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First Order Difference Equation Implementation for Image Filtering

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Abstract: This study introduces an image filtering algorithm for degraded images with Gaussian noise. The stability, observability and controllability concepts from automatic control are formalized for evaluating image quality. The implementation of the developed algorithm is mainly based on first order difference equation. The proposed scheme is applied to gray scale and color degraded images by Gaussian noise. We also give a comparison between the proposed scheme and lowpass filtering technique. The results indicate that our method gives good performance with smoothing the background noise while preserving the edges and the fine details with less bullring, for gray scale and color images rather than the performance of lowpass filter.

Key words: Control system, controllability, gaussian noise, image filtering, observability, stability

INTRODUCTION

Control systems are currently extensively used. The basic ingredients of a control system can be described by objectives of a control, control system components and results or outputs[1,2]. In general, the goal of a control system is to control the outputs in some prescribed manner. One of the most important tasks in the analysis and design of control systems is the mathematical modelling of the systems, this is based on analytical techniques, experimental techniques and combined techniques. There are different forms of mathematical models that describe the control system such as static functional relations, ordinary linear differential equations, non linear differential equations, partial differential equations, difference equations and state space equations[1].

The image processing is an interesting topic of study, because of the variety of applications that use image processing or analysis techniques such as radar, remote sensing, microscopy, medical imaging and oceanography. Image enhancement, is the task of applying certain transformations to input such as obtaining a visually pleasant, more detail, or less noisy output image. Images can be enhanced by reducing the noise that may be present using image filters that can be defined as mathematical algorithms implemented in software to the entire data set to remove noise and background artifacts[3-5].

A type of separation principle in image processing and control technique fields has been applied, where the control techniques and image processing aspects of a problem are treated independently[6]. Image processing may not be independently enough to provide information on industrial processes, but together with other available sensor measurements, such as controllability, observability and stability, it could be a vital part of control system design.

There are recent but few publications on automatic control systems for image processing. P.R. Roesser[7] generalized the linear-discrete state-space model from single-dimensional time to two-dimensional space. In other words, Ito, *et al.*[8] developed real time and accurate filters for nonlinear filtering problems based on the Gaussian distributions. However, Dey, *et al.*[9] introduced finite-dimensional optical risk-sensitive filters and smoothers for discrete-time non linear-systems. Munteanu, *et al.*[10] introduced a new automatic image enhancement technique driven by an evolutionary optimization process. Also, they proposed a new objective criterion for enhancement and attempt finding the best image according to the respective criterion.

In practice, the input-output relation of a system with discrete-data is often described by a difference equation. The implementation of second order difference equation for image filtering was introduced[11]. So, this study provides a method for image filtering based on first order difference equation rather than second, third and fourth order difference equations that need much more elaboration and most of the time the solution is multivalued which makes choice rather dubious. The automatic control concepts such as stability, observability and controllability are used to evaluate the quality of the processed image.

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Continuous-data control system for image filtering: The degradation image can be divided into two components:

- Undesired components such as noise
- Desired components such as edges, features and textures. The desired components require to increase its contrast. Then the degradation image can be expressed by the following relation:

$$D = C+N \tag{1}$$

Where D is the degraded image, C is the desired component and N is the noise component. Assuming the image in each state of enhancement is represented as a function of C, N and t. Such that t represents the time domain. Also, C and N are functions of t. Figure 1 shows the desired and undesired components of the degraded image put together as parallel inputs to the control system.

Assumning the relationship between the desired component (input 1) and the processed image P (output) is represented by the following n_1th order differentional equation with constant real coefficients.

$$\frac{d^{n_1}P(t)}{dt^{n_1}} + \alpha_{n_1-1}\frac{d^{n_1-1}P(t)}{dt^{n_1-1}} + \ldots + \alpha_1\frac{dP(t)}{dt} + \alpha_0 P(t) = \frac{d^{m_1}C(t)}{dt^{m_1}} + a_{m_1-1}\frac{d^{m_1-1}C(t)}{dt^{m_1-1}} + \ldots + a_1\frac{dC(t)}{dt} + a_0 C(t) \tag{2}$$

Such $n_1>m_1$ that the transfer function of the system is defined as the Laplace transform of the impulse response, with all the initial conditions set to zero through the following relation:

$$\phi_1(s) = \frac{P(s)}{C(s)} \tag{3}$$

Where P(s) and C(s) are the Laplace transforms of P(t) and C(t), respectively. To obtain the transfer function of the system represented by Eq. 2, we simply take the Laplace transform for both sides of Eq. 2 and assuming zero initial conditions. Then the result is

$$(s^{n_1} + \alpha_{n_1-1}s^{n_1-1} + \ldots + \alpha_1 s + \alpha_0)P(s) = (a_{m_1}s^{m_1} + a_{m_1-1}s^{m_1-1} + \ldots + a_1 s + a_0)C(s) \tag{4}$$

