International Journal of System Signal Control and Engineering Application 1 (1): 49-63, 2008
ISSN: 1997-5422


# A Survey on $H_\infty$ Robust Stabilization and Filtering Design in Control, Signal Processing, Communications, Systems and Synthetic Biology

Bor-Sen Chen and Yu-Te Chang
Department of Electrical Engineering, National Tsing Hua University, Hsinchu, Taiwan

**Abstract:** Inherently, the real physical systems suffer from intrinsic parameter uncertainties and external disturbance. $H_\infty$ robust stabilization and filtering design have been developed to efficiently tolerate parameter uncertainties and to attenuate the effect of external disturbances so that the designed systems can work reliably. In the past 2 decades, the $H_\infty$ robust designs have been developed from linear systems to nonlinear systems and nonlinear stochastic systems. Their applications are also extended from control systems to signal processings, communication systems and bio-molecular systems. This study presents a survey on the development of $H_\infty$ robust stabilization and filtering theories and their potential applications to the robust designs of robotics, spacecraft, missile, quantum system, systems biology, synthetic gene network, filter bank and wireless communication. The perspectives of $H_\infty$ robust stabilization and filtering design in future are also discussed, especially, in the nascent filed of synthetic biology.

**Key words:** $H_\infty$ stabilization, $H_\infty$ filtering, parameter uncertainties, external disturbance, $H_\infty$ robust control design, systems biology, robust synthetic biology

## INTRODUCTION

In control system design, the controlled system suffers from intrinsic parameter uncertainties and external disturbances. They are due to changes of operation points, unmodeled dynamics, truncation of high order modes, environmental disturbances etc. A primary objective of feedback is to reject disturbances and parameter uncertainties and to provide good tracking facilities. This is the most important topic in system control design to perform more reliably to achieve design purpose.

At the beginning of 1980's, Zames (1981) employed a minimum sensitivity design from the minimum $L_2$-gain perspective and began the research of the topic. The minimum $L_2$-gain from the external disturbance to the reference tracking error in linear time-invariant systems is just equivalent to the minimum $H_\infty$ norm of sensitivity function from the external disturbance to the reference tracking error from the Hardy space perspective. This is the so-called the optimal $H_\infty$ disturbance attenuation design in the frequency domain (Francis, 1987). Then this $H_\infty$ robust control design is combined with the Youla controller (Youla *et al*., 1976a, b), a generalized linear controller based on the algebraic coprime factorization, to efficiently solve the robust stabilization and disturbance rejection design problem via a so-called generalized interpolation technique in frequency domain (Chang and Pearson, 1984; Chen and Kung, 1984; Francis, 1987; Safonov and Chen, 1982; Weinmann, 1991). These techniques have been also employed to treat the robust filtering design problems in signal processing (Grimble and El Sayed, 1990; Peng and Chen, 1997). The main shortages of $H_\infty$ robust design problem in the frequency domain are that first, it need to solve a spectral factorization problem, second, the order of controller or filter is very high so that it is not easy to implement, third, it is not easy to extend their applications to nonlinear systems case. To remedy the first two shortages, a $H_\infty$ robust PID control design has been developed for practical designs via genetic algorithm (Chen and Cheng, 1998; Chen *et al*., 1995; Lee *et al*., 2004).

In the late 1980's, the $H_\infty$ control design was developed based on the state space model in time domain by solving a corresponding Riccati-like equation (Basar and Bernhard, 1995; Doyle *et al*., 1994; Doyle *et al*., 1989; Khargonekar *et al*., 1991; Ravi *et al*., 1991; Stoorvogel, 1992; Xie *et al*., 1992). In order to efficiently solve the robust stabilization and disturbance rejection, the optimal $H_\infty$ control design can be solved by the linear matrix inequality (LMI) method from the suboptimal perspective, which can be efficiently solved by LMI toolbox in Matlab via a convex optimization method (Boyd *et al*., 1994).

**Corresponding Author:** Bor-Sen Chen, Department of Electrical Engineering, National Tsing Hua University, Hsinchu, Taiwan

At the beginning of 1990's, the $H_\infty$ robust control design is extended to the nonlinear system with external disturbance (Ball *et al*., 1993; Isidori and Astolfi, 1992; Van der Schaft, 1992). The nonlinear $H_\infty$ control design needs to solve a nonlinear Hamilton Jacobi inequality (HJI). In the same period, the mixed $H_\infty$ control design in linear uncertain systems was developed when both $H_2$ optimal control and $H_\infty$ attenuation performance are considered simultaneously, which need to solve two Riccati-like equations, one for $H_2$ optimal control and another for $H_\infty$ disturbance attenuation (Chen and Chang, 1997; Chen *et al*., 1995; Hung and Chen, 2000; Limebeer *et al*., 1994).

In general, it is very difficult to solve HJI for nonlinear $H_\infty$ robust control design analytically or numerically, except some mechanical systems in robotics (Aguilar *et al*., 2003; Chang, 2005; Tseng, 2005) and spacecraft (Chen *et al*., 2000; Gershon *et al*., 2007; Wu and Chen, 2001; Wu *et al*., 1999), in which the HJI can be transformed to a corresponding LMI by an adequate choice of Lyapunov function and the use of skew symmetry property (Chang and Chen, 1997; Chen and Chang, 1997; Chen *et al*., 1994; Wu and Chen, 2001; Wu *et al*., 1999). Therefore, the application of nonlinear $H_\infty$ control is limited by the difficulty of solving HJI. In this period, the fuzzy models have been employed to interpolate several linear systems via fuzzy bases to efficiently approximate the nonlinear uncertain systems. Therefore, Takaig-Sugeno (T-S) model (Takagi and Sugeno, 1985) was used to solve the nonlinear $H_\infty$ stabilization and disturbance rejection problem (Chen *et al*., 1994, 1999, 2000; Tanaka and Wang, 2001), which only need to solve a set of LMIs and can be efficiently solved by the LMI toolbox in Matlab (Boyd *et al*., 1994). Hence, by fuzzy interpolation method, the nonlinear $H_\infty$ control design can be easily applied to robust control designs of uncertain robotic systems (Chang and Chen, 1997; Chen *et al*., 1994, 2000; Tseng and Chen, 2001), non-holonomic mechanical systems (Chang and Chen, 2000), spacecraft systems (Chen *et al*., 2000; Wu and Chen, 2001; Wu *et al*., 1999) and missile systems (Chen *et al*., 2002; Uang and Chen, 2002) and quantum systems with external disturbances (Chen *et al*., 2008).

At the beginning of 2000's (Chen and Zhang, 2004; Hinrichsen and Pritchard, 1998; Zhang and Chen, 2006), the $H_\infty$ robust control design was extended to linear and nonlinear stochastic systems, in which the random parameter variations can be modeled as state-dependent noises. Stochastic Nash game was employed to treat the $H_\infty$ robust control or mixed $H_2/H_\infty$ robust control (Chen and Zhang, 2004; Zhang and Chen, 2006; Zhang *et al*., 2006) or filtering design problem (Zhang *et al*., 2005) in stochastic systems with external disturbances. Ito formula should be employed to treat diffusion terms due to stochastic parametric fluctuations. At the same time, the $H_\infty$ robust filtering has been developed for robust system identification (Chen *et al*., 1994), channel tracking (Chen *et al*., 1994) and signal reconstruction in signal processings (Chen *et al*., 2001; De Souza *et al*., 1995; Fridman and Shaked, 2001; Hung and Chen, 2000; Shaked *et al*., 2001; Xie *et al*., 2002) and robust equalization designs in communication systems (Chen *et al*., 2003; Lee *et al*., 2006) with channel disturbances due to multipath channel fading, shadowing and Doppler effect.

Recently, the study and design of gene networks have become a hot topic in systems biology and synthetic biology (Andrianantoandro *et al*., 2006). In the nano-scale, a gene network is inherent with random intrinsic fluctuation and environmental molecular noises (Andrianantoandro *et al*., 2006). In recent years, the robustness and filtering properties of biochemical or gene networks have attracted much attention of molecular biologists (Chen and Wang, 2006; Chen *et al*., 2005). More recently, the stability robustness and filtering ability of gene networks under intrinsic parameter fluctuations and external disturbances have been discussed from the nonlinear stochastic $H_\infty$ stabilization and filtering perspective (Chen *et al*., 2008; Chen and Wu, 2008; Chen *et al*., 2007), which can also be considered as the underlying principles of network evolution under nature selection (Chen and Wu, 2007). Furthermore, some robust circuit designs in gene network under intrinsic parameter fluctuations and external molecular noises have been developed via nonlinear stochastic $H_\infty$ robust stabilization and filtering technique, which has much potential to gene therapy and drug design (Chen *et al*., 2007, 2008; Chen and Wu, 2008).

