International Journal of Soft Computing 11 (6): 343-350, 2016
ISSN: 1816-9503

L(0,1) and L(1,1) Labeling Problems on Circular-Arc Graphs

S.K. Amanathulla and Madhumangal Pal
Department of Applied Mathematics with Oceanology and Computer Programming,
Vidyasagar University, 721102 Midnapore, India

Abstract: An L(0, 1)-labeling of a graph G = (V, E) is a function f from the vertex set V(G) to the set of non-negative integers such that /f(x)-f(y)/≥0 if d(x, y) = 1 and /f(x)-f(y)/≥1 if d(x, y) = 2. The L(0, 1)-labeling number of a graph G, denoted by $\lambda_{0,1}$ (G) is the difference between highest and lowest labels used. Similarly, L(1, 1)-labeling of a graph G = (V, E) is a function f from its vertex set V to the set of non-negative integers such that /f(x)-f(y)/≥1 if d(x, y) = 1 or 2. The span of an L(1, 1)-labeling f of G is max{f(v): v∈V}. The L(1, 1)-labeling number $\lambda_{1,1}$ (G) of G is the smallest non-negative integer k such that G has a L(1, 1)-labeling of span k. In this study, for any circular-arc graph G, we have shown that $\lambda_{0,1}(G)\leq\Delta$ and $\lambda_{1,1}(G)\leq 2$ where Δ represents the degree of the graph G. Also two algorithms are designed to label a circular-arc graph by maintaining L(0, 1)-and L(1, 1)-labeling conditions. The running time of these algorithms are $O(n\Delta^2)$ and $O(n\Delta)$, respectively where n represent the number of vertices of G.

Key words: Frequency assignment, L(0, 1)-labeling, L(1, 1)-labeling, circular-arc, graph, span

INTRODUCTION

The channel assignment problem is a problem where the task is to assign a channel (non-negative integer) to each F.M radio station such that there is no interference between stations and the span of the assigned channels is minimized. Hale (1980) formulated this into a graph vertex coloring problem. In 1988, Roberts proposed a variation of the frequency assignment problem in which 'closed' transmitter must receive different frequency and 'very closed' transmitter must receive a frequency at least two apart. To convert this problem into graph theory, the transmitters are represented by the vertex of a graph; two vertices x and y are said to be 'very closed' if the distance between them is 1 and 'closed' if the distance between x and y is 2. We denote d(x, y) to represent the shortest distance (i.e., the minimum number of edges between x and y) between the vertices x and y. The aim of this problem is to minimize the span. The minimum span over all possible labeling functions of L(0, 1)-labeling is denoted by $\lambda_{0,1}(G)$ and is called $\lambda_{0,1}$- number of G. And the minimum span over all possible labeling functions for L(1, 1)-labeling is denoted by $\lambda_{1,1}(G)$ and is called $\lambda_{1,1}$-number of G.

In the area of graph labeling problem there are two spacial type of collisions (frequency interference), namely direct collisions and hidden collisions. In direct collisions, a radio station and its neighbors must have different frequencies, so their signals will not collide. This is nothing but vertex coloring or L(1, 0)-labeling problem. In hidden collisions, a radio station cannot receive signals of the same frequencies from any of its neighbors. Thus the only requirement here is that for every station, all its neighbors must have distinct frequencies or labels but there is no requirement on the label of the station itself. Bertossi and Bonuccelli (1995) studied the case of avoiding hidden collision in the multihop radio networks. To avoid the hidden collision from its adjacent stations, we require distinct labels for its intermediate adjacent stations. Here, we suppose that there is a little direct collision in the system. Direct collision is so weak that we can ignore it. Hence we allow the same labels for two adjacent stations. Therefore this problem can be formulated as L(0, 1)-labeling problem. The span of an L(0, 1)-labeling f of G is max{f(v): v∈V}. The L(0, 1)-labeling number, $\lambda_{0,1}$ (G), of G is the smallest non-negative integer k such that G has a L(0, 1)-labeling of span k.

Any network is easily modeled as a graph with the networks nodes as vertices and communication links as edges. The explicit scheduling of a network can then be seen as a kind of graph coloring where each vertex is "colored" with the segment of the medium assigned to it by the schedule. Assuming communication links are symmetric as we will throughout the study the broadcast scheduling problem is solved via an explicit schedule arising from a coloring of the corresponding graph such that any vertex is colored differently than any other vertex at distance one or two. Such an coloring is called

Corresponding Author: Madhumangal Pal, Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, 721102 Midnapore, India

L(1, 1)-labeling or alternately a distance coloring. So an L(1, 1)-labeling of a graph G = (V, E) is a function f from its vertex set V to the set of non-negative integers such that /f(x)-f(y)/≥1 if d(x, y) or 2. The span of an L(1, 1)-labeling f of G is max{f(v): v∈V}. The L(1, 1)-labeling number $\lambda_{1,1}$ (G) of G is the smallest non-negative integer k such that G has a L(0, 1)-labeling of span k.

Frequency assignment problem has been widely studied in the past (Bertossi and Pinotti, 2007; Calamoneri and Petreschi, 2004; Calamoneri, 2006; Chang and Lu, 2003; Chiang and Yan, 2008; Das *et al*., 2006; Golumbic, 2004; Griggs and Yeh, 1992; Hale, 1980; Olariu, 1991; Pal, 1995, 2013; Pal and Bhattacharjee, 1995, 1996, 1997; Pal and Pal, 2009; Saha *et al*., 2007; Sakai, 1994; Wan, 1997). We focus our attention on L(0, 1)-labeling and L(1, 1)-labeling of circular-arc graphs. Different bounds for $\lambda_{0,1}$ (G) and $\lambda_{1,1}$ (G) were obtained for various type of graphs. The upper bound of $\lambda_{0,1}$(G) of any graph G is Δ^2-Δ where, Δ is the degree of the graph. The upper bound of $\lambda_{1,1}$(G) of any graph G is Δ^2-2. Bodlaender *et al.* (2004) compute upper bounds for graphs of treewidth bounded by t proving that $\lambda_{0,1}$ (G). They also shown that L(0, 1)-labelling number of a permutation graph not exceed 2Δ-2. Recently, Pal and Pal (2009) shown that L(0, 1)-labeling number of a permutation graph does not exceed Δ-1. Khan *et al.* (2012) shown that Δ-1 ≥ $\lambda_{0,1}$(G)≤Δ for cactus graph. Pal (2013) intersection graphs are discussed. Chang and Kua (1996), the NP-completeness result for the decision version of the L(0, 1)-labeling problem is derived when the graph is planar by means of a reduction from 3-vertex coloring of straight-line planer graph. An exhaustive survey on L(h, k)-labeling is available in (Calamoneri, 2011). For a bipartite graph $\lambda_{0,1}$ (G)≥ Δ^2/4 (Bodlaender *et al*., 2014). Later this lower bound has been improved by a constant factor of 1/4 (Alon and Mohar, 2002). On L(d, 1)-labeling of Cartesian product of cycles and path is done by Chiang and Yan (2008). This problem was introduced by Griggs and Yen (1992) and Wan (1997) in connection with the problem of assigning frequency in a multiple radio network.

