International Journal of Signal System Control and Engineering Application 2 (3-4): 45-50, 2009
ISSN: 1997-5422

On the Design of Optimal Output Regulators for Linear Time-Invariant Multivariable Systems

E.C. Obinabo and O. Omorogiuwa
Department of Electrical and Electronic Engineering,
Ambrose Alli University, P.M.B. 14 Ekpoma, Edo State, Nigeria

Abstract: This study presents the design of an optimal output regulator for application to a linear time-invariant multivariable system based on the use of a vector of inputs and outputs and a matrix of operations characterizing the input-output relations, assuming the system is both controllable and observable under arbitrary state feedback. The stability property of the solution which is desirable from the view point of feedback control design was clarified.

Key words: Stability functions, multivariable systems, interconnected electric power generating plants, optimal output regulators, decouplable linear systems, clarified

INTRODUCTION

In the past few decades, there has been considerable interest in the design of linear multivariable time-invariant systems via state or output feedback. One of the major difficulties in the study of multivariable systems is the fact that each input variable can in principle, affect each output variable. This has led rather naturally to representations for multivariable systems which are based on the use of a vector of inputs, a vector of outputs and a matrix of operations characterizing the input-output relations. It has been shown (Wang, 1970) that for the subclass of decouplable linear systems satisfying the criterion:

$$m + \sum_{i-1}^{m} d_i = n$$

where, m is the number of inputs and outputs and n the dimension of the state space, a minimum order realization can be found such that the set $\{d_i\}; i=1,...,m$ defines a necessary and sufficient condition for decoupling the system by output feedback (Philips and Harbor, 2000; Safanov, 1980). This condition implies that the system is both controllable and observable under arbitrary state feedback. For this class of systems, it can be easily seen that the set $\{d_i\}; i=1,...,m$ is a complete set of independent invariants which thus, defines a unique canonical form (Silverman and Payne, 1971; Sato and Lopresti, 1971) for finding a control law such that the subsets of the output vector of the closed-loop system are controllable by subsets of the input vector in an independent manner. The possible variations of this approach pose the problem of finding existence of a decoupling control law and characterization of the class of decoupling control law (Majumdar and Choudhury, 1973; Cremer, 1971; Morse and Wonham, 1970), thereby satisfying the necessary conditions for acceptable dynamic response namely, steady state accuracy and asymptotic decoupling (Falb and Wolovich, 1967; Wolovich and Falb, 1969; Anderson and Moore, 1990).

The discrete-time deterministic optimal control problem is well established in the existing literature for linear time-invariant dynamic systems. For this system, the performance criterion is related to the smallness of a norm represented as follows:

$$\begin{aligned} J &= \frac{1}{2}\int_{t_0}^{t_1}\left(x^T Qx + u^T Ru\right)dt + \frac{1}{2}x^T(t_1)Gx(t_1) \\ &= \int_{t_0}^{t^1} f_0(x,u)dt + \frac{1}{2}x^T(t_1)Gx(t_1) \end{aligned} \quad (1)$$

The requirement here is an optimal control function u (t), t∈$[t_0, t_1]$, to minimize J. The matrices Q(t) and R(t) are here assumed systematic and nonnegative and positive definite respectively and G is a non-negative definite matrix. For optimal solution, the Hamiltonian:

$$\begin{aligned} H(x,p,t,u) &= \frac{1}{2}\left(u^T Ru + x^T Qx\right) + P^T(Ax + Bu) \\ &= p^T(t)f(x,u) + f_o(x,u) \end{aligned} \quad (2)$$

is defined where represent adjoint or costate variables.

Corresponding Author: E.C. Obinabo, Department of Electrical and Electronic Engineering, Ambrose Alli University, P.M.B. 14 Ekpoma, Edo State, Nigeria

The solution to discrete and continuous-time feedback control problem is possible using spectrum decomposition approach (Obinabo, 2008) or algebraic approach over a ring of polynomials (Morse and Wonham, 1970) and the linear quadratic optimal control approach (Krichman *et al*., 2001). In the solution to the class of decouplable linear systems considered in this study, we quantified the cost savings using a derived performance criterion

Which resulted in a controller that minimizes this function and gives the optimal solution to the problem. In this study, the output regulator problem and pant description applicable to a linear model of an electric power system (Obinabo, 2008) are investigated for stability where the process is assumed detectable and controllable.