More recently, the nascent filed, synthetic biology, is anticipated to have important applications in biotechnology and medicine and to contribute significantly to better understanding of the functioning of complex biological systems. At present, the development of gene network under some prescribed design specifications is still difficult and most newly created gene networks are non-functioning due to intrinsic parameter uncertainties and environmental disturbance (Andrianantoandro *et al*., 2006). How to design a synthetic gene network with desired behavior under some specified allowable intrinsic parameter uncertainties and attenuable external disturbance is the most important topic for synthetic gene networks. Obviously, the robust $H_\infty$ stabilization and filtering design will play an important role in the robust synthetic gene network design methods (Chen and Chang, 2008; Chen and Wu, 2008). It will be a future research work of our group.

## BRIEF REVIEW OF $H_\infty$ CONTROL AND FILTER DESIGN IN FREQUENCY DOMAIN

Consider the multivariable linear system with external disturbance in Fig. 1.

$$y(s) = S(s)d(s) + (I - S(s))^{-1} r(s) \quad (1)$$

where, d(s) denotes the external disturbance, r(s) denotes the desired reference signal and the sensitivity matrix S(s) is defined as:

$$S(s) = (I + G(s)C(s))^{-1} \quad (2)$$

in which G(s) denotes the plant and C(s) denotes the controller.

Therefore, the optimal $H_\infty$ disturbance rejection is to specify controller C(s) to achieve the following sensitivity minimization (Francis *et al*., 1984; Weinmann, 1991; Zames, 1981):

$$\min_{C(s)} \| w(s)S(s) \|_\infty \quad (3)$$

Where,

$$\| A(s) \| \triangleq \max_{W(S)\in[0,\infty)} \sigma_{max}(S(j\omega))$$

and $\sigma_{max}$ (*) denote the maximum singular value of * and w(s) denotes some weighting function to reflect some knowledge about the noise spectrum. This design is useful for robust disturbance attenuation when the statistics of d(s) are unavailable.

Similarly, the robust stabilization condition under plant uncertainties $\Delta G(s)$ with $|\Delta G(s)| \le |l(s)|$ in frequency domain (Fig. 2) is given as follows (Safonov and Chen, 1982; Weinmann, 1991):

$$\| l(s)(I - S(s)) \|_\infty < 1 \quad (4)$$

Therefore, the optimal $H_\infty$ stabilization is to specify controller C(s) to achieve the following $H_\infty$ optimization (Chang and Pearson, 1984; Chen and Kung, 1984; Safonov and Chen, 1982; Weinmann, 1991):

$$\min_{C(s)} \| w(s)(I - S(s) \|_\infty \quad (5)$$

The Youla controller is employed to treat the above optimal $H_\infty$ stabilization and $H_\infty$ disturbance attenuation problem (Youla *et al*., 1976a, b). Suppose the plant can be factorized as the following left and right coprime factorization

$$G(s) = A^{-1}(s)B(s) = B_1(s)A_1^{-1}(s) \quad (6)$$

where, A, B, $A_1$, $B_1$ are polynomial matrices, which satisfy the following coprime factorizations, respectively (Youla *et al*., 1976a, b)

$$A(s)X(s) + B(s)Y(s) = I_r, \quad X_1(s)A_1(s) + Y_1(s)B_1(s) = I_m \quad (7)$$

for some polynomial matrices X(s), Y(s), $X_1(s)$.

In Youla's study (Youla *et al*., 1976b), the stabilizing controller for the systems in Fig. 1 and 2 is given by:

$$C(s) = (Y(s) + A_1(s)Z(s))(X(s) - B_1(s)Z(s))^{-1} \quad (8)$$

for any rational matrix X(s).

In this situation, the optimal $H_\infty$ sensitivity design in Eq. 4 and the optimal $H_\infty$ stabilization in Eq. 5 become how to specify $H_\infty$ such that the following $H_\infty$ optimizations are achieved, respectively (Chang and Pearson, 1984; Chen and Kung, 1984; Safonov and Chen, 1982; Weinmann, 1991):

$$\min_{Z(s)} \| w(s)(X(s) - B_1(s)Z(s))A(s) \|_\infty \quad (9)$$

and

$$\min_{Z(s)} \| l(s)(Y(s) + A_1(s)Z(s) \|_\infty \quad (10)$$

Many $H_\infty$ optimization techniques have been developed to solve the above $H_\infty$ robust stabilization and disturbance rejection problems from the frequency domain

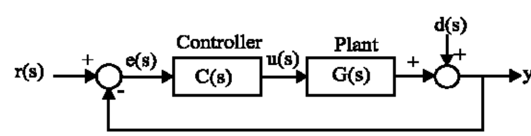


Fig. 1: Robust control with C(s) to achieve $H_\infty$ disturbance rejection of external disturbance d(s)

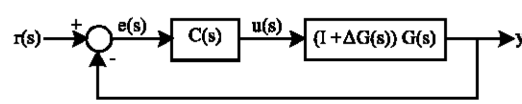


Fig. 2: Robust control with C(s) to achieve $H\infty$ stabilization under plant uncertainties $\Delta G(s)$

point of view. The shortages of $H_\infty$ robust control design in frequency domain are given as follows:

- The design procedure is very complicated, especially, for multivariable nonmimum phase systems.
- The order of controller in Eq. 8 is very high so that it is not easy to implement in practical systems.
- They can not be easily applied to nonlinear system.

To remedy the shortages and the PID controller is employed to treat the robust stabilization and disturbance rejection problems (Chen and Cheng, 1998; Chen *et al*., 1995; Gao, 2005; Lee *et al*., 2004). However, we can not obtain a closed-form solution to these $H_\infty$ robust control designs. Some searching algorithms like GA algorithm and evolution algorithm should be employed to solve a suboptimal solution. These simple robust $H_\infty$ PID designs are much potential for practical application.

For the $H_\infty$ filter design of signal processing with external disturbance in Fig. 3, the reconstruction error is given by:

$$\begin{aligned} e(z) &= x(z) - \hat{x}(z) \\ &= (I - F(z)H(z))x(z) - F(z)v(z) \\ &= \begin{bmatrix} I - F(z)H(z) & -F(z) \end{bmatrix} \begin{bmatrix} x(z) \\ v(z) \end{bmatrix} \end{aligned} \quad (11)$$

Then the $H_\infty$ filter design F(z) is to specify F(z) such that the following (Grimble and El Sayed, 1990; Hung and Chen, 2000)

$$\min_{F(z)} \| [w_1(z)(I - F(z)H(z) \quad -w_2(z)F(z)] \|_\infty \quad (12)$$

where, the weightings $w_1(z)$ and $w_2(z)$ are chosen to reflect some knowledge of the spectra of input signal x(z) and external disturbance v(z).

Other optimal $l_1$ filters (induced $l_\infty$-gain) are also designed with the similar design procedure to reject some noises or disturbances whose statistics are not available (Chen *et al*., 1994; Peng and Chen, 1997). Similarly, these robust filter design algorithms can be employed to specify some FIR filter with a minimax $H_\infty$ error in frequency domain. If the order is fixed, then we can mot get a closed-form solution for $H_\infty$ robust filter design. In this situation, a searching algorithm like genetic algorithm is necessary to find a suboptimal $H_\infty$ filter design (Hung and Chen, 2000).

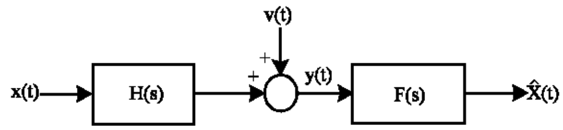


Fig. 3: Robust filter design with filter or equalizer F(s) to construct the transmitted signal x(t) under external disturbance v(t)

## BRIEF REVIEW OF LINEAR $H_\infty$ CONTROL AND FILTER DESIGN: STATE SPACE APPROACH

**$H_\infty$ robust control case:** Consider the following linear uncertain system

$$\dot{x} = (A + \Delta A)x + Bu + v \quad (13)$$

where $\Delta A$ denotes the parameter uncertainties and $v \in L_2$ denotes the external disturbance.

Then the $H_\infty$ control for perturbative system in Eq. 13 is to specify a feedback control u = Kx such that the following $H_\infty$ control performance is satisfied (Boyd *et al*., 1994):

$$\frac{\int_0^\infty x^T Q x + u^T R u \, dt}{\int_0^\infty v^T v \, dt} \le \rho^2$$

or

$$\int_0^\infty x^T Q x + u^T R u \, dt \le \rho^2 \int_0^\infty v^T v \, dt \quad (14)$$

where, Q>0 and R>0, for a prescribed attenuation level $\rho$ in the case x(0) = 0.

