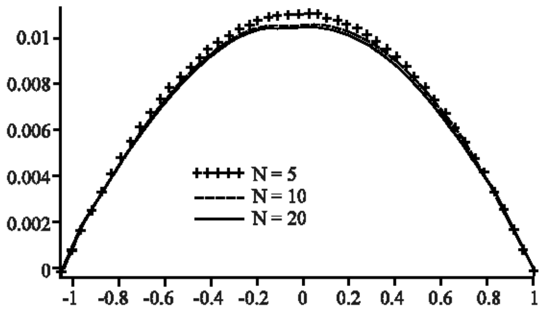

Fig. 1: The magnitude current I(y) for l = λ/2

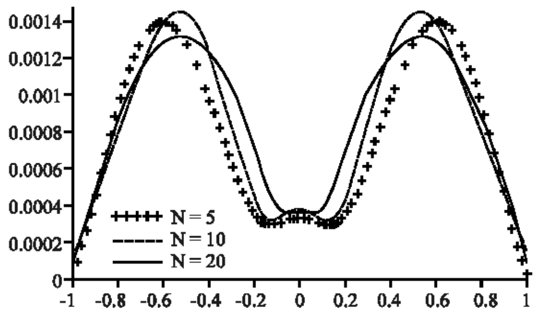

Fig. 2: The magnitude current I(y) for l = λ/2

behaved, by using cell-averaging approximation of integrals in Eq. 11, we approximate the Eq. 28 as follows

$$\frac{1}{2}\sum_{j=0}^{N} I_j w_j K_n\left(\frac{1}{2}x_k, \frac{1}{2}y_j\right) + \frac{1}{2}\sum_{j=0}^{N}\left(I_j - I_k\right) w_j K_s\left(\frac{1}{2}x_k, \frac{1}{2}y_j\right)$$

$$\frac{1}{2} I_k H\left(x_k\right) = f\left(x_k\right),\ k = 0,1,\cdots,N \qquad (29)$$

By solving the system of linear Eq. 27, we can find I_j for j = 0, 1, • • • , N- 1.

Numerical results: In this study, a numerical example is represented to illustrate the validity and the merits of this technique. In this example data are given for two selected wire length so that they include special cases of practical interest, e.g l λ/2 and l = λ. The magnitude of currents I(y) are shown for N = 5, 10 and 20 in Fig. 1 and 2, respectively for and. We can see in Fig. 1 and 2 that by increasing the values of N the solution converges rapidly.

CONCLUSION

In this study, we have investigated the application of Chebyshev pseudospectral method for the solution of Hallen's integral equation. The results given, show the superiority of this method in comparison with other methods.

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