The problem is simple for P_n of n vertices. It is easily verified that $\lambda_{0,1}(p_1)$ = $\lambda_{0,1}$ (p_2) = 0, $\lambda_{0,1}$ (p_n) = 1for n≥3 (Makansi, 1987). When the first and last vertices of P_n are marged then P_n becomes C_{n-1}. Bertossi and Bonuccelli (1995) showed that $\lambda_{0,1}$ (C_n) is equal to 1 if n is multiple of 4 and 2 otherwise. For path $\lambda_{1,1}$ (P_2) = 1 and $\lambda_{1,1}$ (P_n) = 2 for each n≥3 and $\lambda_{1,1}$ (C_n) is 2 if n is a multiple of 3 and it is 3 otherwise (Battiti *et al*., 1999). Calamoneri *et al.* (2009) proved that an interval graph G can be L(h, k)-labeled with span at most max (h, 2k)Δ, also they shown that for circular-arc graphs $\lambda_{h,k}$(G)≤max (h, 2k)Δ+hw. Paul shows that $\lambda_{2,1}$(G)≤Δw for interval graph and they shows that $\lambda_{2,1}$ (G)≤Δ3w for circular-arc graph where w represents the size of the maximum clique. Very recently Paul *et al.* investigated the problem of L(0, 1)-labeling on interval graph (Pal, 2013) and also L(2, 1)-labeling on interval graphs and L(2, 1)-labeling of circular-arc graph (Alon and Mohar, 2002). Recently, Amanathulla and Pal (2006) have studied L(3, 2, 1)- and L(4, 3, 2, 1)-labeling of circular arc graph and have proved that $\lambda_{3,2,1}$(G)≤9Δ-6 and $\lambda_{4,3,2,1}$(G)≤16Δ-12 for circular-arc graphs. In this study we have investigated L(0, 1)-labeling and L(1, 1)-labeling problems on circular-arc graphs and we obtain $\lambda_{0,1}$(G)≤Δ and $\lambda_{1,1}$(G)≤2Δ, this result is tighter than the previous available results is tighter than the previous available results $\lambda_{0,1}$(G)≤2Δ and $\lambda_{1,1}$ (G)≤2Δ+ω (Calamoneri *et al*., 2009).

MATERIALS AND METHODS

Preliminaries and notations: The graphs used in this research are simple, finite without self loop or multiple edges. A graph G = (V, E) is called an intersection graph for a finite family F of a non-empty set if there is a one-to-one correspondence between F and V such that two sets in F have non-empty intersection if and only if there corresponding vertices in V are adjacent to each other. We call F an intersection model of G. For an intersection model F, we use G(f) to denote the intersection graph for F. Depending on the nature of the set F one gets different intersection graphs. For a survey on intersection graph see (Pal, 2013)

The class of circular-arc graphs is a very important subclass of intersection graphs. A graph is a circular-arc graph if there exists a family I of arcs around a circle and a one-to-one correspondence between vertices of G and arcs I such that two distinct vertices are adjacent in G if and only if there corresponding arcs intersect in I. Such a family of arcs is called an arc representation for G. A circular-arc graph and its corresponding circular-arc representation is shown in Fig. 1.

A graph G is a Proper Circular-Arc (PCA) graph if there exists an arc representation for G such that no arc is properly included in another. The circular-arc graphs used in this work may or may not be proper. It is assumed that all arcs must cover the circle, otherwise the circular-arc graph is nothing but an interval graph. The degree of the vertex v_k corresponding to the arc 1_k is denoted by $d(v_k)$ and is defined by the maximum number of arcs which are adjacent to 1_k. The maximum degree or the degree of a circular-arc graph G, denoted by Δ(G) or by Δ is the maximum degree of all vertices corresponding to the arcs of G. Let I = {I_1, I_2, I_3, .., I_n} be a set of arcs around a circle.

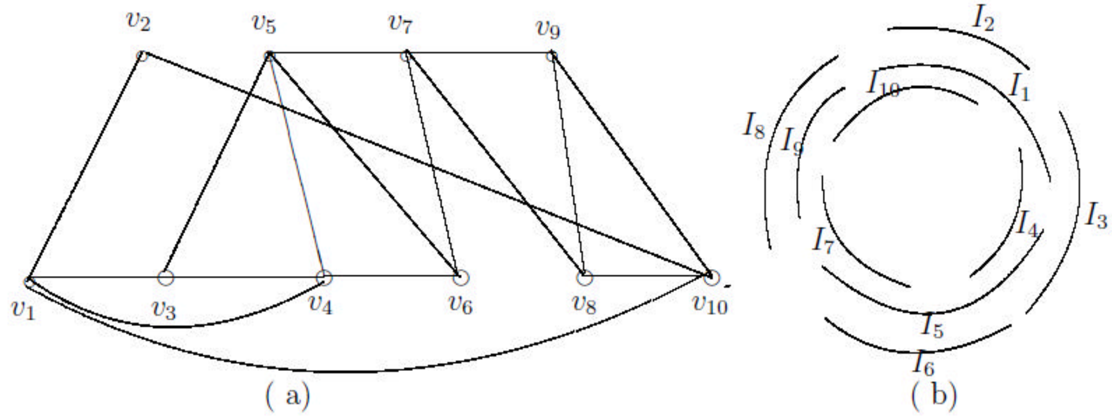

Fig. 1: A circular-arc graph and its corresponding circular-arc representation

While going in a clockwise direction, the point at which we first encounter an arc is called the starting point of the arc. Similarly, the point at which we leave an arc is called the finishing point of the arc. An arc I_j is denoted by a closed interval $[h_j, t_j]$, j = 1, 2, .., n where h_j is the counter clockwise end, i.e., starting point and t_i is the clockwise end, i.e. finishing point. A set $C \subseteq V$ is called a clique if for every pair of vertices of C has an edge. The number of vertices of the clique represents its size. A clique is called maximal if there is no clique of G which properly contains C as a subset. Again a clique with r vertices is called r-clique. A clique is called maximum if there is no clique of G of larger cardinality. The size of the maximum clique is denoted by w(G) or by w. Also, it is observed that an arc I_k of I and a vertex v_k of V are one and same thing.

