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A Twelfth-Order Method to Solve Systems of Nonlinear Equations

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Abstract: In this study, we present and analyze an iterative method of three steps using predictor corrector technique for solving systems of nonlinear equations. Our aim is to achieve high order of convergence with few Jacobian and functional evaluations. The analysis of convergence demonstrates that the order of convergence for this method is twelve. We use the concerned the flops-like efficiency index and the classical efficiency index in order to compare the obtained method with the previous literature. In addition, the proposed method has been tested on a series of examples and has shown good results when compared it with the previous literature.

Key words: Twelfth-order method, nonlinear system, iterative method, high order method, Newton's method, efficiency index, flops-like efficiency index

INTRODUCTION

Suppose we have system of nonlinear equations of the following form:

$$\begin{aligned} f_1(x_1, x_2, \cdots, x_m) &= 0 \\ f_2(x_1, x_2, \cdots, x_m) &= 0 \\ &\vdots \\ f_m(x_1, x_2, \cdots, x_m) &= 0 \end{aligned}$$

where and f_i: $R^m \to R$ the functions f_i is differentiable up to any desired order can be thought of as mapping a vector $x = (x_1, x_2, ..., x_m)^T$ of the n-dimensional space R^m, into the real line R. The system can alternatively be represented by defining a functional F_i:$R^m \to R^m$ by

$$F(x_1, x_2, \ldots x_m) = [f_1(x_1, x_2, \ldots x_m), \ldots, f_m(x_1, x_2, \ldots x_m)]^T$$

Using vector notation to represent the variables, the previous system then assumes the form:

$$F(x) = 0$$

In recent years, there are many approaches to solve the system (Mohamed and Hafiz, 2012; Hafiz and Alamir, 2014; Hafiz and Bahgat, 2012a, b) modified Householder and Halley iterative methods for solving systems of nonlinear equations. He also shows that this new method includes two famous cases of Newton's metho (Hafiz and Bahgat, 2012a, b; Khirallah and Hafiz, 2012, 2013a, b) modified some iterative schemes to get new classes of Jarratt-type methods for solving systems of nonlinear equations. Consequently, Hafiz and Alamir (2014) combined the Halley method with Householder method and used the predictor-corrector technique to construct new high-order iterative methods for solving systems of nonlinear equations.

In quest of more fast algorithms, researchers have also proposed fifth, sixth and higher order methods in cost of applying further functional evaluations alongside the computation of further matrix inverses per computing cycle. Motivated by the recent developments in this area we here propose an efficient method with higher convergence orders. In numerical analysis, an iterative method is regarded as computationally efficient if it attains high computational speed using minimal computational cost. The main elements which contribute towards the total computational cost are the evaluations of functions, derivatives and inverse operators. Among the evaluations, the evaluation of an inverse operator is the most obvious barrier in the development of an efficient iterative scheme since it is expensive from a computational point of view. Therefore, it will turn out to be judicious if we use as small possible number of such inversions. In this work, we present and analyze an iterative method of 3 steps and use the predictor-corrector technique for solving systems of nonlinear equations. Our aim is to achieve high order of convergence with few Jacobian and functional evaluations. Some illustrative examples have been presented in order to demonstrate our methods and the results are compared with those derived from the previous methods. All test problems reveals the accuracy and fast convergence of the new methods. With these considerations, we here construct iterative schemes for nonlinear systems.

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Keeping in view the features of a computationally efficient method for nonlinear systems, we begin with the following iterative two-step Jarratt type of the fourth order method, proposed in (Khattri and Abbasbandy, 2011; Chun *et al*., 2012):

$$y_i = x_i - \frac{2}{3}F'(x_i)^{-1}F(x_i)$$

$$x_{i+1} = x_i - \frac{1}{2}\left[3F'(y_i) - F'(x_i)\right]^{-1} \left[3F'(y_i) + F'(x_i)\right]F'(x_i)^{-1}F(x_i) \quad (5)$$

Darvishi and Barati (2007) developed the third-order method which is written as:

$$x_{i+1} = x_i - F'(x_i)^{-1}(F(x_i) + F(x^*_{i+1})), \quad (6)$$

Where:

$$x^*_{i+1} = x_i - F'(x_i)^{-1}F(x_i)$$

Motivated and inspired by the on-going activities in this direction, we construct a modification (based on the above Darvishi and Barati's method) of Jarratt's method with higher-order convergence for solving the nonlinear system of equations. It has been shown that this 3-step iterative method is twelfth-order convergence. Several numerical examples are given to illustrate the efficiency and the performance of the new iterative methods. Our results can be viewed as an improvement and refinement of the previously known results.