Then, the transfers function between P(t) and C(t) is given by

$$\phi_1(s) = \frac{a_{m_1}s^{m_1} + a_{m_1-1}s^{m_1-1} + \ldots + a_1 s + a_0}{s^{n_1} + \alpha_{n_1-1}s^{n_1-1} + \ldots + \alpha_1 s + \alpha_0} \tag{5}$$

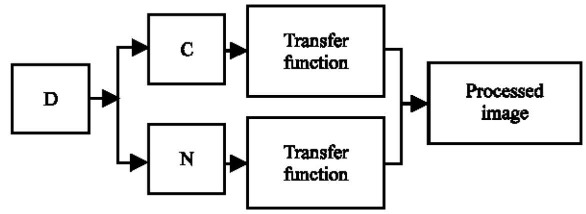

Fig. 1: The control system for image filtering

In the same manner, assuming the relation between the noise component (input 2) and the processed image P (output) is represented by the following n_2th order differential equation with constant real coefficients.

$$\frac{d^{n_2}P(t)}{dt^{n_2}} + \beta_{n_2-1}\frac{d^{n_2-1}P(t)}{dt^{n_2-1}} + \ldots + \beta_1\frac{dP(t)}{dt} + \beta_0 P(t) = \frac{d^{m_2}N(t)}{dt^{m_2}} + b_{m_2-1}\frac{d^{m_2-1}N(t)}{dt^{m_2-1}} + \ldots + b_1\frac{dN(t)}{dt} + b_0 N(t) \tag{6}$$

Such that $n_2>m_2$ then, the transfer function can be expressed as follows

$$\phi_2(s) = \frac{P(s)}{N(s)} \tag{7}$$

Where N(s) is the Laplace transform of N(t). By taking the Laplace transform for both sides of Eq. 6 and assuming zero initial conditions. Then the result is

$$(s^{n_2} + \beta_{n_2-1}s^{n_2-1} + \ldots + \beta_1 s + \beta_0)P(s) = (b_{m_2}s^{m_2} + b_{m_2-1}s^{m_2-1} + \ldots + b_1 s + b_0)N(s) \tag{8}$$

The transfer function between P(t) and N(t) is given by

$$\phi_2(s) = \frac{b_{m_2}s^{m_2} + b_{m_2-1}s^{m_2-1} + \ldots + b_1 s + b_0}{s^{n_2} + \beta_{n_2-1}s^{n_2-1} + \ldots + \beta_1 s + \beta_0} \tag{9}$$

Thus, the two components are parallel connection, then the transfer function for all control system can be written as

$$\phi(s) = \sum_{i=1}^{2}\phi_i(s) \tag{10}$$

and

$$\phi\ (s) = \frac{(v_g s^g + \ldots + vs + v_0) + (u_q s^q + \ldots + us + u_0)}{s^n + h_{n-1}s^{n-1} + \ldots + h_1 s + h_0} \tag{11}$$

Where $n = n_1+n_2$, $g = m_1+n_2$, $q = m_2+n_1$ and $g,q<n$ However, h_i, v_j and u_k are functions of α and β. $i = 0,1,\ldots, n-1$, $j= 0,1,\ldots, g$ and $k = 0,1,\ldots,q$. The characteristic equation

of a system is defined as the equation obtained by setting the denominator polynomial of the transfer function to zero. Then the characteristic equation of the system is described as

$$s^n + h_{n-1}s^{n-1} + \ldots h_1 s + h_0 = 0 \quad (12)$$

One of the foremost important considerations is the state-variable formulation. The basic characteristic of the state-variable formulation is that linear and nonlinear systems, time-invariant and time-varying systems and single-variable and multivariable systems can all be modeled in a uniform manner. The n state equations of nth order dynamic system can be represented as

$$\frac{dx_i(t)}{dt} = y_i[x_1(t), x_2(t), \ldots x_n(t), C(t), N(t)] \quad (13)$$