Notations: Let G be a circular-arc graph with arcs set I. We define the following objects:

- $L_0(I_k)$: the set of labels which are used before labeling the arc I_k for any arc $I_k \in I$
- $L(I_k)$ the set of labels which are used to label the vertices at distance one from the arc I_k, before labeling the arc I_k, $I_k \in I$
- $L_2(I_k)$: the set of labels which are used to label the vertices at distance two from the arc I_k, before labeling the arc I_k, I_k, $I_k \in I$
- $L_{1v2}(I_k)$: the set of labels which are used to label the vertices at distance either one or two from the arc I_k, before labeling the arc I_k, for any arc $I_k \in I$
- f_j: the label of the arc I_j, $I_j \in I$
- L: the label set, i.e., the set of labels used to label the circular-arc graph G completely
- It can be verified that $L_{1v2}(I_k) = L_1(I_k) \cup L_2(I_k)$ for any arc $I_k \in I$

Defination 1: For a circular-arc graph G we define a set of arcs S_{ij}, for each $I_j \in I$ such that:

- All arcs of S_{ij} are adjacent to I_j
- No two arcs of S_{ij} are adjacent
- Each S_{ij} is maximal

The adjacency between two arcs (vertices) $I_j = [h_j, t_j]$ and $I_k = [h_k, t_k]$ can be tested by using the following well known lemma.

Lemma 1: If two arcs I_j and $[h_j, t_j]$ and $Ik = [h_k, t_k]$ are adjacent then one of the following conditions is true:

$$h_j < h_k < t_j < t_k$$

or:

$$h_j < h_k < t_k < t_j$$

or:

$$h_k < h_j < t_j < t_k$$

L(0, 1)-labeling of circular arc graphs: In this study, we present some lemmas related to the proposed algorithm. Also an algorithm is designed to solve L(0, 1)-labeling problem on circular-arc graphs, along with time complexity.

Lemma 2: If $L_1(I_j)-L_2(I_j) \neq \phi$ then $f_j = I$ where $I \in L_1(I_j)-L_2(I_j)$ for any arc $I_j \in I$.

Proof: If $L_1(I_j)-L_2(I_j) \neq \emptyset$ then the set $L_1(I_j)-L_2(I_j)$ contains some integers which are not used to label the vertices at distance two from the arc I_j. So any label $1 \in L_1(I_j)-L_2(I_j)$ is a valid label for I_j, i.e., we can assign I to the arc I_j, since it satisfies L(0, 1)-labeling condition.

Lemma 3: If $L_0(I_j)-L_2(I_j)\neq\varnothing$ then f_j = I where $I\in L_0(I_j)-(L_2(I_j))$ for any arc $I_j\in I$.

Proof: If $L_0(I_j)-L_2(I_j)\neq\varnothing$ then the set $L_0(I_j)-L_2(I_j)$ contains some unused integers to label the vertices at distance two from the arc I_k. So any label $l\in L_0(I_j)-L_2(I_j)$ is a valid label for I_j, i.e., $f_{J=1}$.

Lemma 4: If $L_0(I_j)-L_2(I_j)=\varnothing$ then $f_j\neq$ where $I\in L0(I_j)-L_2(I_j)$ and f_i = m where m = max$\{L_0(I_j)\}$+1 for any arc $I_j\in I$.

Proof: If $L_0(I_j)-L_2(I_j)=\varnothing$ then all the integers in $L_0(I_j)$ are already used to label the vertices at distance two from the arc I_j before labeling I_j. So no integer is available in $L_0(I_j)$ that can be used to label the arc I_j satisfying L(0, 1) labeling condition, i.e., $f_j\neq I$ for $l\in L_0(I_j)-L_2(I_j)$. So, we must label the arc I_j by a new integer m where m = max$\{L_0(I_j)\}$+1, otherwise the condition of labeling does not satisfied. Now we discuss about the bounds of $\lambda_{0,1}(G)$ for a circular-arc graphs.

Theorem 1: For any circular-arc graph G $\lambda_{0,1}(G)\geq k-1$ where, $k=\max_{I_j\in I}|S_{I_j}|$, j = 1, 2, 3, ..., n.

Proof: Let G be a circular-arc graph and I = $\{I_1, I_2, I_3, ..., I_n\}$. Let $I_\alpha\in I$ such that $|S_{I_\alpha}|=\max_{I_j\in I}|S_{I_j}|=k$, then clearly forms a subgraph of G. Thus, when we label this subgraph by L(0, 1)-labeling, then any one member of $S_{I\alpha}$ and I_α takes same label and all other members get distinct labels. Thus, exactly k labels (namely 0, 1, 2,...., k-1) are needed to label the subgraph $S_{I_\alpha}\cup\{I_\alpha\}$. Hence, $\lambda_{0,1}(G)\geq k-1$.

Lemma 5: For any circular-arc graph G, $L_2(I_k)\subseteq L_0(I_k)$ for any arc $I_k\in 1$.

Proof: Any label used to label a circular-arc graph G belongs to $L_0(I_k)$. Thus, any label $l\in L_2(I_k)$implies $l\in L_0(I_k)$. Hence $L_2(I_k)\subseteq L_0(I_k)$

Lemma 6: For a circular-arc graph G $|L_2(I_k)|\leq\Delta$ for any arc $l\in L_k$.

Proof: Let G be a circular-arc graph and L_k be any arc of G and let $|L_2(I_k)|$ = m. This implies that m number of distinct labels are used to label the arcs which are in distance two from the arc I_r before labeling the arc I_k.

Since G is a circular-arc graph, there must exists an arc say I_k which is adjacent to at least m arcs of G. Therefore, the degree of the vertex corresponding to the arc I_r is at least m. Hence the degree of the graph G is at least m. Hence, $m\leq\Delta$, i.e., $L_2(I_k)|\leq\Delta$.