If $x(0) \neq 0$, the $H_\infty$ control performance is in should be modified as follows

$$\begin{aligned} &\int_0^\infty x^T Q x + u^T R u \, dt \\ &\le x^T(0) P x(0) + \rho^2 \int_0^\infty v^T v \, dt \end{aligned} \quad (15)$$

The physical meaning of $H_\infty$ control performance in Eq. 14 is that the effect of external disturbance v on the control performance should be attenuated below a prescribed level $\rho$ from the energy point of view. The optimal $H_\infty$ control could be achieved by minimizing $\rho^2$ in Eq. 14. Let us chose a Lyapunov function $V(x) = x^T Px$, then we get the following result.

**Theorem 1 (Boyd *et al.*, 1994):**

- In the perturbative system Eq. 13, if and the following matrix inequality has a positive solution $P = P^T > 0$

$$A^TP + PA + K^TB^TP + PBK + Q + K^TRK + \alpha^2 I + PP + \frac{1}{\rho^2}PP < 0$$

or, equivalently, the following LMI has a positive solution $W = P^{-1} > 0$

$$\begin{bmatrix} \Xi & Y^T & W & I \\ Y & -R^{-1} & 0 & 0 \\ W & 0 & -(Q+\alpha^2 I)^{-1} & 0 \\ I & 0 & 0 & -\rho^2 I \end{bmatrix} < 0 \quad (16)$$

where $\Xi$, $WA^T + AW + Y^TB^T + BY + I$, then the $H_\infty$ control performance in is achieved by $K = YW^{-1} = YP$ and the worst-case disturbance is by $v^* = (1/\rho^2)\,Px$.

- The optimal $H_\infty$ robust control can be design by Boyd *et al.* (1994)

$$\min_{W,Y} \rho^2 \quad (17)$$

subject to Eq. 16

If the state x can not measured directly, we can measure from output

$$\begin{aligned} \dot{x} &= (A + \Delta A)x + Bu + v \\ y &= (C + \Delta C)x + n \end{aligned} \quad (18)$$

where, y is the measurement output and n is the measurement noise. The bound of uncertainties $\Delta A$ and $\Delta C$ are defined as $\|\Delta A\|<\alpha$ and $\|\Delta A\|<\beta$. In this situation, the following observer-based output feedback controller is employed to treat the $H_\infty$ control design problem

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \\ u &= K\hat{x} \end{aligned} \quad (19)$$

Let us denote the state estimation error and the augmented vector, respectively, as follows:

$$\tilde{x} = x - \hat{x},\ \ \bar{x} = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix},\ \ \bar{v} = \begin{bmatrix} v \\ n \end{bmatrix} \quad (20)$$

then the augmented system can be expressed in the following form

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{L}\Delta\bar{A}\bar{x} + \bar{B}_{\bar{v}}\bar{v} \quad (21)$$

Where,

$$\bar{A} = \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix},\ \Delta\bar{A} = \begin{bmatrix} \Delta A & 0 \\ \Delta C & 0 \end{bmatrix},$$

$$\bar{L} = \begin{bmatrix} I & 0 \\ 0 & -L \end{bmatrix},\ \bar{B}_{\bar{v}} = \begin{bmatrix} I & 0 \\ 0 & -L \end{bmatrix}$$

And the $H_\infty$ robust control performance in Eq. 14 for the observer-based output feedback system in Eq. 18 and 19 should be modified as Boyd *et al.* (1994) and Chen *et al.* (2000):

$$\int_0^\infty \bar{x}^T\bar{Q}\bar{x} + u^TRu\,dt \le \rho^2 \int_0^\infty \bar{v}^T\bar{v}\,dt \quad (22)$$

Let us chose a Lyapunov function $V(\bar{x}) = \bar{x}^TP\bar{x}$, then we get the following result.

**Theorem 2 (Boyd *et al.*, 1994, Chen *et al.*, 2000):** In the perturbative control system Eq. 18 and 19 if $\|\Delta A\|<\alpha$, $\|\Delta A\|<\beta$ and we specify K and L such that the following matrix inequality has a positive solution $P = P^T > 0$

$$\begin{aligned} &\bar{A}^TP + P\bar{A} + \bar{Q} + \bar{K}^TR\bar{K} + \Phi \\ &+ P\bar{L}\bar{L}^TP + \frac{1}{\rho^2}P\bar{B}_{\bar{v}}\bar{B}_{\bar{v}}{}^TP < 0 \end{aligned} \quad (23)$$

Where,

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix},\ \bar{K} = \begin{bmatrix} K & -K \end{bmatrix}$$

And

$$\Phi = \begin{bmatrix} (\alpha^2 + \beta^2 I & 0 \\ 0 & 0 \end{bmatrix}$$

then the $H_\infty$ robust control performance Eq. 22 is achieved.

**Remark:** In Chen *et al.* (2000), a decoupling technique has been developed so that the robust $H_\infty$ observer-based control design problem can be treated by two-stage design procedure, i.e., solving the observer gain L at first and then solving the control gain K.

Note that, for the linear time-delay systems, the robust $H_\infty$ control design could be found in papers (Mahmoud and Zribi, 1999; Xie and Desouza, 1992).

If the parameter variations $\Delta A$ in Eq. 13 are due to several random sources, especially in nano-scale systems (Chen *et al*., 2008) or bio-molecular systems (Chen and Wang, 2006), as follows:

$$\Delta A = \sum_{i=1}^{m} D_i n_i \tag{24}$$

where, $D_i$ denotes the location and magnitude of the ith parameter variation and the random part is absorbed to the standard white noise $n_i$ with unit variance. Then the perturbative system in Eq. 13 is represented by the following Ito stochastic system (Chen and Zhang, 2004; Hinrichsen and Pritchard, 1998):

$$\begin{aligned} dx &= (Ax + Bu + v)dt + \sum_{i=1}^{m} D_i x dW_i \\ y &= Cx + n \end{aligned} \tag{25}$$

where, $W_i$ denotes a standard Wiener process or Brownian motion.

In the stochastic system Eq. 25, the $H_\infty$ control performance in and should be modified, respectively, as (Chen and Zhang, 2004; Hinrichsen and Pritchard, 1998):

$$\begin{aligned} &E\int_0^\infty x^T Q x + u^T R u \, dt \\ &\quad \le E x^T(0) P x(0) + \rho^2 E\int_0^\infty v^T v \, dt \end{aligned} \tag{26}$$

and

$$\begin{aligned} &E\int_0^\infty \bar{x}^T \bar{Q} \bar{x} + u^T R u \, dt \\ &\quad \le E\bar{x}^T(0) \bar{P} \bar{x}(0) + \rho^2 E\int_0^\infty \bar{v}^T \bar{v} \, dt, \end{aligned} \tag{27}$$

where, E denotes the expectation. The robust $H_\infty$ control for Eq. 25 is to specify the observer-based controller in Eq. 19 such that the $H_\infty$ control performance Eq. 27 is satisfied for a prescribed disturbance attenuation level $\rho$. Let us chose a Lyapunov function $V(\bar{x}) = E\bar{x}^T P \bar{x}$, then we get the following result.

**Theorem 3 (Chen and Zhang, 2004):** For the linear stochastic system Eq. 25 if we specify K and L such that the following inequality matrix has a positive solution $P = P^T > 0$

$$\begin{aligned} &P\bar{A} + \bar{A}^T P + \bar{D}^T P \bar{D} + \bar{Q} \\ &\quad + \frac{1}{\rho^2} P \bar{B}_{\bar{v}} \bar{B}_{\bar{v}}^T P + \bar{K}^T R \bar{K} < 0 \end{aligned} \tag{28}$$

then the robust $H_\infty$ control performance in Eq. 27 is achieved and the worst-case disturbance

$$\bar{v}^* = (1/\rho^2) \bar{B}_{\bar{v}} P x$$

In the mixed $H_2/H_\infty$ control design case (Chen and Zhang, 2004), the worst-case disturbance

$$\bar{v}^* = (1/\rho^2) \bar{B}_{\bar{v}} P x$$

should be substituted into Eq. 13 for the state feedback case (Limebeer *et al*., 1994) or Eq. 18 for the observer-based control case (Chen and Zhang, 2004). Since, the worst-case disturbance are function of state x, then the following optimal $H_2$ control performance can be designed (Chen and Zhang, 2004):

$$\min_K E\int_0^\infty \bar{x}^T \bar{Q} \bar{x} + u^T R u \, dt \tag{29}$$

Then we get following result.

**Theorem 4 (Chen and Zhang, 2004):** If we specify K and L such that the following matrix inequalities have a positive common solution P>0

$$\begin{aligned} &P\bar{A} + \bar{A}^T P + \bar{D}^T P \bar{D} + \bar{Q} \\ &\quad + \frac{1}{\rho^2} P \bar{B}_{\bar{v}} \bar{B}_{\bar{v}}^T P + \bar{K}^T R \bar{K} < 0 \end{aligned} \tag{30}$$

And

$$\begin{aligned} &PP + P\bar{A} + \bar{A}^T P \\ &\quad + \bar{D}^T P \bar{D} + \bar{Q} + \bar{K}^T R \bar{K} < 0 \end{aligned} \tag{31}$$

then the mixed $H_2/H_\infty$ performance Eq. 27 and 29 are achieved simultaneously.

