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Implementation of an Order Seven Self-Starting Multistep Methods Using Scilab and Fortran Codes

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Abstract: Self-starting methods such as the Runge-Kutta methods are often used for the solution of engineering problems on a computer because they need no special starting method and are therefore easy to program. Also the Runge-Kutta methods give an automatic procedure for adjusting the step-size, which again makes for easy use by a non-specialist. However, there are three major disadvantages of the Runge-Kutta methods. An important disadvantages of this method is that, the form of the error term is extremely complicated and can not easily be used for the estimation of the truncation error or determine a suitable step-size. Another undesirable feature of this method is the large number of derivative evaluation required per-step, this makes it time-consuming. The third problem associated with the method is that, it has proved inadequate for stiff systems. Fatokun, in earlier reports proposed a continuous approach for deriving self-starting multistep methods for solving initial value problems of ordinary differential equations. In this study we present an implementation procedure of an order seven integration method for initial value problems. Using the SCILAB to resolve the generated matrices and consequently a FORTRAN code was used to run the block method. The result shows that the method is convenient and easy to use. It needs no other single step method for the implementation of the seven-step method. The graphs of the region of Absolute Stability are also presented for each grid point.

Key words: Complicated, estimation, turneation, inadequate, integration

INTRODUCTION

Consider the initial value problem

$$y'(x) = f(x, y(x)), y(0) = y_0,; y_0 \in R^n \tag{1}$$

Classical multistep methods to solve the ivp (1) are the basis of some important codes for non-stiff differential equations as discussed in many texts such as Dahlquist and Bjorek (1974). The integral k-step method is written as

$$\sum_{j=1}^{k} \alpha_j y_{n+1} = h \sum \beta_j f_{n+j} \quad \alpha_k \neq 0 \tag{2}$$

where α_k, β_k are real parameters, $\alpha_k \neq 0$, $|\alpha_0| + |\beta 0| > 0$ whenever$\beta_k = 0$ in (2), the method is said to be explicit otherwise it is implicit. Explicit multistep algorithms are based on grid frames were proposed by Crouch and Grossman. For an application of (2), we require a starting procedure to compute the approximates

$$y_1, y_2, ..., y_{k-1} \text{ to } y(x_0 + h), y(x_0 + 2h), ..., y(x_0 + (k-1)h).$$

This has been the standard approach to implement multistep methods. All computer codes will have to integrate as a subroutine codes for single step methods like the Runge-Kutta methods. Butcher introduced the concept of multistep Runge-Kutta methods. In contrast to single-step methods, where numerical solution is obtained solely from the differential equation and the initial starting values at each stage of the iteration: using the Taylor's series expansion of the exact solution, using any single step method such as Runge-Kutta methods, or using a low order Adams methods.

Derivation of the order seven method by collocation approach: Using the power series described in Fatokun (2006) to the power series function $f(x)$=

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... \tag{3}$$

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and using this as the interpolation equation for the integration together with the respective derivatives for j = 0 (j) 7 and substituting into

$$\sum_{j=0}^{k} a_j Q_j(x) = f(x,y) + \tau P_k(x) \tag{4}$$

Where $P_k(x)$ is the Legendre polynomial of degree k and valid in $x_n \le x \le x_{n+k}$ and τ is a parameter we have the following:

$$\begin{aligned} &a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 \\ &+6a_6x^5 + 7a_7x^6 = f(x,y) + \tau P_7(x) \end{aligned} \tag{5}$$

Now collocating (5) at x_{n+j}, j = 0(1)7 and interpolating (4) at $x = x_{n+6}$, we obtain a system of eight equations with α_j, j = 0 (j) and parameter τ which can be represented inthe matrix form below:

$$\begin{bmatrix} 1 & x_{n+6} & x_{n+6}^2 & x_{n+6}^3 & x_{n+6}^4 & x_{n+6}^5 & x_{n+6}^6 & x_{n+6}^7 & 0 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 1 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 & \frac{84425}{823543} \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 & \frac{47127}{823543} \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 & \frac{-212183}{823543} \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 & \frac{212183}{823543} \\ 0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 & 5x_{n+5}^4 & 6x_{n+5}^5 & 7x_{n+5}^6 & \frac{-47127}{823543} \\ 0 & 1 & 2x_{n+6} & 3x_{n+6}^2 & 4x_{n+6}^3 & 5x_{n+6}^4 & 6x_{n+6}^5 & 7x_{n+6}^6 & \frac{-84425}{823543} \\ 0 & 1 & 2x_{n+7} & 3x_{n+7}^2 & 4x_{n+7}^3 & 5x_{n+7}^4 & 6x_{n+7}^5 & 7x_{n+7}^6 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ \tau \end{bmatrix} = \begin{bmatrix} y_{n+6} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+7} \end{bmatrix} \tag{6}$$