Lemma 7: For any circular-arc graph G, $L_0(I_k)$-$L_2(I_k)$ where $L_2(I_k)$ = $\{0, 1, 2, ..., \Delta\}$.

Proof: Here $L_0(I_k)$= $\{0, 1, 2, ..., \Delta\}$. If possible let $L_0 = \Delta+1$. At every stage of labeling $L_2(I_k)\subseteq L_0(I_k)$, by Lemma 4. Since $L_0(I_k)$-$L_2(I_k)$ = $\varnothing$, we must have $L_2(I_k)|$ = $L_0(I_k)|$ which contradicts $L_2(I_k)|\leq\Delta$, (Lemma 6). Hence $L_0(I_k)$-$L_2(I_k)\neq\varnothing$

Theorem 2: For any circular-arc graph G, $\lambda_{0,1}(G)\leq\Delta$ where Δ is the degree of the graph G.

Proof: Let the total number of arcs of the circular-arc graph G be n and the set of arcs I = $\{I_1, I_2, I_3, ..., I_n\}$. Let $L_0(I_k)$ = $\{0, 1, 2, ..., \Delta\}$ where, $I_k\in I$. Then $(G)\leq\Delta$, if we can prove that no extra label is necessary to label all the arcs of G. At every stage of labeling $L_2(I_k)\subseteq L_0(I_k)$, by Lemma 5. And we need extra label only when $L_0(I_k)$-$L_2(I_k)$ = $\varnothing$. But by Lemma 7, $L_0(I_k)$-$L_2(I_k)$ = $\varnothing$. So, f_k =1 where, $l\in L_0(I_k)$-$L_2(I_k)$. Therefore, extra label is not needed to label the arc I_k. Since, I_k is arbitrary, therefore one can conclude that $\lambda_{0,1}(G)\leq\Delta$.

Algorithm for L(0, 1)-labeling: In this subsection, we design an algorithm to L_0-label a circular-arc graphs.

Algorithm L01
Input: A set of ordered arcs I of a circular-arc graph
//assume that the arcs are ordered with respect to left end points (i.e., in clockwise
direction namely $I_1, I_2, I_3,..., I_n$) where I = $\{I_1, I_2, I_3,..., I_n\}$//
Output: f_j, the L(0, 1)-label of Ij j = 1, 2, 3..., n
Initialization: $f_1 = 0$
$L_0(I_O) = \{0\}$;
for each j = 2 to n-1 compute $L_1(I_j)$ and $L_2(I_j)$
if $L_1(I_j)$-$L_2(I_j)\neq\varnothing$ then f_j = l and set $L_0(I_{j+1}) = L_0(I_j)$
for any $l\in L_1(I_j$-$L_1(I_j)$
else if $L_1(I_j)$-$L_2(I_j)\neq\varnothing$ then f_j = m and set $L_0(I_{j+1}) = L_0(I_j)$
where $m\in L_0(I_j)$-$L_2(I_j)$;
else f_j = p where p = max$\{L_0(I_j)\}$+1 and set
end for
if $L_1(I_n)-L_2(I_n)\neq\varnothing$ then f_n = q where $q\in L_1(I_n)-L_2(I_n)$
else if $L_0(I_n)-L_2(I_n)\neq\varnothing$ then f_n = r where $r\in L_0(I_n)-L_2(I_n)$
else f_n = s where S = max$\{L_0(I_n)\}$+1
end L01

Theorem 3: The Algorithm L01 correctly labels the vertices of a circular-arc graph using L(0, 1)-labeling condition. The maximum label used by this algorithm is Δ.

Proof: Let I = $\{I_1, I_2, I_3, ..., I_n\}$, also let $f_1 = 0$,.$L_0(I_k) = \{0\}$.

Case 1: If the set $L_0(I_k)$ is sufficient to label the whole graph then the result is obviously true and $\lambda_{0,1}(G)$.

Case 2: If we use extra label then we have to show that the set $L_0(I_k)$ is not sufficient to label the graph. Suppose we label the arc I_k. This arc can not be labeled by a label from the set Li_k. In this case, obviously $L_1(I_k)-L_2(I_k)=\varnothing$, otherwise by Lemma 2, f_k = I where, $l\in L_1(I_k)-L_2(I_k)\subseteq L_0(I_k)$. In this case $L_2(I_k)-L_2(I_k)=\varnothing$, otherwise by Lemma 3, f_k = m where $m\in L_0(I_k)-L_2(I_k)\subseteq L_0(I_k)$. So, all the labels in $L_0(I_k)$ are already use to label the arcs which are at distance two from the arc I_k before labeling I_k. So, there is no scope to label the arc I_k by a label from the set $L_0(I_k)$. So by Lemma

4, we must label the arc I_k by an extra label I_k, i.e., $f_k = m$ where, $m = L_0(I_k+1)$; otherwise the condition of L(0, 1)-labeling is violated.

If $L_0(I_k) = \{0, 1, 2, ..., \Delta\}$, then by Lemma 7, $L_0(I_k)-L_2(I_{kl}) \neq \varnothing$. According to our proposed algorithm, we need additional label if $L_0(I_j)-L_2(I_j)\neq\varnothing$. But $L_0(I_j)-L_2(I_j)\neq\varnothing$, so additional label is not required to label the arc I_k. This is true for any arc I_k. Hence $\lambda_{0,1}(G)\leq\Delta$. It may be noted that after completion of the entire circular arc graph the set L becomes $L_0(I_n)\cup\{f_n\}$ and $\lambda_{0,1}(G) = \max\{L\}$.

Theorem 4: Any circular-arc graph can be L(0, 1)-labeled using $O(n\Delta^2)$? time where n and Δ represents the number of vertices and the degree of the graph G.

Proof: Let L be the label set and |L| be its cardinality. According to the algorithm L01, $|L_1(I_k)|\leq|L|$ and $|L_2(I_k)|\leq|L|$ for any $I_k\in I$. So, $L_0(I_j)-L_2(I_j)$ can be computed using at most $|L|.|L| = |L|^2$ time. Here $L_2(I_k)\subseteq L_0(I_k)$ and both $L_0(I_j)$ and $L_2(I_j)$ are subsets of $\{0, 1, 2, ..., |L|-1\}$, so using Algorithm $L_0(I_j)-L_2(I_j)$ diff B $L_0(I_j)-L_2(I_j)$ can be computed using |L| time. Again, the union of the set $L_0(I_j)$ and a singleton set can be done in unit time, since the sets are disjoint. This process is repeated for n-1 times. So, the total time complexity for the algorithm L01 is $O((n-1)|L|^2) = O(n|L|^2)$. Since, $|L|\leq\Delta$, therefore the running time for the algorithm L01 is $O(n\Delta^2)$.