**Remark:** In Chen and Zhang (2004), for the $H_\infty$ and mixed $H_2/H_\infty$ observer-based control design problem of linear stochastic system, a two-stage convex optimization design procedure has been developed to solve the coupling problem of control gain K and observer gain L.

**Remark:** The problems of robust $H_\infty$ control for stochastic systems with parameter uncertainty and time delay have been studied in Xu and Chen (2002).

**$H_\infty$ robust filter design case:** In signal processing case, the signal transmission system can modeled as:

$$x(k+1) = Ax(k) + Bv(k) \tag{32}$$

$$y(k) = Cx(k) + n(k) \tag{33}$$

where statistics of v(k) and n(k) are unavailable. In general, the transmitted signal is embedded in the signal model in Eq. 32 and transmission channel is modeled in Eq. 33 so that filter design problem becomes a state estimation problem. Then the $H_\infty$ filter design is to specify the filer gain L in the following state estimator:

$$\hat{x}(k+1) = A\hat{x}(k) + L(y(k) - C\hat{x}(k)) \tag{34}$$

such that the following $H_\infty$ filtering performance is achieved (De Souza *et al*., 1995; Fridman and Shaked, 2001; Shaked and Berman, 1995):

$$\frac{\sum_{k=0}^{\infty} e^T(k)Qe(k)}{\sum_{k=0}^{\infty} \bar{v}^T(k)\bar{v}(k)} \le \rho^2 \tag{35}$$

Where,

$$e(k) = x(k) - \hat{x}(k)$$

denotes the estimation error and

$$\bar{v}(k) = [v(k) \quad n(k)]^T$$

Actually, there are many $H_\infty$ filters can satisfy the $H_\infty$ performance in Eq. 35. If the nominal statistics of v(k) and n(k) are available, for example, we could obtain these values by the averaging method to get the nominal statistics, then we design a $H_2$ filter among these feasible $H_2$ filters by only minimizing

$$E\sum_{k=0}^{\infty} e^T(k)Qe(k)$$

to achieve a $H_2/H_\infty$ filter design (Chen *et al*., 2001). In general, a suboptimal $H_\infty$ design is employed in the mixed $H_2/H_\infty$ filter design problem (Duan *et al*., 2006). The $H_\infty$ filtering problem of the continuous-time linear system is also discussed in Duan *et al*. (2006). In Gao and Dian (2007) and Pila *et al*. (1999), a $H_\infty$ filtering design is proposed for the continuous time linear system with time-delay.

## NONLINEAR $H_\infty$ CONTROL DESIGN

In general, physical systems are almost nonlinear. In this situation, the perturbative system in Eq. 13 is modified as the following the nonlinear system (Isidori and Astolfi, 1992):

$$\dot{x} = f(x) + g(x)u + v \tag{36}$$

where all the perturbations and uncertainties are included into external disturbance v.

The nonlinear $H_\infty$ control design is to specify a nonlinear control u = k(x) in Eq. 36 such that the $H_\infty$ control performance in Eq. 14 is achieved for a prescribed disturbance attenuation level ρ. Then we get the following result.

**Theorem 5 (Ball *et al*., 1993; Isidori and Astolfi, 1992):** If the following HJI has a positive Lyapunov solution V(x)> 0

$$\begin{aligned}&\left(\frac{\partial V(x)}{\partial x}\right)^T (f(x) + g(x)k(x)) \\ &+ x^TQx + k^T(x)Rk(x) \\ &+ \frac{1}{\rho^2}\left(\frac{\partial V(x)}{\partial x}\right)^T \left(\frac{\partial V(x)}{\partial x}\right) < 0\end{aligned} \tag{37}$$

then the $H_\infty$ disturbance attenuation in Eq. 14 or 15 is achieved. The worst-case disturbance can be obtained as follows:

$$v^* = \frac{1}{\rho^2}\frac{\partial V(x)}{\partial x}$$

If the state x is unavailable and could be estimated from the output measurement y, i.e.,

$$\begin{aligned}\dot{x} &= f(x) + g(x)u + v \\ y &= h(x) + n\end{aligned} \tag{38}$$

then the following observer-based control is employed to achieve the robust $H_\infty$ control performance Eq. 22.

$$\begin{aligned}\dot{\hat{x}} &= f(\hat{x}) + g(\hat{x})u + l(\hat{x})(y - h(\hat{x})) \\ u &= k(\hat{x})\end{aligned} \tag{39}$$

i.e., to specify $l(\hat{x})$ and $k(\hat{x})$ to achieve the $H_\infty$ control performance Eq. 22. In general, it needs to solve 2 coupled HJIs (Isidori and Astolfi, 1992), one for $H_\infty$ controller gain $k(\hat{x})$ and another for $H_\infty$ observer gain $l(\hat{x})$ (Isidori and Astolfi, 1992). In general, it is very difficult to solve HJIs for this $H_\infty$ control problem except very simple system. However, in some mechanical systems (Chang and Chen, 2000; Chen *et al*., 1996, 1994, 2000; Wu *et al*., 1999), if we choose an adequate Lyapunov function X(x) in Eq. 37 and make use of some properties of mechanical systems, for example, the skew symmetry property. In this situation, the HJI in Eq. 37 becomes easy to solve. Therefore, $H_\infty$ robust control design can be applied to robotic systems (Chang and Chen, 1997; Chen *et al*., 1994, 2000; Tseng and Chen, 2001), spacecraft systems (Chen *et al*., 2000; Gadewadikar and Lewis, 2006; Nichols *et al*., 1993) and some non-holonomic systems (Chang and Chen, 2000), to efficiently attenuate the external disturbance to achieve a desired model reference tracking.

**Remark:** In the model reference tracking (Chang, 2000; Chen *et al*., 1996; Tseng and Chen, 2001), a model reference model $\dot{x}_r = A_r x_r + r$ should be given at first to specify a desired reference response $x_r$. The $H_\infty$ tracking control performance should be modified as follows:

$$\int_0^\infty (x-x_r)^T Q(x-x_r) + u^T Ru\,dt \le x^T(0)Px(0) + \rho^2 \int_0^\infty v^T v\,dt$$

In the nonlinear stochastic system case, the stochastic system in Eq. 25 should be modified as follows (Zhang and Chen, 2006; Zhang *et al*., 2006):

$$dx = (f(x) + g(x)u + v)dt + \sum_{i=1}^{m} D_i(x)dW_i \tag{40}$$

Then the stochastic nonlinear $H_\infty$ control design is to specify a controller u = k(x) such that the $H_\infty$ control performance in Eq. 26 is achieved. Therefore, we get the following result.

**Theorem 6 (Zhang and Chen, 2006):** For the nonlinear stochastic system in Eq. 40, if the following second-order HJI has a positive solution V(x)>0

$$\begin{aligned}&\left(\frac{\partial V(x)}{\partial x}\right)^T (f(x) + g(x)k(x)) \\ &\quad + x^T Qx + k^T(x)Rk(x) \\ &\quad + \frac{1}{\rho^2}\left(\frac{\partial V(x)}{\partial x}\right)^T \left(\frac{\partial V(x)}{\partial x}\right) \\ &\quad + \frac{1}{2}\sum_{i=1}^{m} D_i^T(x)\frac{\partial^2 V(x)}{\partial x^2}D_i(x) < 0\end{aligned} \tag{41}$$

then the $H_\infty$ robust control performance Eq. 26 can be achieved.

For output measurement system,

$$\begin{aligned}dx &= (f(x) + g(x)u + v)dt + \sum_{i=1}^{m} D_i(x)dW_i \\ y &= h(x) + n\end{aligned} \tag{42}$$

It is more difficult to design the observer-based control design in Eq. 39 to achieve the following stochastic $H_\infty$ performance:

$$E\int_0^\infty \bar{x}^T \bar{Q}\bar{x} + u^T Ru\,dt \le \rho^2 E\int_0^\infty \bar{v}^T \bar{v}\,dt \tag{43}$$

because we need to solve 2 coupled second-order HJIs, one for a stochastic $H_\infty$ controller and other for a stochastic $H_\infty$ estimator (Zhang *et al*., 2005).

The stochastic state estimator is designed for Eq. 42 by a $H_\infty$ robust filter in Zhang *et al*. (2005), which is to specify an estimator gain $l(\hat{x})$ in the following estimator,

$$d\hat{x} = (f(\hat{x}) + g(\hat{x})u)dt + l(\hat{x})(y - h(\hat{x})) \tag{44}$$

Such that the following $H_\infty$ filtering performance is achieved for a prescribed filtering level ρ.

$$E\int_0^\infty e^T e\,dt \le \rho^2 E\int_0^\infty \bar{v}^T \bar{v}\,dt \tag{45}$$

In general, it is very difficult to solve the HJI in the nonlinear $H_\infty$ control and filtering design problem except some special cases. Therefore, the application of $H_\infty$ robust control and filtering designs will be limited by the difficulty in solving HJI. A remedy of this shortage is to employ the fuzzy approximation method to simplify the design procedure.