Solving for the coefficients α_j 's (j = 0(j) 7) and τyields the following:

$$\begin{aligned} \tau = &-\frac{-823543}{17297280}(f_{n+7} - 7f_{n+6} + 21f_{n+5} - \\ &35f_{n+4} + 35f_{n+3} - 21f_{n+2} + 7f_{n+1} - f_n) \end{aligned}$$

$$\begin{aligned} a_7 = &\frac{1}{5040h^6}(f_{n+6} - 6f_{n+5} + 15f_{n+4} \\ &-20f_{n+3} + 15f_{n+2} - 6f_{n+1} + f_n - \frac{8648640}{823543}\tau) \end{aligned}$$

$$\begin{aligned} a_6 = &\frac{1}{720h^5}(f_{n+5} - 5f_{n+4} + 10f_{n+3} - 10f_{n+2} \\ &+5f_{n+1} - f_n - 5040x_nh^5a_7 - 12600h^6a_7 \\ &+\frac{4102560}{823543}\tau) \end{aligned}$$

$$a_5 = \frac{1}{120h^4}(f_{n+4} - 4f_{n+3} + 6f_{n+2} - 4f_{n+1} + f_n - 720x_n h^4 a_6 - 1440h^5 a_6 - 2520x_n^2 h^4 a_7 - 10080x_n h^5 a_7 - 10920h^6 a_7 - \frac{1829520}{823543}\tau)$$

$$a_4 = \frac{1}{24h^3}\left(\begin{array}{l} f_{n+3} - 3f_{n+2} + 3f_{n+1} - f_n - 180h^4 a_5 - 120x_n \\ h^3 a_5 - 360x^2{}_n h^3 a_6 - 1080x_n h^4 a_6 - 900h^5 \\ a_6 - 840x^3{}_n h^3 a_7 - 3780x^2{}_n h^4 a_7 - 6300x_n h^5 \\ a_7 - 3780h6a + \frac{923832}{823543}\tau \end{array}\right)$$

$$a_3 = \frac{1}{6h^2}(f_{n+2} - 2f_{n+1} + f_n - 24h^3 a_4 - 24x_n h^2 a_4 + 60x_n^2 h^2 a_5 - 120x_n h^3 a_5 - 360x_n^2 h^3 a_6 - 210x^4 h^2 a_7 - 840x_n^3 h^3 a_7 - 1470x_n^2 h^4 a_7 - 1260x_n h^5 a_7 - 70h^4 a_5 - 120x_n^2 h^3 a_6 - 360x_n^2 h^3 a_6 - 434h^6 a_7 - 420x_n h^4 a_6 - 180h^5 a_6 + \frac{701820}{825343}\tau)$$

$$a_2 = \frac{1}{2h}(f_{n+1} - f_n - 6x_n h a_3 - 3h^2 a_3 - 4h^3 a_4 - 12x_n^2 h\ a_4 - 12x_n h^2 a_4 - 20hx_n^3 a_5 - 30x_n^2 h^2 a_5 - 6h^5 a_6 - 42x_n^5 h\ a_7 - 105x_n^4 h^2 a_7 - 140x_n^3 h^3 a_7 - 105x_n^2 h^4 a_7 - 42x_n h^5 a_7 - 7h^6 a_7 - 20x_n h^3 a_5 - 5h^4 a_5 - 30x_n^4 h a_6 - 60x_n^3 h^2 a_6 - 60x_n^2 h^3 a_6 - 30x_n h^4 a_6 + \frac{739118}{823543}\tau)$$

$$a_1 = f_n - 2x_n a_2 - 3x_n^2 a_3 - 4x^3_n a_4 - 5x_n^4 a_5 - 6x_n^5 a_6 - 7x_n^6 a_7 - \tau$$

$$a_0 = y_{n+6} - x_{n+6} a_1 - x_{n+6}^2 a_2 - x_{n+6}^3 a_3 - x_{n+6}^4 a_4 - x_{n+6}^5 a_5 - x_{n+6}^6 a_6 - x_{n+6}^7 a_7 \quad (7)$$