Illustration of the algorithm L01: Let us, consider the circular-arc graph of Fig. 2 to illustrate the algorithm L01. Now $I = \{I_1, I_2, I_3, ..., I_{10}\}$ and also $\Delta = 4$; f_j = the label of the arc I_j, for j = 1, 2, 3, ..., 10

$f_1 = 0$, $L_0(I_2) = \{0\}$

$L_0(I_j)-L_2(I_j) = \{0\}-\varnothing = \{0\}\neq\varnothing$

So $f_2 = 0$, $L_0(I_3) = L_0(I_2) = \{0\}$

$L_1(I_3)-L_2(I_3) = \{0\}-\{0\} = \varnothing$

Also $L_0(I_3)-L_2(I_3) = \{0\}-\{0\} = \varnothing$

Therefore:

$f_3 = \max\{L_0(I_3)\}+1 = 0+1 = 1,\quad L_0(I_4) = L_0(I_3)\cup\{1\} = \{0\}\cup\{1\} = \{0,1\}$

$L_1(I_4)-L_2(I_4) = \{0,1\}-\{0\} = \{1\}\neq\varnothing$

So, $f_4 = 1$, $L_0(I_5) = L_0(I_4) = \{0,1\}$

$L_1(I_5)-L_2(I_5) = \{1\}-\{0\} = \{1\}\neq\varnothing$

So $f_5 = 1$, $L_0(I_6) = L_0(I_5) = \{0,1\}$

$L_1(I_6)-L_2(I_6) = \{1\}-\{1,0\} = \varnothing$

Also $L_0(I_6)-L_2(I_6) = \{0,1\}-\{1,0\} = \varnothing$

Therefore:

$f_6 = \max\{L_0(I_6)\}+1 = 1+1 = 2, L_0(I_7) = L_0(I_6)\cup\{2\} = \{0,1\}\cup\{2\} = \{0,1,2\}$

$L_1(I_7)-L_2(I_7) = \{1,2\}-\{1\} = \{2\}\neq\varnothing$

So $f_7 = 2$, $L_0(I_8) = L_0(I_7) = \{0,1,2\}$

$L_1(I_8)-L_2(I_8) = \{2\}-\{1,2,0\} = \varnothing$

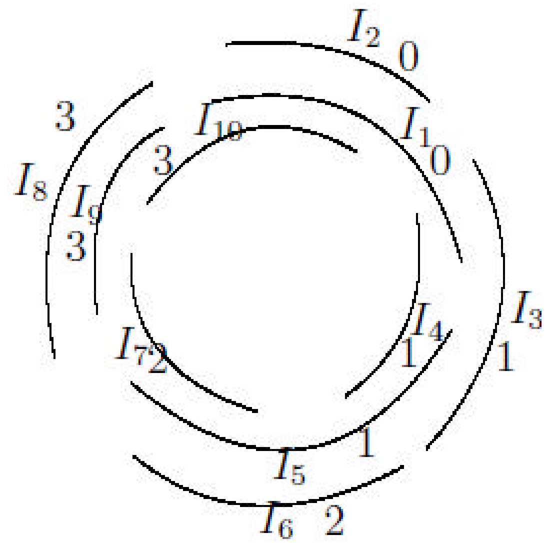

Fig. 2: Illustration of Alogrithm L01

Also, $L_0(I_8)-L_2(I_8) = \{0,1,2\}-\{1,2,0\} = \varnothing$

Therefore:

$f_8 = \max\{L_0(I_8)\}+1 = 2+1 = 3$

$L_0(I_9) = L_0(I_8)\cup\{3\} = \{0,1,2\}\cup\{3\} = \{0,1,2,3\}$

$L_1(I_9)-L_2(I_9) = \{3,2\}-\{0,1,2\} = \{3\}\neq\varnothing$

So $f_9 = 3$, $L_0(I_{10}) = L_0(I_9) = \{0,1,2,3\}$

$L_1(I_{10})-L_2(I_{10}) = \{3,0\}-\{2,1\} = \{3,0\}\neq\varnothing$

So $f_{10} = 3$

Therefore the label set $L = L_0(I_{10})\cup\{f_{10}\} = \{0,1,2,3\}\cup\{3\} = \{0,1,2,3\}$ and $\lambda_{0,1}(G) = \max\{L\} = 3$

RESULTS AND DISCUSSION

L(1, 1)-labeling of circular-arc graphs: By extending the idea of L(0, 1)-labeling, we design an algorithm for L(1, 1)-labeling of circular-arc graphs. In this section we present some lemmas related to our work, upper bound of L(1, 1)-labeling, an algorithm L11 and time complexity of the proposed algorithm L11.

Lemma 8: If $L_0(I_j)-L_{1\vee2}(I_j)\neq\varphi$, then $f_j = l$ where $l\in L_0(I_j)$ for any $I_j\in I$.

Proof: If $L_0(I_j)-L_{1\vee2}(I_j)\neq\varnothing$, then the set $L_0(I_j)-L_{1\vee2}(I_j)$ contains some integers which are not used to label the arcs at distance one or two from the arc I_j before labeling I_j. So any label $l\in L_0(I_j)-L_{1\vee2}(I_j)$ is the valid label for I_j, i.e., $f_j = l$, since it satisfies L(1, 1)?-labeling condition.

Lemma 9: If $L_0(I_j)-L_{1\vee2}(I_j)=\varnothing$, then $f_j\neq l$ for any $l\in L_0(I_j)-L_{1\vee2}(I_j)$ but $f_j = m$ where $m = \max\{L_0(I_j)\}+1$.

Proof: If $L_0(I_j)$-$L_{1v2}(I_j)$ = ∅, then all the labels in $L_0(I_j)$are already used to label the arcs at distance one or two from the arc I_j before labeling the arc I_j. So no integer in $L_0(I_j)$is available that can be used to label the arc I_j, satisfying L(1, 1)-labeling condition, i.e., $f_j \neq I$, for any $I \in L_0(I_j) - L_{1v2}(I_j)$. So in this case we must label the arc I_j by m, i.e., f_j = m where, $m = \max\{L_0(I_j)\} + 1$, otherwise the condition of L(1, 1)-labeling does not satisfied. Now we discuss about the upper bound of $\lambda_1 G_1$ of a the circular-arc graphs.