## FUZZY $H_\infty$ ROBUST CONTROL AND FILTERING DESIGN FOR NONLINEAR SYSTEMS

In order to overcome the difficulty of solving HJI in nonlinear $H_\infty$ control and filter designs, the fuzzy interpolation method is employed to simplify the design procedure. The nonlinear system in Eq. 36 could be represented by the following fuzzy model (Takagi and Sugeno, 1985; Tanaka and Wang, 2001)

$$\begin{aligned}&\text{If } z_1 \text{ is } F_{i1}, \text{ and} \ldots, \text{ and } z_g \text{ is } F_{ig} \\ &\text{then } \dot{x} = A_i x + B_i u + v\end{aligned} \tag{46}$$

for i = 1, 2, ..., L, where, $F_{ij}$ is the fuzzy set, $A_i$ and $B_i$ are linear system parameters. L is the number of If-then rules and $z_1$,... $z_g$ are the premise variables. The overall fuzzy system is inferred as follows (Takagi and Sugeno, 1985; Tanaka and Wang, 2001):

$$\dot{x} = \sum_{i=1}^{L} h_i(z)(A_i x + B_i u) + v \tag{47}$$

Where,

$$h_i(z)=\frac{\mu_i(z)}{\sum_{k=1}^{L}\mu_k(z)},\quad \mu_i(z)=\prod_{j=1}^{g}F_{ij}(z_j)$$

Therefore, the nonlinear system can be approximated by fuzzy system in Eq. 47

$$\begin{aligned}\dot{x}&=f(x)+g(x)u+v\\&=\sum_{i=1}^{L}h_i(z)(A_ix+B_iu)+\Delta f(x)+\Delta g(x)+v\end{aligned}\qquad(48)$$

Where,

$$\Delta f(x)=f(x)-\sum_{i=1}^{L}h_i(z)A_ix(+B_iu),$$

$$\Delta g(x)=g(x)u-\sum_{i=1}^{L}h_i(z)B_iu$$

The fuzzy control rule is proposed as follows:

Control Rule j:

$$\begin{aligned}&\text{If } z_1 \text{ is } F_{j1},\ \text{and}\ldots,\ \text{and } z_g \text{ is } F_{jg}\\&\text{then } u=K_jx \qquad \text{for } j=1,2,\ldots,L\end{aligned}\qquad(49)$$

Hence, the overall fuzzy controller is given by:

$$u=\sum_{j=1}^{L}h_j(z)K_jx\qquad(50)$$

Substituting into Eq. 48, we get:

$$\begin{aligned}\dot{x}&=f(x)+g(x)u+v\\&=\sum_{i=1}^{L}\sum_{j=1}^{L}h_i(z)h_j(z)(A_i+B_iK_j)x\\&\quad+\Delta f(x)+\Delta g(x)+v\end{aligned}\qquad(51)$$

The identification of $A_i$ and $B_i$ can be found in Takagi and Sugeno (1985), i.e., we could interpolate L linear systems to approximate a nonlinear systems. Suppose the approximation error $\Delta f(x)$ and $\Delta g(x)$ are assumed to be bounded by the bounding matrices $\Delta A_i$ and $\Delta B_i$, respectively, as follows (Chen *et al*., 1999):

$$\|\Delta f(x)\|\le\|\sum_{i=1}^{L}h_i(z)\Delta A_ix\|\qquad(52)$$

And

$$\|\Delta g(x)\|\le\|\sum_{i=1}^{L}\sum_{j=1}^{L}h_i(z)h_j(z)\Delta B_iK_jx\|\qquad(53)$$

The bounding matrices $\Delta A_i$ and $\Delta B_i$ can be described by

$$\begin{bmatrix}\Delta A_i\\\Delta B_i\end{bmatrix}=\begin{bmatrix}\delta_iA_p\\\eta_iB_p\end{bmatrix}$$

where, $\|\delta_i\|\le 1$ and $\|\eta_i\|\le 1$ for i = 1,...L (Chen *et al*., 1999). Let us chose a Lyapunov function $V(x)=x^TPx$, then, we get the following fuzzy $H_\infty$ control result.

**Theorem 7 (Chen *et al*., 1999):** If the following LMIs have common positive solution W>0

$$\begin{bmatrix}\Xi & Y_j^T & (B_pY_j)^T & W\\Y_j & -R^{-1} & 0 & 0\\B_pY_j & 0 & -I & 0\\W & 0 & 0 & \Omega\end{bmatrix}<0\qquad(54)$$

Where,

$$\Xi=WA_i^T+A_iW+Y_j^TB_i^T+B_iY_j+(2+(1/\rho^2)I$$

And

$$\Omega=-(A_p{}^TA_p+Q)^{-1}$$

for i, j = 1, ... L, then the $H_\infty$ control performance in Eq. 14 is achieved for a prescribed attenuation level ρ and the fuzzy controller parameters in Eq. 50 are given by $K_j=Y_jW^{-1}$ and $W=P^{-1}$. Therefore, the fuzzy $H_\infty$ optimal control is to solve the following constrained optimization

$$\begin{aligned}&\min\rho^2\\&\text{subject to } W=W^T>0,\ \text{Eq. 54}\end{aligned}\qquad(55)$$

In the fuzzy $H_\infty$ control design, we replace the HJI in Eq. 57 by a set of LMIs in Eq. 54, which can be efficiently solved by LMI toolbox in Matlab.

**Remark:**

- The nonlinear $H_\infty$ output feedback control design by nonlinear observer-based control design could be also represented by a fuzzy observer-based fuzzy control design in Chen *et al.* (1999), Huang and Nguang (2006), in which a set of LMIs are used to replace two HJIs in Isidori and Astolfi (1992). Since, the infeasible HJI is transformed to an equivalent set of feasible LMIs, several complex nonlinear control design problem like non-holonomic mechanical systems (Chang and Chen, 2000; Tseng and Chen, 2003), missile systems (Chen *et al*., 2002; Uang and Chen, 2002) etc. could be easily solved by fuzzy $H_\infty$ robust design method (Chen *et al*., 1999). For the uncertain fuzzy system with time delay, the output feedback $H_\infty$ control design is proposed in Lee *et al*. (2000).

- The mixed $H_2/H_\infty$ fuzzy control design could also simplify the nonlinear $H_2/H_\infty$ control design problem through fuzzy interpolation method (Chen *et al*., 2000; Tseng *et al*., 2001). We also replace two HJIs, one for H∞ optimal control and other for $H_\infty$ robust control, by a set of LMIs to simplify the design procedure. The observer-based fuzzy $H_2/H_\infty$ control design is also found in Chen *et al*. (2000).

The nonlinear stochastic system can also be represented by fuzzy interpolation system as follows (Chen *et al*., 2008; Chen and Wu, 2007, 2008):

$$\begin{aligned} dx &= (f(x)+g(x)u+v)dt+\sum_{k=1}^{m} D_k(x)dW_k \\ &= \sum_{i=1}^{L} h_i(z)\left[(A_i x + B_i u + v)dt + \sum_{k=1}^{m} D_{i,k} dW_k\right] \\ &\quad + \Delta f(x)dt + \Delta g(x)udt + \sum_{k=1}^{m} \Delta D_k(x)dW_k \end{aligned} \quad (56)$$

The above nonlinear stochastic system has been used to model the gene network and biochemical network under intrinsic parameter variations and extrinsic molecular noise. The robust $H_\infty$ stabilization and $H_\infty$ filtering ability have been employed to discuss why the gene network and biochemical network can work reliably under large parameter fluctuations and extrinsic disturbances in the nano-scale molecular systems (Chen *et al*., 2007, 2008; Chen and Wang, 2006, 2007; Chen and Wu, 2007). The robustness and the filtering ability of these nonlinear stochastic bio-molecular systems are also measured by the locations of eigenvalues of $A_i$ in Eq. 56 and the attenuation level ρ in Eq. 26 by fuzzy approximation method (Takagi and Sugeno, 1985; Tanaka and Wang, 2001) or global linearization method (Boyd *et al*., 1994). The robust $H_\infty$ stabilization and filtering ability can be used to interpret why gene network or biochemical networks can work reliably in large parameter fluctuation in the noisy gene expression process, thermal fluctuations, alternative RNA splicing and environmental molecular noises (Chen and Wang, 2006). Furthermore, if a gene network or a biochemical network has not enough robustness to tolerate parametric fluctuations or to attenuate more external disturbance, some gene circuit could be implemented through transfection and transformation biotechnologies (Chen *et al*., 2007, 2008; Chen and Wu, 2007, 2008). More recently, the synthetic biology becomes an important topic in the biology and engineering (Andrianantoandro *et al*., 2006). How to synthesize a gene network or biochemical network by the engineering methods to work reliably under some design specifications such as an allowable kinetic parameter range, the tolerable variances of parametric fluctuations and a prescribed filtering ability of external disturbances is the most important work of synthetic biology at present. The synthetic gene network could be represented by the nonlinear stochastic system as follows (Chen *et al*., 2008; Chen and Wu, 2007, 2008):

$$dx = (f(x,k)+v)dt+\sum_{i=1}^{m} D_i(x)dW_i \quad (57)$$

where, k denotes the kinetic parameters to be designed,

$$\textstyle\sum_{i=1}^{m} D_i(x)dW_i$$

denote the specified (to be tolerated) parameter fluctuations and ν denotes the external disturbance to be attenuated. In the robust synthetic network design, some specifications of design target are given as follows:

- The desired behavior $\dot{x}_r = A_r x_r + r$.
- The covariances of tolerable intrinsic parameter variations.
- The permittable control parameter range $k \in [k_1, k_2]$.
- The disturbance attenuation level ρ (Chen and Chang, 2008; Chen and Wu, 2008).