MATERIALS AND METHODS

The proposed method and analysis of convergence: This section contains the new method of this study. We aim at having an iteration method to have high order of convergence with an acceptable efficiency index. Hence, in order to reach the twelfth order of convergence without imposing the computation of further frechet derivatives let us introduce now a new Jarratt-type scheme of three steps which we denote as M12. From Khirallah and Hafiz (2013a, b) and Hafiz and Alamir (2012), we construct a novel iterative method:

$$y_i = x_i - \frac{2}{3}F'(x_i)^{-1}F(x_i)$$

$$z_i = x_i - \frac{1}{2}\left[3F'(y_i) - F'(x_i)\right]^{-1}, \left[3F'(y_i) + F'(x_i)\right]F'(x_i)^{-1}F(x_i) \quad (7)$$

$$x_{i+1} = z_i - F'(z_i)^{-1}\{F(z_i) + F[z_i - F'(z_i)^{-1}F(z_i)]\}$$

Per computing step of the new method (Khattri and Abbasbandy, 2011) for not large-scale problems, we may use the LU decomposition to prevent the computation of the matrix inversion which is costly. Simplifying method (Khattri and Abbasbandy, 2011) for the sake of implementation yields in:

$$y_i = x_i - \frac{2}{3}V(x_i)$$

$$z_i = x_i - \frac{1}{2}M(x_i)V(x_i) \quad (8)$$

$$x_{i+1} = z_i - W(z_i)$$

$$F'(x_i)V(x_i) = F(x_i)$$

Where in:

$$\left[3F'(y_i) - F'(x_i)\right]M(x_i) = 3F'(y_i) + F'(x_i)$$

and:

$$F'(z_i)W(z_i) = F(z_i) + F[z_i - V(z_i)]$$

The convergence-order of (Khattri and Abbasbandy, 2011) is twelve and the whole scheme requires two functional evaluations of F, three evaluations of Jacobian and their inversions at different points.

In the next result, we prove the local order of convergence of the M12 method by using the Taylor expansions and obtaining the error equation.

Theorem 1: The iterative method (Khattri and Abbasbandy, 2011) has a local order of convergence at least nine with the following error equation:

$$\frac{2}{729}c_2^2(9c_2^3 - 9c_2c_3 + c_4)^3 e_i^{12} + O(e_i^{13})$$

Proof: The proof of this theorem can be followed by writing the Taylor expansions of F around the simple root x^* of Eq. 1. Using a same methodology of (Cordero *et al*., 2009, 2010) for $x^*+ h\in\Omega$ lying in the neighborhood of a solution x^* of and assume that, $F'(x) \neq 0$, we first have:

$$F(x^* + h) = F(x^*)\left[h + \sum_{q=2}^{p-1} C_q h^q\right] + O(h^p) \quad (9)$$

where, $c_q = \frac{1}{q!}[F'(x^*)]^{-1}F^{(q)}(x^*), q \geq 2$ we observe that $C_q h^q \in \Omega$, Since $F^{(q)}(x^*) \in L(\Omega\times...\times\Omega)$ and $[F'(x^*)]^{-1} \in L(Y\Omega)$. In addition, we can express the Jacobian matrix F' as:

$$F'(x^*+h)=F'(x^*)\left[I+\sum_{q=2}^{p-1} qc_q h^{q-1}\right]+O(h^p), \quad (10)$$

where, i is the identity matrix. Therefore $qc_q h^{q-1} \in$? (Y) L(Y, Ω). From (Darvishi and Barati, 2007), we assume

$$[F'(x^*+h)]^{-1}=\left[I+\sum_{q=2}^{p-1} x_q h^{q-1}\right][F'(x^*)]^{-1}+O(h^p), \quad (11)$$