Lemma 10: For any circular-arc graph G, $L_{1v2}(I_k) \subseteq L_0(I_k)$ for any arc I_k of G.

Proof: In this case, any label used to label a circular-arc graph G belongs to $L_0(I_k)$. So any label $I \in L_{1v2}(I_k)$ implies $I \in L_0(I_k)$. Hence, $L_{1v2}(I_k) \subseteq L_0(I_k)$.

Lemma 11: For any circular-arc graph G, $L_0(I_k)$- $L_{1v2}(I_k)$ ≠ ∅, for any arc I_k of G where $L_0(I_k) = \{0, 1, 2, \ldots, 2\Delta\}$.

Proof: Here $L_0(I_k) = \{0, 1, 2, \ldots, 2\Delta\}$. If possible let $L_0(I_k)$- $L_{1v2}(I_k)$ ≠ ∅. At every stage of labeling $L_{1v2}(I_k) \subseteq L_0(I_k)$, by Lemma 10. Since, $L_0(I_k)$-$L_{1v2}(I_k) \neq \emptyset$, we must have $|L_{1v2}(I_k)| = |L_0(I_k)| = 2\Delta + 1$.

Now $|L_{1v2}(I_k)| = |L_0(I_k)| = 2\Delta + 1$. This implies $|L_1(I_k) \cup L_2(I_k)| = 2\Delta + 1$. Again $|L_1(I_k)| \leq \Delta$, otherwise it contradicts that the degree of the graph is Δ. So, $|L_2(I_k)| \geq \Delta + 1$ which contradicts $|L_2(I_k)| \leq \Delta$, (Lemma 6). So our assumption is wrong. Therefore, $L_0(I_k)$- $L_{1v2}(I_k) \neq \emptyset$. Hence, the lemma.

Theorem 5: For any circular-arc graph G, $\lambda_1(G) \leq 2\Delta$, where Δ is the degree of the graph G.

Proof: Let the total number of arcs of the circular-arc graph G? be n and the set of arcs I = $\{I_1, I_2, I_3, \ldots, I_n\}$. Let $L_0(I_k) = \{1, 2, \ldots, 2\Delta\}$ where, $I_k \in I$. Then $\lambda_1(G) \leq 2\Delta$, if we can prove that no extra label is necessary to label all the arcs of G. At every stage of labeling $L_{1v2}(I_k) \subseteq L_0(I_k)$, by Lemma 10. And we need extra label only when $L_0(I_k)$-$L_{1v2}(I_k) \neq \emptyset$. But by Lemma 11, $L_0(I_k)$-$L_{1v2}(I_k) \neq \emptyset$. So, f_k = l where $I_k \in I$ $L_0(I_k)$-$L_{1v2}(I_k)$. Therefore, extra label is not needed to label the arc I_k. Since, I_k is arbitrary, therefore one can conclude that $\lambda_1(G) \leq 2\Delta$.

Algorithm for L(1,1)-labeling: In this subsection we present an algorithm to L(1, 1)-label a circular-arc graphs.

Algorithm L(1, 1)
Input: A set of ordered arcs ? of a circular-arc graph
//assume that the arcs are ordered with respect to left end points (namely $I_1, I_1, I_3 \ldots I_n$) where $I_1, I_1, I_3 \ldots I_n$ }//
Output: F_J the L(1, 1) label of J_J J = 1, 2, 3, ..., n.
Initialization: $f_1 = 0$;
$L_0(I_2) = \{0\}$;
for each J = 2 to n-1 compute $L1_{1V2?}(I_J)$
if $L(I_J) \neq \emptyset$ then f_J = 1 and set $L_0(I_{J+1}) = L_0(I_J)$
where I∈L $0(I_J)$-$L_1V_2(I_J)$;
else f_I = m where $m = \max\{L_0(I_j)\} + 1$ and set
$L_0(I_{j+1}) = L_0(I_j) \cup \{m\}$
end for
if $L_0(I_n) - L_{1v2}(I_n) \neq \emptyset$ Then f_n = P, where P is any integer of the set $L_0(I_n) - L_{1v2}(I_n)$
else f_n = P where $q = \max\{L_0(I_n)\} + 1$
end L11

Theorem 6: The Algorithm L11 correctly labels the vertices of a circular-arc graph using L(1, 1)-labeling condition. The maximum label used by this algorithm is 2Δ.

Proof: Let $I = \{I_1, I_2, I_3, \ldots, I_n\}$, also let f_1, $L_0(I_2) = \{0\}$. If the graph has only one vertex then obviously, $\lambda_{1,1}G = 0$. If the graph contains more than one vertex then the set $L_{1v2}(I_k)$ is insufficient to label the whole graph. Now we are going to label the arc I_k. We $m = \{L_0(I_k)\} + 1$ can not label the arcI_k by the label in the set $L_0(I_k)$-$L_{1v2}(I_k)$. In this case obviously $L_0(I_k)$-$L_{1v2}(I_k)$= ϕ, otherwise by Lemma 8, f_k = I where $I \in L_0(I_k) - L_{1v2}(I_k) \subseteq L_0(I_k)$. So all the labels in $L0(I_k)$ are already used to label the arcs which are in distance one or two from the arc I_k before labeling the arc I_k. So, there is no scope to label the arc I_k by the label in the set $L_0(I_k)$.

Hence by Lemma 9, we label the arc I_k by an additional label m, i.e., f_k where $m = \{L_0(I_k)\} + 1$; otherwise the condition of L(1,1)-labeling is violated. If $L_0(I_K)$, then by Lemma 11, $L_0(I_k)$-$L_{1v2}(I_k)$ = ϕ According to our proposed algorithm, we need additional label. If $L_0(I_k)$-$L_{1v2}(I_k)$ = ϕ. But $L_0(I_k)$-$L_{1v2}(I_k) \neq \emptyset$, so additional label is not required to label the arc I_k. This is true for any arc I_K. Hence, $\lambda_{1,1}(G) \leq 2\Delta$. It may be noted that after completion of the labeling of the entire circular arc graph the set L becomes $L_0(I_n) \cup \{f_K\}$ and $\lambda_{1,1}(G) = \max\{L\}$.