After the synthetic gene network with intrinsic parametric fluctuations and external disturbances is modeled as Eq. 57. Then the robust synthetic gene network could be formulated as a robust nonlinear stochastic $H_\infty$ tracking problem (Chen and Chang, 2008; Chen and Wu, 2008). Then the synthetic biology design is to specify kinetic parameters $k \in [k_1, k_2]$ in an allowable range so that:

$$E\int_0^{\infty}(x-x_r)^T(x-x_r)dt \le \rho^2 E\int_0^{\infty} v^T v\, dt$$

is achieved for a prescribed disturbance attenuation level ρ. Obviously, the robust $H_\infty$ stabilization and filtering designs in nonlinear stochastic system are useful for robust designs of synthetic biology (Chen *et al.*, 2008; Chen and Wu, 2007, 2008).

Let us denote

$$\bar{x}=\begin{bmatrix} x_r \\ x \end{bmatrix}, \bar{f}(\bar{x},k)=\begin{bmatrix} A_r \\ f(x,k) \end{bmatrix},$$
$$\bar{v}=\begin{bmatrix} r \\ v \end{bmatrix}, \bar{D}_i(\bar{x})=\begin{bmatrix} 0 \\ D_i(x) \end{bmatrix}$$

then we get the following result

**Theorem 8 (Chen and Chang, 2008; Chen and Wu, 2008):**

- If the design specifications are given, then the robust synthetic gene network become how to specify $k\in[k_1, k_2]$ in Eq. 57 such that the following HJI has a positive solution $V(\bar{x}) > 0$

$$\begin{aligned}&\left(\frac{\partial V(\bar{x})}{\partial \bar{x}}\right)^T \bar{f}(\bar{x},k)+\bar{x}^T Q\bar{x} \\ &+\frac{1}{\rho^2}\left(\frac{\partial V(\bar{x})}{\partial \bar{x}}\right)^T\left(\frac{\partial V(\bar{x})}{\partial \bar{x}}\right) \\ &+\frac{1}{2}\sum_{i=1}^{m} D_i^T(\bar{x})\frac{\partial^2 V(\bar{x})}{\partial \bar{x}^2}D_i(\bar{x})<0\end{aligned} \quad (58)$$

- The optimal robust $H_\infty$ synthetic gene network design is reduced to solving the following constrained optimization

$$\min_{k\in[k_1,k_2]} \rho^2$$
$$\text{sbject to } V(\bar{x})>0, \text{Eq.58}$$

In general it is not easy to solve the HJI in Eq. 58 and 59 for robust $H_\infty$ synthetic gene network to satisfy design specifications. If the fuzzy interpolation in Eq. 46 is employed to treat the robust $H_\infty$ synthetic gene network, then we get the following result. Let us denote

$$\bar{A}_i=\begin{bmatrix} A_r & 0 \\ 0 & A_i(k) \end{bmatrix}, \bar{D}_i=\begin{bmatrix} 0 \\ D_i \end{bmatrix}$$

where, $A_i$ (k) denotes a matrix A containing k as its elements.

**Theorem 9 (Chen and Chang, 2008; Chen and Wu, 2008):**

- If the design specification are given, then the robust synthetic gene network become how to specify $k\in[k_1, k_2]$ in Eq. 57 such that the following matrix inequality has a positive solution $P = P^T>0$

$$\begin{aligned}&\bar{A}_i^T(k)P+P\bar{A}_i(k)+\bar{Q} \\ &+\frac{1}{\rho^2}PP+\sum_{j=1}^{m}\bar{D}_j^T P\bar{D}_j<0\end{aligned} \quad (60)$$

or, equivalently, the following LMIs hold

$$\begin{bmatrix} \bar{A}_i^T(k)P+P\bar{A}_i(k)+\bar{Q}+\sum_{j=1}^{m}\bar{D}_j^T P\bar{D}_j & P \\ P & -\rho^2 I \end{bmatrix}<0 \quad (61)$$

- The optimal robust $H_\infty$ synthetic gene network design is reduced to solving the following constrained optimization

$$\min_{k\in[k_1,k_2]} \rho^2 \quad (62)$$
$$\text{subject to } P>0, \text{Eq.61}$$

Similarly, the nonlinear $H_\infty$ filter design for the following nonlinear stochastic system (Tseng and Chen, 2001)

$$\begin{aligned}\dot{x}&=f(x)+g(x)u+v \\ &=\sum_{i=1}^{L}h_i(z)(A_i x+B_i u)+\Delta f(x)+\Delta g(x)u+v \\ y&=\sum_{i=1}^{L}h_i(z)C_i x+\Delta h(x)+n\end{aligned} \quad (63)$$

could be simplified by the fuzzy $H_\infty$ filter

$$\dot{\hat{x}}=\sum_{i=1}^{L}\sum_{j=1}^{L}h_i(z)h_i(z)(A_i\hat{x}+B_i u+L_i(y-C_j\hat{x})) \quad (64)$$

The $H_\infty$ fuzzy filter design is employed in Tseng and Chen (2001) by using LMI technique to replace HJI in the conventional nonlinear $H_\infty$ filtering design in Zhang *et al.*

(2005). Because the design procedure is simplified significantly, this method has been employed for equalization design of communication systems with nonlinear channel (Chen *et al*., 2003). A robust mixed $H_2/H_{\infty}$ filtering problem subject to parameter uncertainties and multiple time-varying delays is discussed in Lin and Lo (2006). In Zhang *et al*. (2007), a fuzzy $H_{\infty}$ filter design is proposed for a nonlinear discrete-time system with time-delay.

**Remark:** If the parameters of nonlinear systems are uncertain or unknown, the robust adaptive fuzzy $H_{\infty}$ control design is also employed to achieve the robust control design with addition of some adaptive learning law (Chen *et al*., 1996). These adaptive $H_{\infty}$ designs can be applied to some robotic system (Chang and Chen, 1997, 2000; Hoa *et al*., 2007; Lian *et al*., 2002; Lin and Mon, 2003), spacecraft system (Chen *et al*., 2000; Wu and Chen, 2001) and missile systems (Uang and Chen, 2002) with unknown or uncertain parameters.

## CONCLUSION

The robust $H_{\infty}$ design is the most important topic for a designed system to work reliably under intrinsic parameter variations and external disturbances. This study presents a brief survey on the analysis and design method of $H_{\infty}$ robust control and filtering methods in the past decades. At the beginning, $H_{\infty}$ robust control and filter design was developed in the frequency domain. Then, these $H_{\infty}$ design methods were extended to the state-space system, then nonlinear systems and finally the stochastic nonlinear systems. In order to overcome the difficulty of solving infeasible HJI for nonlinear $H_{\infty}$ robust control or filter design, the fuzzy $H_{\infty}$ control or filter design via solving a set of feasible LMIs is an efficient method to simplify the design procedure. Recently, nano-systems, systems biology, synthetic biology and quantum systems become hot topics in the field of system engineering design. Because these systems are inherently nonlinear stochastic with large intrinsic parameter fluctuations external disturbance, in order to make these systems perform reliably, nonlinear stochastic $H_{\infty}$ control and filtering (estimation) designs will become the most important design topics in the future. More efforts are still needed to develop a simple but suitable $H_{\infty}$ robust control design or filter to achieve this robust design purpose.

## REFERENCES

Aguilar, L.T., Y. Orlov and L. Acho, 2003. Nonlinear $H_{\infty}$-control of nonsmooth time-varying systems with application to friction mechanical manipulators. Automatica, 39: 1531-1542.

Andrianantoandro, E., S. Basu, D.K. Karig and R. Weiss, 2006. Synthetic biology: New engineering rules for an emerging discipline. Mol. Syst. Biol., 1: 1-14.