Taking into account that $[F'(x^*+h)]^{-1}F'(x^*+h) = F'(x^*+h)[F'(x^*+h)]^{-1} = I$

$$x_S=-\sum_{j=2}^{S} jx_{S-j+1}, \quad S=2,3,...,$$

where x_1= I. We remark that if we denote $e_n = x_n - x^*$ the error in the n th iteration, the equation:

$$e_{n+1}=Me_n^p+O(e_n^{p+1}),$$

where M is a p-linear function M∈L (Ω × ... × Ω, Y) is called the error equation and p is the order of convergence. Notice that is e_n^p $(\overbrace{e_n, e_n, ..., e_n}^{p})$

$$y_i-x^*=\frac{e_i}{3}[I+2c_2e_i-4(c_2^2-c_3)e_i^2+2(4c_2^3-7c_2c_3+3c_4)e_i^3-4(4c_2^4-10c_2^2c_3+3c_3^2+5c_2c_4-2c_5)e_i^4]+O(e_i^6)$$

$$z_i-x^*=(c_2^3-c_2c_3+\frac{c_4}{9})e_i^4+O(e_i^5)$$

$$e_{i+1}=\frac{2}{729}c_2^2(9c_2^3-9c_2c_3+c_4)^3e_i^{12}+O(e_i^{13})$$

The classical efficiency index: Using the definition of classical efficiency indexes $p^{1/C}$ due to Ostrowski (1966), where "p" is the order of convergence and "C" stands for the total computational cost per iteration in terms of the number of functional evaluations. For a system of m equations in m unknown, the first Frechet derivative ΔF is a matrix with m^2 evaluations. At m = 2, we compare our proposed method M12 which requires 8m function evaluations and $3m^2$ of its first Frechet derivative with order of convergence which is twelve. Therefore, the efficiency index of our proposed Method M12 is 121/18≈1.14803 which is im-proved as compared to efficiency index $3^{1/14}$ ≈ 1.08163 of Noor and Waseem method (Noor and Waseem, 2009; Noor *et al.,* 2013). Darvishi and Barati (2007) obtained a fourth order method which used 2 m function evaluation and $3(m)^2$ of its first Frechet derivative that has efficiency index $4^{1/16}$ ≈ 1.090508. Similarly, Hafiz and Alamir (2014) obtained a fourth order method which used 2 m function evaluation and $2(m)^2$ of its first Frechet derivative and another Frechet m+m2 derivative which has efficiency index $4^{1/18}$~1.08006. Cordero *et al*. (2009) obtained a fourth order method which used 2 m function evaluation and $2(m)^2$ of its Frechet derivative which has efficiency index at $4^{1/12}$ ≈ 1.12246. We also notify a fact here that the efficiency of our proposed method M12 is same as that of Newton's method but the order and efficiency have improved. The proposed method M12 is very easy to work with as compared to other well-known iterations methods for solving systems of nonlinear equations. It is highly practical to use this scheme. It is also highly practical to work with Newton's method.

Concerning the Flops-Like efficiency index: Now the implementation of Method M12 depends on the involved linear algebra problems. For large-scale problems, one may apply the GMRES iterative solver which is well known for its efficiency for large sparse linear systems.

The interesting point in M12 is that three linear systems should be solved per computing step but all have the same coefficient matrix. Hence, one LU factorization per full cycle is needed which reduces the computational load of the method when implemented.

Generally speaking, the number of scalar products, matrix products, LU decompositions of the first derivatives and the resolution of the triangular linear systems are of great importance in assessing the real efficiency of such schemes. As a result, we take into account the number of main operations per cycle i.e., LU decompositions along with the cost of solving two triangular systems, based on flops. In this case, we remark that the flops for obtaining the LU factorization is $(2m^3)/3$

To solve two triangular systems, the flops would be $2m^2$ Note that if the right hand side is a matrix, then the cost (flops) of the two triangular systems is see $2m^3$ (Khan *et al*., 2015).