Substituting the coefficients α_j s (j = 0(j) 7) into (1.20) we obtain the continuous scheme as follows:

$$\bar{y}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7$$

$$= y_{n+6} + (x - x_{n+6})a_1 + (x^2 - x_{n+6}^2)a_2 + (x^3 - x_{n+6}^3)a_3 + (x^4 - x_{n+6}^4)a_4 + (x^5 - x_{n+6}^5)a_5 + (x^6 - x_{n+6}^6)a_6 + (x^7 - x_{n+6}^7)a_7 \quad (8)$$

Collocating at the grid point, $x = x_{n+6}$ we obtain the discrete method

$$y_{n+7} = y_{n+6} + \frac{h}{665280}(5350487f_{n+7} + 19380797f_{n+6} - 15564141f_{n+5} + 14519969f_{n+4} - 9574171f_{n+3} + 4081527f_{n+2} - 1005583f_{n+1} + 108395f_n) \quad (9)$$

This is the proposed seventh order economized discrete method

The Self-starting approach: Following the approach of Fatokun (2006), we collocate (9) at the grid point $x = x_{n+5}$, $x = x_{n+4}$, $x = x_{n+3}$, $x = x_{n+2}$, $x = x_{n+1}$, $x = x_{n+0}$ and x = xn. On simplifying, we have:

$$y_{n+5} = y_{n+6} + \frac{h}{17297280}\{390645f_{n+7} - 8193397f_{n+6} - 10418487f_{n+5} - 384729f_{n+4} + 2946431f_{n+3} - 2423199f_{n+2} + 929283f_{n+1} - 143827f_n\} \quad (10)$$

$$y_{n+4} = y_{n+6} + \frac{h}{8648640}\{90337f_{n+7} - 3238391f_{n+6} - 11007243f_{n+5} - 3237299f_{n+4} + 123331f_{n+3} - 50661f_{n+2} + 28327f_{n+1} - 5681f_n\} \quad (11)$$

$$y_{n+3} = y_{n+6} + \frac{h}{5765760}\{69329f_{n+7} - 2248493f_{n+6} - 6883851f_{n+5} - 5414929f_{n+4} - 3174509f_{n+3} + 420537f_{n+2} - 70681f_{n+1} + 5317f_n\} \quad (12)$$

$$y_{n+2} = y_{n+6} + \frac{h}{4324320}\{85113f_{n+7} - 1904527f_{n+6} - 4582419f_{n+5} - 4736139f_{n+4} - 3903349f_{n+3} - 2583597f_{n+2} + 376143f_{n+1} - 48505f_n\} \quad (13)$$

$$y_{n+1} = y_{n+6} + \frac{h}{3459456}\{39325f_{n+7} - 1337765f_{n+6} - 4150575f_{n+5} - 3199625f_{n+4} - 3199625f_{n+3} - 4150575f_{n+2} - 1337765f_{n+1} + 39325f_n\} \quad (14)$$

$$y_n = y_{n+6} + \frac{h}{2882880}\{14705f_{n+7} - 947207f_{n+6} - 4139067f_{n+5} - 1070659f_{n+4} - 5086349f_{n+3} - 864789f_{n+2} - 4344937f_{n+1} - 858977f_n\} \quad (15)$$

Some mathematical analysis of the methods: Following Lambert (1990), Dahlquist (1974), Atkinson (1987), the discrete methods thus derived in section 3.0 above were analyzed to determine the convergence, stability and the error constants. The Table 1 shows the results.

It can be observed that the methods are all of uniform order seven. Thus it is expected that if implemented as a block method, it is most likely that we shall have an accuracy of order seven. The graph of the Region of Absolute stability of the method is as shown in Fig. 1-4.

A numerical experiment: The method was applied in block form to solve the initial value problem y' - y = 0, with the theoretical solution given as y (x)= e^x; y(0) = 1, h = 0.01. By applying the seven discrete methods (9) with (10-15) to the ivp, we generate a system of seven equations. Using SCILAB to solve for the zeroes, we then wrote a FORTRAN code using FORCE 2.0 to run the iterations. The Table 2 shows the result with accuracy of order seven and also a considerate level of stability.