Ball, J., J.W. Helton and M. Walker, 1993. $H_{\infty}$ control for nonlinear systems with output feedback. IEEE. Trans. Automatic Control, 38: 546-559.

Basar, T. and P. Bernhard, 1995. $H_{\infty}$ Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach, Birkhauser.

Boyd, S., L. El Ghaoui, E. Feron and V. Balakrishnan, 1994. Linear Matrix Inequalities in System and Control Theory. Society for Industrial Mathematics.

Chang, B.C. and J. Pearson, 1984. Optimal disturbance reduction in linear multivariable systems. IEEE. Trans. Automatic Control, 29: 880-887.

Chang, Y.C. and B.S. Chen, 1997. A nonlinear adaptive $H_{\infty}$ tracking control design in robotic systems via neural networks. IEEE. Trans. Control Syst. Technol., 5: 13-29.

Chang, Y.C. and B.S. Chen, 2000. Robust tracking designs for both holonomic and nonholonomic constrained mechanical systems: Adaptive fuzzy approach. IEEE. Trans. Fuzzy Syst., 8: 46-66.

Chang, Y.C., 2000. Robust tracking control for nonlinear MIMO systems via fuzzy approaches. Automatrica, 36: 1535-1545.

Chang, Y.C., 2005. Intelligent robust control for uncertain nonlinear time-varying systems and its application to robotic systems. IEEE. Trans. Syst. Man and Cybernetics, Part B, 35: 1108-1119.

Chen, B.S., T.S. Lee and W.S. Chang, 1996. A robust H model reference tracking design for non-holonomic mechanical control systems. Int. J. Control, 63: 283-306.

Chen, B.S., Y.T. Chang and Y.C. Wang, 2008. Robust $H_{\infty}$-stabilization design in gene networks under stochastic molecular noises: Fuzzy-interpolation approach. IEEE. Trans. Syst. Man and Cybernetics, Part B: Cybernetics (Special Issue for Systems Biology), 38: 25-42.

Chen, B.S. and C.C. Kung, 1984. The robustness optimization of a multivariable feedback system in Hankel norm space. Int. J. Control, 39: 1211-1228.

Chen, B.S. and C.H. Wu, 2008. On the systematic design of robust synthetic gene network, Submitted.

Chen, B.S. and W.H. Zhang, 2004. Stochastic $H_2/H_{\infty}$ control with state-dependent noise. IEEE. Trans. Automatic Control, 49: 45-57.

Chen, B.S. and W.S. Wu, 2007. Underlying principles of natural selection in network evolution: Systems biology approach. Evolutionary Bioinformatics, 3: 245-262.

Chen, B.S. and W.S. Wu, 2008. Robust filtering circuit design for stochastic gene networks under intrinsic and extrinsic molecular noises. Mathe. Biosci., 211: 342-355.

Chen, B.S. and Y.C. Chang, 1997. Nonlinear mixed $H_2/H_\infty$ control for robust tracking design of robotic systems. Int. J. Control, 67: 837-857.

Chen, B.S. and Y.C. Wang, 2006. On the attenuation and amplification of molecular noise in genetic regulatory networks. BMC Bioinformatics, 7: 1-14.

Chen, B.S. and Y.M. Cheng, 1998. A structure-specified $H_\infty$ optimal control design for practical applications: A genetic approach. IEEE. Trans. Control Syst. Technol., 6: 707-718.

Chen, B.S. and Y.T. Chang, 2008. Robust tracking design of synthetic biology network: Fuzzy dynamic game approach (in Press).

Chen, B.S., C.H. Lee and Y.C. Chang, 1996. H tracking design of uncertain nonlinear SISO systems: Adaptive fuzzy approach. IEEE. Trans. Fuzzy Syst., 4: 32-43.

Chen, B.S., C.L. Tsai and D.S. Chen, 2003. Robust H and mixed $H_2/H_\infty$ filters for equalization designs of nonlinear communication systems: Fuzzy interpolation approach. IEEE. Trans. Fuzzy Syst., 11: 384-398.

Chen, B.S., C.L. Tsai and Y.F. Chen, 2001. Mixed $H_2/H_\infty$ filtering design in multirate transmultiplexer systems: LMI approach. IEEE. Trans. Signal Processing, 49: 2693-2701.

Chen, B.S., C.S. Tseng and H.J. Uang, 1999. Robustness design of nonlinear dynamic systems via fuzzy linear control. IEEE. Trans. Fuzzy Syst., 7: 571-585.

Chen, B.S., C.S. Tseng and H.J. Uang, 2000. Mixed $H_2/H_\infty$ fuzzy output feedback control design for nonlinear dynamic systems: An LMI approach. IEEE. Trans. Fuzzy Syst., 8: 249-265.

Chen, B.S., C.S. Wu and H.J. Uang, 2000. A minimax tracking design for wheeled vehicles with trailer based on adaptive fuzzy elimination scheme. IEEE. Trans. Control Syst. Technol., 8: 418-434.

Chen, B.S., C.S. Wu and Y.W. Jan, 2000. Adaptive fuzzy mixed $H_2/H_\infty$ attitude control of spacecraft. IEEE. Trans. Aerospace Elect. Syst., 36: 1343-1359.

Chen, B.S., J.M. Chen and S.C. Shern, 1994. An ARMA robust system identification using a generalized lp norm estimation algorithm. IEEE. Trans. Signal Processing, 42: 1063-1073.

Chen, B.S., T.S. Lee and J.H. Feng, 1994. A nonlinear $H_\infty$ control design in robotic systems under parameter perturbation and external disturbance. Int. J. Control, 59: 439-461.

Chen, B.S., W.H. Chen and F. Hsu, 2008. Robust $H_\infty$ tracking control of quantum systems under disturbances. IEEE Transactions on Control Systems Technology (revised).

Chen, B.S., W.S. Wu, Y.C. Wang and W.H. Li, 2007. On the robust circuit design schemes of biochemical networks: Steady-state approach. IEEE. Trans. Biomed. Circuits Syst., 1: 91-104.

Chen, B.S., Y.C. Wang, W.S. Wu and W.H. Li, 2005. A new measure of the robustness of biochemical networks. Bioinformatics, 21: 2698-2705.

Chen, B.S., Y.M. Cheng and H.C. Lee, 1995. A genetic approach to mixed H2/H PID Control. IEEE. Trans. Control Syst. Mag., 15: 51-60.

Chen, B.S., Y.Y. Chen and C.L. Lin, 2002. Nonlinear fuzzy $H_\infty$ guidance law with saturation of actuators against maneuvering targets. IEEE. Trans. Control Syst. Technol., 10: 769-779.

De Souza, C.E., U. Shaked and M. Fu, 1995. Robust $H_\infty$ filtering for continuous time varying uncertain systems with deterministic input signals. IEEE. Trans. Signal Processing, 43: 709-719.

Doyle, J., K.M. Zhou, K. Glover and B. Bodenheimer, 1994. Mixed H-2 and H-∞ Performance-Objectives. I. Robust performance analysis. IEEE. Trans. Automatic Control, 39: 1564-1574.

Doyle, J.C., K. Glover, P.P. Khargonekar and B.A. Francis, 1989. State-space solutions to standard $H_2$ and $H_\infty$ control problems. IEEE. Trans. Automatic Control, 34: 831-847.

Duan, Z., J. Zhang, C. Zhang and E. Mosca, 2006. Robust H-2 and H-∞ filtering for uncertain linear systems Automatica, 42: 1919-1926.

Francis, B., J. Helton and G. Zames, 1984. $H_\infty$-optimal feedback controllers for linear multivariable systems. IEEE. Trans. Automatic Control, 29: 888-900.

Francis, B.A., 1987. A course in H control theory. Berlin, Germany: Springer-Verlag.

Fridman, E. and U. Shaked, 2001. A new H filter design for linear time delay systems. IEEE. Trans. Signal Processing, 49: 2839-2843.

Gadewadikar, J. and F.L. Lewis, 2006. Aircraft flight controller tracking design using H-∞ static output-feedback. Trans. Inst. Measurement and Control, 28: 429-440.

Gao, Z.W. and S.X. Dian, 2007. State and Disturbance Estimator for Time-Delay Systems With Application to Fault Estimation and Signal Compensation. IEEE. Trans. Signal Processing, 55: 5541-5551.

Gao, Z.W., 2005. PD observer parameterization design for descriptor systems. J. Franklin Inst., 342: 551-564.

Gershon, E., U. Shaked and I. Yaesh, 2007. Static output-feedback of state-multiplicative systems with application to altitude control. Int. J. Control, 77: 1160-1170.

Grimble, M.J. and A. El Sayed, 1990. Solution of the $H_\infty$ optimal linear filtering problem for discrete-time systems. IEEE. Trans. Signal Processing, 38: 1092-1104.

Hinrichsen, D. and A.J. Pritchard, 1998. Stochastic $H_\infty$, SIAM J. Control Optim, 36: 1504-1538.