RESULTS AND DISCUSSION

Numerical results: Here and from now on, SM, HM and M12 denote (Sharma *et al*., 2013; Homeier, 2005) and the present method in (Chun *et al*., 2012), respectively. In this study, we test M12 with some sparse systems with m unknown variables. In the examples 1-5, we compare the SM method with the proposed method HM focusing on iteration numbers. Where Sharma's method is:

Table 1: Comparison of efficiency indices for different methods

Iterative methods	SM	HM	M12
Number of steps	2	2	3
Rate of convergence	4	3	12
Number of functional evaluations	m+2m²	m+2m²	3m+2m³
The classical efficiency Index	$4^1/(2m^2+2)$	$3^1/(2m^2+m)$	$12^1/(3m^2+3m)$
Number of LU Factorizations	2	1	3
Cost of LU factorizations (based on flops)	$1/3m^3$	$1/3m^3$	$1/3m^3$
Cost of linear systems (based on flops)	$16/3m^3+2m^2$	$5/5m^3+4m2$	4m2(m+1)
Flops-like efficiency Index	$4^1/(16/3m^3+4m^2+m)$	$3^{1/(4/3m3+6m2+m)}$	$12^1/(4/m^3+4m^2+3m)$

Table 2: Number of iterations for different methods

$\epsilon = 10^{-13}$	F_1	F_2	F_3	F_4	F_5	F_1	F_2	F_3	F_4	F_5
Number of variables			3					7		
SM	6	7	51	5	4	6	7	52	6	4
HM	4	4	27	4	4	4	4	27	4	4
M12	3	3	18	3	3	3	3	19	3	3

$$y_i = x_i - \frac{2}{3}F'(x_i)^{-1}F(x_i)$$

$$x_{i+1} = x_i - \frac{1}{2}\left[-I + \tfrac{9}{4}F'(y_i)^{-1}F'(x_i) + \tfrac{3}{4}F'(x_i)^{-1}F'(y_i)\right] F'(x_i)^{-1}F(x_i)$$

and Homey's method is:

$$y_i = x_i - F'(x_i)^{-1}F(x_i)$$

$$x_{i+1} = x_i - \frac{1}{2}\left[F'(x_i)^{-1}F(x_i) + F'(y_i)^{-1}F(x_i)\right]$$

Here, numerical results are performed by Maple 15 with 200 digits but only 3 digits are displayed. In Table 1and 2, we list the results obtained by SM, HM, M12 which are introduced in the present study. The following stopping criterion is used for computer programs:

$$\| x_{n+1} - x_n \| + \| F(x_n) \| < 10^{-15}$$

where, n denoted to the number of iterations and (COC); the computational order of convergence can be approximated using in equation below:

$$COC \approx \frac{\ln\left(\| x_{n+1} - x_i \| / \| x_n - x_{n-1} \|\right)}{\ln\left(\| x_n - x_{n-1} \| / \| x_{n-1} - x_{n-2} \|\right)}$$

Table 2 shows the number of iterations, the Computational Order of Convergence (COC) $\|X_{n+1}-X_n\|$ and the norm of the function $F(x_n)$ which are also shown in Table 2 for various methods.

Example 1: Consider the following system of nonlinear equations (Hafiz and Bahgat, 2012a, b):

$$F_1 : f_i = e^{x_i} - 1, \quad i = 1,2,\ldots,m$$

The exact solution of this system is $x^* = [0,0,\ldots,0]^T$. To solve this system, we set $x_0 = [0.5,0.5,\ldots, 0.5]^T$ as an initial value.

Example 2: Consider the following system of nonlinear equations:

$$F_2 : f_i = x_i^2 - \cos(x_i - 1), \quad i = 1,2,\ldots,m$$

One of the exact solutions of this system is $x^* = [1,1,\ldots,1]^T$. To solve this system, we set $x_0 = [2,2.,\ldots,2]^T$ as an initial value.