Where i = 0,1...,n. The ith state variable is represented by $x_i(t)$. However, the output variables are function of the state variables and the input variables. The output equations can be expressed as:

$$p = g[x_1(t), x_2(t), \ldots x_n(t), C(t), N(t)] \quad (14)$$

The state Eq. 15 and the output Eq. 16 as sets are called the dynamic equations of the system and are written as:

$$\frac{dX(t)}{dt} = HX(t) + EC(t) + FN(t) \quad (15)$$

$$p(t) = QX(t) + rC(t) + wN(t) \quad (16)$$

It is convenient to represent the dynamic equations in vector-matrix form such that X(t) represents the sate vector, H, E , F and Q are the following matrices:

$$H = \begin{bmatrix} h_{11} & h_{12} & .. & h_{1n} \\ h_{21} & h_{22} & .. & h_{2n} \\ . & . & . & . \\ . & . & . & . \\ h_{n1} & h_{n2} & .. & h_{nn} \end{bmatrix}, \quad X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ . \\ x_n(t) \end{bmatrix} \quad E = \begin{bmatrix} e_1 \\ e_2 \\ . \\ . \\ e_{n_1} \\ 0 \\ . \\ . \\ 0 \end{bmatrix},$$

$$F = \begin{bmatrix} f_1 \\ f_2 \\ . \\ . \\ f_{n_2} \\ 0 \\ . \\ . \\ 0 \end{bmatrix}, \quad Q = [q1 \quad q2 \quad \ldots \quad q_n]$$

and r, w are scalars.

Image Filtering Controllability Canonical Form (IFCCF): Consider the dynamic equations given by Eq. 15 and 16. The characteristic equation of His

$$|sI - H| \equiv s^n + h_{n-1}s^{n-1} + \ldots + h_1 s + h_0 = 0 \quad (18)$$

The dynamic equations are transformed into IFCCF according to the following transformations

$$\begin{aligned} &\bar{H} = D^{-1}HD, \quad \bar{E} = D_1^{-1}E, \quad \bar{F} = D_2^{-1}F, \\ &\bar{Q} = QD, \quad \bar{r} = r, \quad \bar{w} = w \end{aligned} \quad (19)$$

where

$$\begin{aligned} &D = SM, \quad S = S_1 + S_2, \\ &D_1 = S_1 M, \quad D_2 = S_2 M \end{aligned} \quad (20)$$

such that,

$$S_1 = [E \quad HE \quad H^2E \quad \ldots \quad H^{n-1}E] \quad (21)$$

$$S_2 = [F \quad HF \quad H^2F \quad \ldots \quad H^{n-1}F] \quad (22)$$

$$M = \begin{bmatrix} h_1 & h_2 & .. & h_{n-1} & 1 \\ h_2 & h_3 & .. & 1 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ h_{n-1} & 1 & 0 & .. & 1 \\ 1 & 0 & 0 & .. & 0 \end{bmatrix} \quad (23)$$

The system for image filtering is said to be controllable, if the controllability matrix S with size n×n is of rank n.

Image Filtering Observability Canonical Form (IFOCF): A dual form of the transformation of (IFCCF) is the image filtering observability canonical Form (IFOCF) and this transformation is driven as

$$\bar{H} = K^{-1}HK, \quad \bar{E} = K^{-1}E, \quad \bar{F} = K^{-1}F, \quad \bar{Q} = QK, \quad \bar{r} = r, \quad \bar{w} = w \tag{24}$$

where

$$K = (MV)^{-1} \tag{25}$$

and

$$V = \begin{bmatrix} Q \\ QH \\ . \\ . \\ . \\ QH^{n-1} \end{bmatrix} \tag{26}$$

The system for image filtering is said to be observable, if the observable matrix Vwith size n×n is of rank n.

The most important requirement for the system is that it should be stable. An unstable system is generally considered to be useless. Basically, the design of control systems may be regarded as a problem of arranging the location of poles and zeros of the system transfer function such that the system will perform according to prescribed specifications. For stability, the roots of the characteristic equation must all lie in the left-half s-plane. Figure 2 shows stable and unstable regions in the s-plane.

Discrete-data control system for image filtering: For discrete-data systems modeled by difference equations, the transfer function is a function of z when the z-transform is used. Assuming that a single-variable linear digital or discrete-data system is described by the following nth order difference equation.

$$p(k+n) + \lambda_{n-1}p(k+n-1) + ... + \lambda_1 p(k+1) + \lambda_0 p(k) = \beta \tag{27}$$

Where β and the coefficients λ_j, j = 0,1,..n, are real constants. Since one of the foremost important considerations in control system design is a state-variable formulation. The problem is to represent Eq. 27 by n state equations and q output equations. The first step involves the defining of state variables as function p(k). Then the state variables of Eq. 27 is

$$\begin{aligned} x_1(k) &= p(k) \\ x_2(k) &= p(k+1) = x_1(k+1) \\ &. \\ &. \\ x_n(k) &= p(k+n-1) \end{aligned} \tag{28}$$

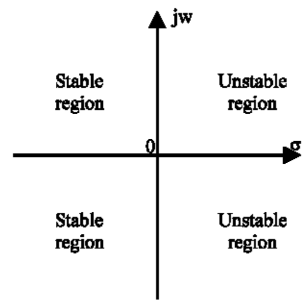

Fig. 2: Stable and unstable regions in the s-plane, where s = σ+jω

After substitution of the relations in Eq. 27 into Eq. 28 and rearranging, the state equations are written as