Table 1: Analyzed to determine the convergence, stability and the error constants

Grid point	Order	Err. Const.	Convergence	R.A.S
$X = X_{n+7}$	7	$-\frac{1}{196}$	convergent	[-0.65,0.73]
$X = X_{n+5}$	7	$-\frac{1}{89}$	convergent	[-5.73,6.44]
$X = X_{n+4}$	7	$-\frac{1}{505}$	convergent	[-0.027,0]
$X = X_{n+3}$	7	$-\frac{1}{505}$	convergent	[-4.09,0.17]
$X = X_{n+2}$	7	$-\frac{1}{89}$	convergent	[-48.58,0.84]
$X = X_{n+1}$	7	$-\frac{1}{8.7*10^8}$	convergent	[0,0]
$X = X_n$	7	$-\frac{1}{196}$	convergent	[-0.46,0.058]

Table 2: The result with accuracy of order seven and also a considerate level of stability

x_n	Exact Y	ComputedY	Error
1.00000E-01	1.1051710E+00	1.1051708E+00	1.1920929E-07
1.10000E-01	1.1162781E+00	1.1162779E+00	1.1920929E-07
1.20000E-01	1.1274968E+00	1.1274967E+00	1.1920929E-07
1.30000E-01	1.1388284E+00	1.1388283E+00	1.1920929E-07
1.40000E-01	1.1502738E+00	1.1502737E+00	1.1920929E-07
1.50000E-01	1.1618342E+00	1.1618341E+00	1.1920929E-07
1.60000E-01	1.1735109E+00	1.1735107E+00	2.3841858E-07
1.70000E-01	1.1853049E+00	1.1853046E+00	2.3841858E-07
1.80000E-01	1.1972173E+00	1.1972171E+00	2.3841858E-07
1.90000E-01	1.2092496E+00	1.2092493E+00	3.5762787E-07
2.00000E-01	1.2214028E+00	1.2214024E+00	3.5762787E-07

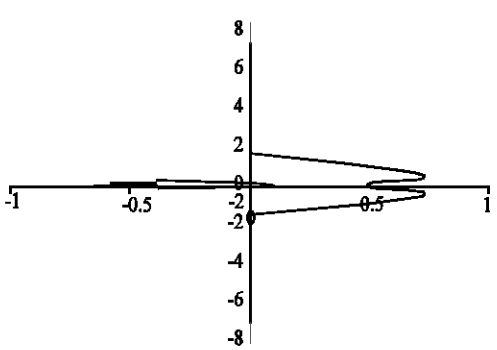

Fig 1: Step seven method alone

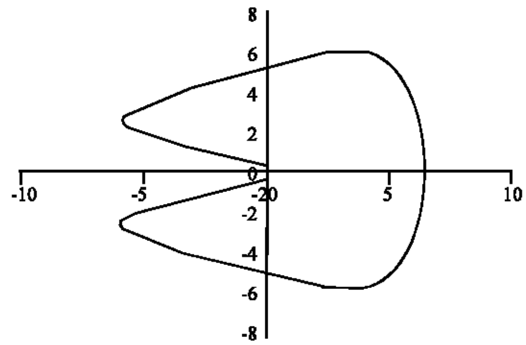

Fig 2: Step seven method at $x = x_{n+5}$

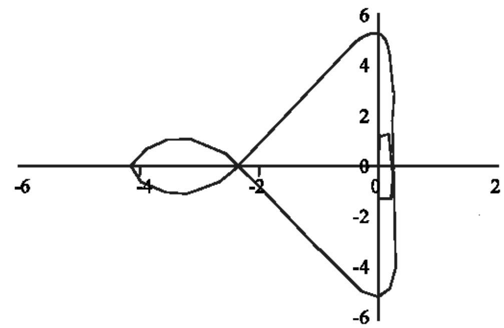

Fig 3: Step seven method at $x = x_{n+3}$

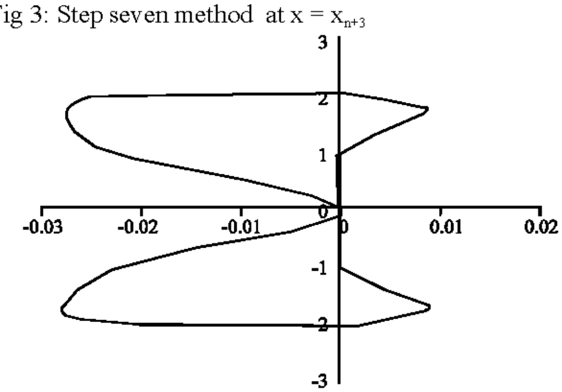

Fig. 4: Step seven method at $x = x_{n+3}$

CONCLUSION

We have presented a new approach of self starting a multistep method without the use of predictors or single step methods. In a follow-up study, we shall consider the behaviour of this method near singular points and for very stiff problems.

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