Example 3: Consider the following system of nonlinear equations:

$$F_3 : f_i = \cos x_i - 1, \quad i = 1,2,\ldots,m$$

One of the exact solutions of this system is $x^* = [0,0,\ldots,0]^T$. To solve this system, we set $x_0 = [2,2.,\ldots,2]^T$ as an initial guess.

Example 4: Consider the following system of nonlinear equations: One of the exact solutions of this system is $x^* = [1,1,\ldots,1]^T$. To solve this system, we set $x_0 = [1.2, 1.2, \ldots, 1.2]^T$ as an initial value:

$$F4 : f_i = x_{i+1}x_i - \cos(x_i - 1),$$
$$i = 1,2,\ldots,m.;x_{m+1} = x_1$$

Example 5: Consider the following system of nonlinear equations:

$$F_5 : f_i = x_{i+1}x_i - 1, \quad i = 1,2,\ldots,m-1.;$$
$$f_m = x_m x_1 - 1$$

For m odd; one of the exact solutions of this system is $x^* = [1,1,\ldots,1]^T$. To solve this system, we set $x_0 = [2,2.,\ldots,2]^T$ as an initial value.

Table 3: Numerical results for Examples 1-5, m = 3

Methods and functions	IT	COC	$\|x_{n+1}-x_n\|$	$\|F(x_n)\|$
F_1				
SM	6	2.001	1.09E-18	0
HM	4	3.001	3.44E-20	1.13E-64
M12	3	4.000	8.89E-46	0
F_2				
SM	7	2.000	1.61E-22	0
HM	4	3.990	1.02E-23	1.68E-99
M12	3	4.000	5.75E-22	0
F_3				
SM	51	1.000	6.66E-14	0
HM	27	1.000	4.82E-14	7.44E31
M12	18	1.000	8.33E-14	0
F_4				
SM	5	2.000	7.15E-15	0
HM	4	4.000	1.37E-55	2.42E-197
M12	3	4.000	5.89E-55	0
F_5				
SM	4	3.99	7.03E-24	0
HM	4	3.99	1.01E-28	4.98E-120
M12	3	4.00	1.01E-28	0

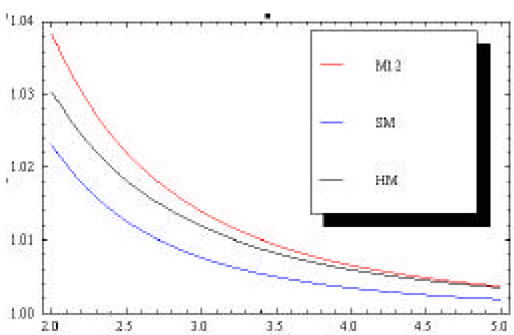

Fig. 1: The curves of the efficiency indices for different methods when m = 2, · · · , 5

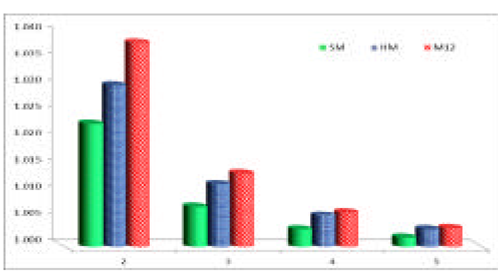

Fig. 2: Efficiency index of the different methods for different sizes of the system

In Tables 2 and 3, we list the results obtained by the modified M12 iteration method. As we see from these tables, it is clear that the result obtained by M12 is very superior to that obtained by HM and SM. In Table 4 the results show that M12 is promising in contrast to the compared methods. Figures 1 and 2 show that M12 with only one matrix inversion per cycle beats its other competitors.

CONCLUSION

The twelfth-order method continues to be an important subject of investigation. In our study we extend the standard iteration in order to obtain robust algorithms based on Darvishi`s method to construct a new twelfth-order iterative method using the predictor–corrector technique. This method is applied for solving nonlinear systems of equations. The numerical examples show that our method is very effective and efficient. Moreover, our proposed method provides highly accurate results in a less number of iterations as compared with Sharma's method and Homeier's method.

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