THE LINEAR MULTIVARIABLE SYSTEM

The multivariable system considered was a linear, time-invariant plant assumed controllable and represented by the state equations:

$$\dot{x}(\underline{t})=A\underline{x}(t)+B\underline{u}(t),\underline{x}(t)\varepsilon R^{n} \quad (3)$$

$$\underline{y}(t)=C\underline{x}(t),y(t)\varepsilon R^{m},\underline{u}(t)\varepsilon R^{m} \quad (4)$$

The system S (A, B, C) was assumed to be square since the actual plant outputs and those defined by Eq. 4 need not necessarily coincide. The matrices B and C have full rank since the system has independent inputs and outputs. In addition, CB has full rank that is rank so that the system is therefore clearly output controllable. The initial condition is given by:

$$\underline{x}(0)=\underline{x}_{0} \quad (5)$$

For controllability, it was desired to take the system to a terminal state $\underline{x}(t_f)=0$, where t_f is the terminal time. This was done by applying a suitable control $\underline{u}(t)=f(\underline{x},t)$ which is optimal in some sense. Consequently, a quadratic performance criterion of the from :

$$J=\frac{1}{2}\underline{x}^{T}S_{f}\underline{x}+\frac{1}{2}\int_{t_0}^{t_f}\left[\underline{x}^{T}Q(t)\underline{x}+\underline{u}^{T}R(t)\underline{u}\right]dt \quad (6)$$

was found and minimized. Without loss of generality, Q(t), R(t) and S_f were assumed to be symmetric matrices. Further, S_f, Q(t) and R(t) are respectively positive semi-definite, positive definite matrices with Q(t) and R(t) being continuous in t. An appropriate choice of S_f Q(t) and R(t) must be made for obtaining acceptable performance of the system.

To obtain a linear feedback law a cost functional of the form given in Eq. 6 was mandatorily desired to keep the problem mathematically tractable. For optimal solution, the Hamiltonian (Lehtomaki *et al*., 1981; Obinabo, 2008):

$$H\left[\underline{x}(t),(t),t\right]=\frac{1}{2}\underline{x}^{T}Q(t)\underline{x}+\frac{1}{2}\underline{u}+ \underline{\lambda}^{T}\left[A(t)\underline{x}+B(t)\underline{u}\right] \quad (7)$$

was defined and by the maximum principle of Pontryagin the following was obtained:

$$\frac{\partial H}{\partial \underline{u}}=0=R(t)\underline{u}(t)+B^{T}(t)\underline{\lambda}(t) \quad (8)$$

$$\frac{\partial H}{\partial \underline{x}}=\lambda=Q(t)\underline{x}(t)+A^{T}(t)\underline{\lambda}(t) \quad (9)$$

Terminal condition:

$$\underline{\lambda}(t_f)=S_f\underline{x}(t_f) \quad (10)$$

Hence from Eq. 5:

$$\underline{u}(t)=-R^{-1}(t)B^{T}(t)\underline{\lambda}(t) \quad (11)$$

Assuming that the solution to λ (t) has the same structure as Eq. 10, λ (t) was represented by:

$$\underline{\lambda}(t)=S(t)\underline{x}(t) \quad (12)$$

Hence,

$$\underline{x}=A(t)\underline{x}(t)-B(t)-B(t)R^{-1}(t)A^{T}(t)\underline{x}(t) \quad (13)$$

$$\underline{\lambda}=S\underline{x}(t)+S(t)\underline{x}=Q(t)\underline{x}(t)-A^{T}(t)S(t)\underline{x}(t) \quad (14)$$

Combining Eq. 13 and 14, the following equation was obtained:

$$S+\underline{s}(t)A(t)+A^{T}(t)S(t)B(t)R^{-1}(t) B^{T}(t)S(t)+Q(t)\underline{x}(t)=0 \quad (15)$$

The result Eq. 15 holds for all non-zero $\underline{x}(t)$ and hence S(t) is an n×n symmetric matrix satisfying the matrix Riccati equation (Riccati, 1724):

$$S=-\underline{s}(t)A(t)-A^{T}(t)S(t)+S(t) B(t)R^{-1}(t)B^{T}(t)S(t)-Q(t) \quad (16)$$

With the terminal condition:

$$S(t_f)=S_f \tag{17}$$

The optimal control law becomes:

$$\underline{u}(t)-(t)\underline{x}(t) \tag{18}$$

Where K(t) is the n×n feedback gain matrix given by:

$$K(t)=R^{-1}(t)B^T(t)S(t) \tag{19}$$

The corresponding optimal cost function was obtained as:

$$J^*=\frac{1}{2}\underline{x}_0 S(t_0)\underline{x}_0 \tag{20}$$

An optimal regulator was then required to bring a subset of the system states to given non-zero constant set points. The expected steady state error between the states and their set points was found to be zero.