Theorem 7: Any circular-arc graph can be L(1,1)-labeled using O(nΔ) time where n and Δ represent number of vertices and the degree of the graph G.

Proof: Let, L be the label set and |L| be its cardinality. Here, $L_0(I_k) \subseteq L_{1v2}(I_k)$ and both $L_0(I_K)$ and $L_{1V2}(I_K)$ are subsets of {1, 2, 3..., |L|-1}, so there difference $L_0(I_K)$-$L_{1V2}(I_K)$ can be computed using |L| time. Also, the maximum of $L_0(I_K)$ can be determined in unit time as it is an ordered set. Again, the union of the set $L_0(I_K)$ and a singleton set can be done in unit time, since the sets are disjoint. This process is repeated for n-1 times. So the total time complexity for the algorithm L11 is $O((n-1)|L|) = O(n|L|)$. Since, $|L| \leq 2\Delta$, therefore the time complexity for the algorithm L11 O(nΔ).

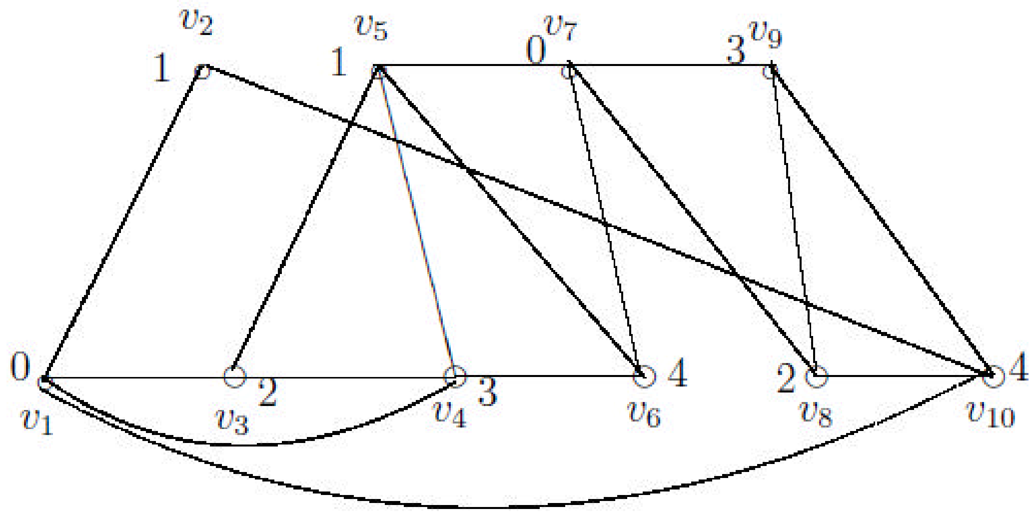

Fig. 3: Illustration of Alogrithm L11

Illustration of the algorithm L11:

To illustrate the algorithm we consider a circular-arc graph of Fig. 3.

Let $I = \{I_1, I_2, I_3, ..., I_{10}\}$ and also $\Delta 4$

f_J The label of the arc $_{IJ}$, for $j = 1,2,3,....,10$

$f_1 = 0 L_0,(I_2) = \{0\}$.

$L0\ (I_2)\text{-}L_{1v2}(I2) = \{0\}\text{-}\{0\} = \varnothing$

Therefore, $f_2 = \max\{L_0(I_2)\} + 1 = 0 + 1 = 1$

$L_0(I_3) = L_0(I_2) \cup \{f_2\} = \{0\} \cup \{1\} = \{0,1\}$

$L_0(I_2) - L_{1v2}(I_2) = \{0\} - \{0\} = \varnothing$

Therefoe,

$f_3 = \max\{L_0(I_3)\} + 1 = 1 + 1 = 2, L_0(I_4) = L_0(I_3) \cup \{f_3\} = \{0,1\} \cup \{2\} = \{0,1,2\}$

$L_0(I_4) - L_{1v2}(I_4) = \{0,1,2\} - \{0,2,1\} = \varnothing$

Therefore:

$f_4 = \max\{L_0(I_4)\} + 1 = 2 + 1 = 3$

$L_0(I_5) = L_0(I_4) \cup \{f_4\} = \{0,1,2\} \cup \{3\} = \{0,1,2,3\}$

$L_0(I_5) - L_{1v2}(I_5) = \{0,1,2,3\} - \{3,2,0\} = \{1\} \neq \varnothing$

So $f_5 = 1$, $L_0(I_6) = L_0(I_5) = \{0,1,2,3\}$

$L_0(I_6) - L_{1v2}(I_6) = \{0,1,2,3\} - \{1,3,2,0\} = \varnothing$

Therefore:

$f_6 = \max\{L_0(I_6)\} + 1 = 3 + 1 = 4$

$L_0(I_7) = L_0(I_6) \cup \{f_6\} = \{0,1,2,3\} \cup \{4\} = \{0,1,2,3,4\}$

$L_0(I_7) - L_{1v2}(I_7) = \{0,1,2,3,4\} - \{1,4,3,2\} = \{0\} \neq \varnothing$

So $f_7 = 0$, $L_0(I_8) = L_0(I_7) = \{0,1,2,3,4\}$.

$L_0(I_8) - L_{1v2}(I_8) = \{0,1,2,3,4\} - \{0,1,4\} = \{2,3\} \neq \varnothing$

So $f_8 = 2$, $L_0(I_9) = L_0(I_8) = \{0,1,2,3,4\}$

So $f_9 = 3$, $L_0(I_{10}) = L_0(I_9) = \{0,1,2,3,4\}$

$L_0(I_{10}) - L_{1v2}(I_{10}) = \{0,1,2,3,4\} - \{0,1,3,2\} = \{4\} \neq \varnothing$

So $f_{10} = 4$

Therefore the label set:

$L = L_0(I_{10}) \cup \{f_{10}\} = \{0,1,2,3,4\} \cup \{4\} = \{0,1,2,3,4\}$

Hence, $\lambda_{1,1}(G) = \max L = 4$

CONCLUSION

In this study, we determine the upper bounds for $\lambda_{0,1}$ and $\lambda_{1,1}$ for a circular-arc graph G and have shown that $\lambda_{0,1}(G) \leq \Delta$ and $\lambda_{1,1}(G) \leq 2\Delta$. These are not the first bounds for the problems on circular-arc graphs but this result is tighter than the previous available results $\lambda_{0,1}(G) \leq 2\Delta$ and $\lambda_{1,1}(G) \leq 2\Delta + \omega$ (Calamoneri *et al.*, 2009). Also, two algorithms are designed to L(0,1)-label and L(1,1)-label for circular-arc graphs. The time complexities for the algorithms are $O(n\Delta^2)$ and $O(n\Delta)$, respectively. But we are unable to find exact value of $\lambda_{0,1}(G)$ and $\lambda_{1,1}(G)$ for these graphs by our proposed algorithms. We feel that it is very difficult to determine the exact value of $\lambda_{0,1}(G)$ and $\lambda_{1,1}(G)$ for circular-arc graphs. Also, we feel that there is a chance for new upper bounds for the problems and the time complexities of the proposed algorithms may be reduced.