Hoa, H.F., Y.K. Wonga and A.B. Radb, 2007. Robust fuzzy tracking control for robotic manipulators. Simulation Modelling Practice and Theory, 15: 801-816 .

Huang, D. and S.K. Nguang, 2006. Robust $H_\infty$ static output feedback control of fuzzy systems: An ILMI approach. IEEE. Trans. Syst. Man and Cybernetics, Part B, 36: 216-222.

Hung, J.C. and B.S. Chen, 2000. Genetic algorithm approach to fixed-order mixed $H_2/H_\infty$ optimal deconvolution filter designs. IEEE. Trans. Signal Processing, 48: 3451-3461.

Isidori, A. and A. Astolfi, 1992. Disturbance attenuation and $H_\infty$-control via measurement feedback in nonlinear systems. IEEE. Trans. Automatic Control, 37: 1283-1293.

Khargonekar, P.P., K.M. Nagpal and K.R. Poolla, 1991. $H_\infty$ control with transients. SIAM J. Control and Optimization, 29: 1373-1393.

Lee, B.K., H.W. Chen and B.S. Chen, 2004. Power control of cellular radio systems via robust Smith prediction filter. IEEE. Trans. Wireless Commun., 3: 1822-1831.

Lee, B.K., Y.H. Chen and B.S. Chen, 2006. Robust $H_\infty$ power control for CDMA cellular communication systems. IEEE. Trans. Signal Processing, 54: 3947-3956.

Lee, K.R., J.H. Kim, E.T. Jeung and H.B. Park, 2000. Output feedback robust $H_\infty$ control of uncertain fuzzy dynamic systems with time-varying delay. IEEE. Trans. Fuzzy Syst., 8: 657-664.

Lian, K.Y., C.S. Chiu and P. Liu, 2002. Semi-decentralized adaptive fuzzy control for cooperative multirobot systems with $H_\infty$ motion/internal force tracking performance. IEEE. Trans. Syst. Man and Cybernetics, Part B: Cybernetics, 32: 269-280.

Limebeer, D.J.N., B.D.O. Anderson and B. Hendel, 1994. A Nash game approach to mixed $H_2/H_\infty$ control. IEEE. Trans. Automatic Control, 39: 69-82.

Lin, C. and Y. Mon, 2003. Hybrid adaptive fuzzy controllers with application to robotic systems. Fuzzy Sets Syst., 139: 151-165.

Lin, Y.C. and J.C. Lo, 2006. Robust Mixed $H_2/H_\infty$ Filtering for Time-Delay Fuzzy Systems. IEEE. Trans. Signal Processing, 54: 2897-2909.

Mahmoud, M. and M. Zribi, 1999. $H_\infty$-controllers for time-delay systems using linear matrix inequalities. J. Optim. Theory Applic., 100: 89-122.

Nichols, R.A., R.T. Reichert and W.J. Rugh, 1993. Gain scheduling for $H_\infty$-controllers: A flight control example. IEEE. Trans. Control Syst. Technol., 1: 69-79.

Peng, S.C. and B.S. Chen, 1997. Deconvolution filter design via l1 optimization technique. IEEE. Trans. Signal Processing, 45: 736-746.

Pila, A.W., U. Shaked and C.E. de Souza, 1999. $H_\infty$-filtering for continuous-time linear systems with delay. IEEE. Trans. Automatic Control, 44: 1412-1417.

Ravi, R., K.M. Nagpal and P.P. Khargonekar, 1991. $H_\infty$ control of linear time-varying systems: A state-space approach. SIAM J. Control Optim., 29: 1394-1413.

Safonov, M.G. and B.S. Chen, 1982. Multivariable stability margin optimization with decoupling and output regulation. 21st IEEE. Conf. Decision and Control, 21: 616-622.

Shaked, U. and N. Berman, 1995. $H_\infty$ nonlinear filtering of discrete-time processes. IEEE. Trans. Signal Processing, 43: 2205-2209.

Shaked, U., L.H. Xie and Y.C. Soh, 2001. New approaches to robust minimum variance filter design. IEEE. Trans. Signal Processing, 49: 2620-2629.

Stoorvogel, A.A., 1992. The $H_\infty$ Control Problem: A State Space Approach, New Yourk: Prentice-Hall.

Takagi, T. and M. Sugeno, 1985. Fuzzy identification of systems and its applications to modeling and control. IEEE. Trans. Syst. Man and Cybernetics, 15: 116-132.

Tanaka, K. and H.O. Wang, 2001. Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach, John Wiley and Sons, Inc. New York, USA.

Tseng, C.S. and B.S. Chen, 2001. $H_\infty$ decentralized fuzzy model reference tracking control design for nonlinear interconnected systems. IEEE. Trans. Fuzzy Syst., 9: 795-809.

Tseng, C.S. and B.S. Chen, 2001. $H_\infty$ fuzzy estimation for a class of nonlinear discrete-time dynamic systems. IEEE. Trans. Signal Processing, 49: 2605-2619.

Tseng, C.S. and B.S. Chen, 2001. Multiobjective PID control design in uncertain robotic systems using neural network elimination scheme. IEEE. Trans. Syst. Man and Cybernetics, Part A, 31: 632-644.

Tseng, C.S. and B.S. Chen, 2003. A mixed $H_2/H_\infty$ adaptive tracking control for constrained non-holonomic systems. Automatica, 39: 1011-1018.

Tseng, C.S., 2005. Mixed $H_2/H_\infty$ adaptive tracking control design for uncertain constrained robots. Asian J. Control, 7: 296-309.

Tseng, C.S., B.S. Chen and H.J. Uang, 2001. Fuzzy tracking control design for nonlinear dynamic systems via T-S fuzzy model. IEEE. Trans. Fuzzy Syst., 9: 381-392.

Uang, H.J. and B.S. Chen, 2002. Robust adaptive optimal tracking design for uncertain missile systems: A fuzzy approach. Fuzzy Sets Syst., 126: 63-87.

Van der Schaft, A.J., 1992. L2-gain analysis of nonlinear systems and nonlinear state-feedback H control. IEEE. Trans. Automatic Control, 37: 770-784.

Weinmann, A., 1991. Uncertain Models and Robust Control, Springer-Verlag, Hong Kong.

Wu, C.S. and B.S. Chen, 2001. Adaptive attitude control of spacecraft: Mixed $H_2/H_\infty$ approach, AIAA. J. Guidance, Control and Dynamics, 24: 755-766.

Wu, C.S., B.S. Chen and Y.W. Jan, 1999. Unified design for $H_2/H_\infty$ and mixed control of spacecraft. AIAA J. Guidance, Control and Dynamic, 22: 884-896.

Xie, L.H. and C.E. Desouza, 1992. Robust $H_\infty$-control for linear-systems with norm-bounded time-varying uncertaint. IEEE. Trans. Automatic Control, 37: 1188-1191.

Xie, L.H., C.L. Du, C.S. Zhang and Y.C. Soh, 2002. $H_\infty$-deconvolution filtering of 2-D digital systems. IEEE. Trans. Signal Processing, 50: 2319-2332.

Xie, L.H., M.Y. Fu and C.E. Desouza, 1992. $H_\infty$-control and quadratic stabilization of systems with parameter uncertainty via output-feedback. IEEE. Trans. Automatic Control, 37: 1253-1256.

Xu, S. and T. Chen, 2002. Robust $H_\infty$ control for uncertain stochastic systems with state delay. IEEE. Trans. Automatic Control, 47: 2089-2094.

Youla, D., H. Jabr and J. Bongiorno, 1976b. Modern Wiener-Hopf design of optimal controllers Part II: The multivariable case. IEEE. Trans. Automatic Control, 21: 319-338.

Youla, D., J. Bongiorno and H. Jabr, 1976a. Modern Wiener-Hopf design of optimal controllers Part I: The single-input-output case. IEEE. Trans. Automatic Control, 21: 3-13.

Zames, G., 1981. Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms and approximate inverses. IEEE. Trans. Automatic Control, 26: 301-320.

Zhang, H., S. Lun and D. Liu, 2007. Fuzzy $H_\infty$ filter design for a class of nonlinear discrete-time systems with multiple time delays. IEEE. Trans. Fuzzy Syst., 15: 453-469.

Zhang, W. and B.S. Chen, 2006. State feedback $H_\infty$ control for a class of nonlinear stochastic systems. SIAM J. Control and Optimization, 44: 1973-1991.

Zhang, W., B.S. Chen and C.S. Tseng, 2005. Robust $H_\infty$ filtering for nonlinear stochastic systems. IEEE. Trans. Signal Processing, 53: 589-598.

Zhang, W., H. Zhang, and B.S. Chen, 2006. Stochastic $H_2/H_\infty$ control with (x, u, v)-dependent noise: Finite horizon case. Automatica, 42: 1891-1898.