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= x_3(k) \\ &. \\ &. \\ x_n(k+1) &= -\lambda_0 x_1(k) - \lambda_1 x_2(k) - ..... - \lambda_{n-1} x_n(k) + \beta \end{aligned} \tag{29}$$

And the output equation is simply

$$p(k) = x_1(k) \tag{30}$$

The dynamic equations of the system are written in vector-matrix form as

$$X(k+1) = AX(k) + B\beta \tag{31}$$

$$p(k) = CX(k) \tag{32}$$

The coefficient matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & ... & 1 \\ -\lambda_0 & -\lambda_1 & -\lambda_2 & ... & -\lambda_{n-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \end{bmatrix}$$

Taking the z-transform on both sides of Eq. 31 and solving for X(z), we get

$$X(z) = (zI - A)^{-1} zX(0) + (zI - A)^{-1} B\beta \tag{33}$$

Substituting Eq. 33 into the z-transform of Eq. 32 yields

$$P(z) = C(zI - A)^{-1}zX(0) + C(zI - A)^{-1}BF(z) + D\beta \quad (34)$$

To get the transfer function, we assume that the initial state X(0) is a null matrix; thus, Eq. 34 becomes

$$P(z) = [C(zI - A)^{-1}B + D]\beta \quad (35)$$

Then the z-transfer function matrix of the system may be written as

$$T(z) = C(zI - A)^{-1}B + D \quad (36)$$

In this section the image filtering is modeled based on first order difference equation as follows

$$p(k+1) + \lambda_0 p(k) = \beta \quad (37)$$

Equation 37 is a linear first order nonhomogenous difference equation. The values of λ_0 and β are computing by solving the following equations

Taking the summation on both sides of Eq. 37

$$\sum_{k=1}^{N-1} p(k+1) + \lambda_0 \sum_{k=1}^{N-1} p(k) = \beta(N-1) \quad (38)$$

Multiply Eq. 37 by p(k+1) and take the summation

$$\sum_{k=1}^{N-1} (p(k+1))^2 + \lambda_0 \sum_{k=1}^{N-1} p(k+1)p(k) = \beta \sum_{k=1}^{N-1} p(k+1) \quad (39)$$

Where N is the number of column in the image. The processed image is obtained by solving the difference Eq. 37 So, it is applied row by row for gray scale images and color images as shown in the following case study. However, the automatic control concepts such as stability, observability and controllability are used to evaluate the image quality.

By computing the values of of λ_0 and β from the noisy image in Fig. 3 using Eq. 38 and 39 then the difference equation becomes

$$p(k+1) + 3.54p(k) = 0.009 \quad (40)$$

The transfer function is

$$T(z) = \frac{1}{z + 3.54}$$

Fig. 3: The result for gray scale image

Fig. 4: The result for color image

The controllability matrix is s = [1] where rank r(s) =1 then the system is controllable. However, The observability matrix is v = [1] where rank r(v) = 1, then the system is observable.

The characteristic equation is

$$R(z) = z + 3.54 = 0$$

And the root of R(z) are z1= -3.54 that is in the left-half of z-plane. Then the system is stable.

By computing the values of of λ_0 and β from the noisy image in Fig. 4 using Eq. 38 and 39 then the difference equation becomes

$$p(k+1) + 4.9p(k) = 0.0075 \quad (41)$$

The transfer function is

$$T(z) = \frac{1}{z + 4.9}$$

The controllability matrix is s = [1] where rank r(s) =1 then the system is controllable. However, The observability matrix is v = [1] where rank r(v) =1, then the system is observable.

The characteristic equation is

$$R(z) = z + 4.9 = 0$$

Table 1: Comparison between the processed image and lowpass filter

Image	Lowpass	Proposed method
Gray scale	0.9887	0.9953
Color	0.97989	0.9815

And, the root of R(z) are z1= -4.9 that is in the left-half of z-plane. Then the system is stable.

In order to illustrate the performance of the proposed method, it is compared with the result of lowpass filter by computing the correlation between processed image and original image as shown in Table 1. The correlation value of lowpass filter is the average between different types of lowpass filter.

CONCLUSION

In this study a scheme based on first order difference equation is introduced for image filtering. The results indicate that the method gives good performance with smoothing the background noise while preserving the edges and the fine details with less bullring, for gray scale and color images degraded with Gaussian noise rather than the performance of lowpass filter. The automatic control concepts such as stability, observability and controllability are used to analyze and evaluate the system for the image quality. The results indicate that the system was controllable, observable and stable. However, the possible extension of this study is to apply in three-dimensional images and old images that are often deteriorated due aging or improper handling.

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