APPENDIX

Here we discuss an algorithm to compute A-B where $A \subseteq B$ and both A and B are subsets of $\{0,1,...,|L|-1\}$. The time complexity of the propose algorithm is $|L|$.

Algorithm A diff B.

Input: The array $A = \{A_0, A_1, A_2, ..., A_p\}$ and $(B = B_0, B_1, B_2, ..., B_p)$.

Output: A-B

Step 1. for $i = 0$ to $|L|-1$

set $a_i = 0, b_i = 0$; //where a_i and b_i are the variables corresponding to A_i and:

B_i respectively.//

for $j = 0$ to p

if $A_j = k$ then $a_k = 1$

Step 2. for $j = 0$ to q

if $B_j = k$ then $b_k = 1$

Step 3. for $i = 0$ to $|L|-1$

$c_i = a_i - b_i$

if $c_i = 1$ then put i to the set A-B

stop

REFERENCES

Alon, N. and B. Mohar, 2002. The chromatic number of graph powers. Comb. Probab. Comput., 11: 1-10.

Amanathulla, S.K. and M. Pal, 2016. L(3, 2, 1)-and L(4, 3, 2, 1)-labeling problems on circular-arc graphs. Int. J. Control Theory Appl., 9: 869-884.

Battiti, R., A.A. Bertossi and M.A. Bounccelli, 1999. Assigning code in wireless networks: Bounds and scaling properties. Wirel. Network, 5: 195-209.

Bertossi, A.A. and C.M. Pinotti, 2007. Approximate L($ò_1$, $ò_2$,....,$ò_t$-coloring of trees and interval graphs Networks, 49: 204-216.

Bertossi, A.A. and M.A. Bonuccelli, 1995. Code assignment for hidden terminal interference avoidance in multihop packet radio networks. IEEE. ACM. Trans. Networking, 3: 441-449.

Bodlaender, H.L., T. Kloks, R.B. Tan and J.V. Leeuwen, 2004. Approximations for -colorings of graphs. Comput. J., 47: 193-204.

Calamoneri, T. and R. Petreschi, 2004. L(h, 1)-labeling of planer graphs. J. Parallel Distrib. Comput., 64: 414-426.

Calamoneri, T., 2006. Optimal L(h, k)-labeling of regular grids. Discret. Math. Theor. Comput. Sci., 8: 141-158.

Calamoneri, T., 2014. The L(h, k)-labeling problem: An updated survey and annotated bibliography. Comput, J., 54: 1-54.

Calamoneri, T., S. Caminiti and R. Petreschi, 2009. On the L(h, k) -labeling of co-comparability graphs and circular-arc graphs. Networks, 53: 27-34.

Chang, G.J. and C. Lu, 2003. Distance two labelling of graphs. Eur. J. Comb., 24: 53-58.

Chang, G.J. and D. Kua, 1996. The L(2, 1)-labeling on graphs. SIAM. J. Discrete Math., 9: 309-316.

Chiang, S.H. and J.H. Yan, 2008. On L(d,1)-labeling of cartesian product of a path. Discrete Appl. Mathe., 156: 2867-2881.

Das, S.K., I. Finocchi and R. Petreschi, 2006. Conflict-free star-access in parallel memory systems. J. Parallel Distrib. Comput., 66: 1431-1441.

Golumbic, M.C., 2004. Algorithmic Graph Theory and Perfect Graphs. 2nd Edn., Elsevier, Amsterdam, Netherlands, ISBN:0-444-51530-5, Pages: 313.

Griggs, J. and R.K. Yeh, 1992. Labeling graphs with a condition at distance two. SIAM. J. Discrete Math., 5: 586-595.

Hale, W.K., 1980. Frequency assignment: Theory and applications. Proc. IEEE., 68: 1497-1514.

Khan, N., M. Pal and A. Pal, 2012. L(0, 1)-lavelling of cactus graphs. Commun. Network, 4: 18-29.

Makansi, T., 1987. Transmitter-oriented code assignment for multihop packet radio. IEEE. Trans. Commun., 35: 1379-1382.

Olariu, S., 1991. An optimal greedy heuristic to color interval graphs. Inf. Process. Lett., 37: 21-25.

Pal, A. and M. Pal, 2009. Interval tree and its applications. Adv. Model. Optim., 11: 211-224.

Pal, M. and G.P. Bhattacharjee, 1995. Optimal sequential and parallel algorithms for computing the diameter and the center of an interval graph. Int. J. Comput. Mathe., 59: 1-13.

Pal, M. and G.P. Bhattacharjee, 1996. An optimal parallel algorithm to color an interval graph. Parallel Process. Lett., 6: 439-449.

Pal, M. and G.P. Bhattacharjee, 1997. A data structure on interval graphs and its applications. J. Circuits Syst. Comput., 7: 165-175.

Pal, M., 1995. Some sequential and parallel algorithms on interval graphs. Ph.D Thesis, Indian Institute of Technology, Kharagpur, India.

Pal, M., 2013. Intersection graphs: An introduction. Anal. Pure Appl. Math., 4: 41-93.

Saha, A., M. Pal and T.K. Pal, 2007. Selection of programme slots of television channels for giving advertisement: A graph theoretic approach. Inf. Sci., 177: 2480-2492.

Sakai, D., 1994. Labeling chordal graphs with a condition at distance two. SIAM. J. Discrete Math., 7: 133-140.

Wan, P.J., 1997. Near-optimal conflict-free channel set assignments for an optical cluster-based hypercube network. J. Combin. Optim., 1: 179-186.