The common trend in the design of optimal regulators places special premium on some aspects of the performance in order to minimize energy losses. This is usually done by quantifying the cost savings using a defined performance index.

The controller that minimizes this function gives the optimal solution to the problem. The advantage of the linear quadratic optimal control approach is that if the state feedback system is realizable, the resulting closed-loop configuration known as the optimal regulator is expected to have some desirable sensitivity and robustness properties as in the case of finite-dimensional systems. The study reported in the existing literature (Elowitz and Leibler, 2000; Tempo *et al.*, 1997) presents fundamental results on the existence and characterization of optimal control by Riccati equations (Riccati, 1724) and criteria for stability of the system with delay in state. The system is described by the linear delay differential equation:

$$\dot{x}(t)=A_0x(t)+A_1x(t-h)+Bu(t) \tag{21}$$

Where z, ϕ is an element in $M^2 = R^n \times L^2([-h, 0]; R^n)$, the time delay h is a positive real number. The constant matrices A_0, A_1 and B have the dimensions n×n, n×n and n×r, respectively. The admissible control u is an element in $L^2([0, \infty]: R^r$. For the state model of the transient heat flow in the continuous casting of steel slabs, the performance index is related to the smallness of a norm represented by the performance index:

$$\begin{aligned} J &=\frac{1}{2}\int_{t_0}^{t_1}\left(x^T Qx+u^T Ru\right)dt+\frac{1}{2}x^T(t_1)Gx(t_1) \\ &=\int_{t_0}^{t_1} f_0(x,u)dt+\frac{1}{2}x^T(t_1)Gx(t_1) \end{aligned} \tag{22}$$

The requirement here is an optimal control function u (t), t ε [t_0, t_1] to minimize J. The matrices Q (t) and R (t) are assumed non-negative and positive definite, respectively and lead to an optimal control which is a linear function of the state and G is a non-negative definite matrix. For optimal solution, the Hamiltonian (Riccati, 1724; Grimble, 1978; Obinabo, 2008):

$$\begin{aligned} H(x, p,t,u) &=\frac{1}{2}\left(u^T Ru+x^T Qx\right)+P^T(Ax+Bu) \\ &=p^T(t)f(x,u)+f_0(x,u) \end{aligned} \tag{23}$$

is defined where P_i represent adjoint or costate variables. Now to make H a minimum:

$$\frac{\partial H}{\partial H}=Ru+B^TP=0$$

giving,

$$u(t)=-R^{-1}(t)B^Tp(t)$$

It was then easily shown that:

$$p(t)=p(t)x(t)$$

giving,

$$u(t)=-R^{-1}(t)B^Tp(t)x(t)$$

From,

$$\dot{p}(t)=-\frac{\partial H}{\partial x}, \dot{x}(t)=\frac{\partial H}{\partial p}$$

We obtain the canonical differential equations:

$$\begin{bmatrix}\dot{x}(t)\\ \dot{p}(t)\end{bmatrix}=\begin{bmatrix}A & -BR^{-1}B^T\\ -Q & -A^T\end{bmatrix}\begin{bmatrix}x\\ p\end{bmatrix}, p(t_1)=0 \tag{24}$$

We now consider a quadratic form defined in the state space M^2 of the system Eq. 21. For an optimal regulator, we let a non-negative definite triplet {P_0, P_1, P_2 be a solution to the infinite-dimensional Riccati Eq. 24 with boundary conditions:

$$x(t=0)=x(t_1),\ P(t=0)=P(t_1)=0,\ p(0)=G$$

and the state feedback control law given by:

$$u(t) = -R^{-1}B'\left\{P_o x(t) + \int_h^0 P_1(\beta)x(t+\beta)d\beta\right\} \quad (25)$$

Theorem: Let system Eq. 21 be stabilizable and detectable with respect to the cost function given by:

$$J(u) = \int_0^\infty \begin{bmatrix} (x(t), x_t)'\{Q_0, Q_1, Q_2\} \\ (x(t), x_t) + u'(t)Ru(t) \end{bmatrix} dt \quad (26)$$

Where $x_t = x\ (t+\xi)$, $-h \le \xi \le 0$. The triplet $\{Q_0, Q_1, Q_2\}$ is non-negative definite and R is a positive definite constant real matrix with dimension r×r. If the Riccati Eq. 24 have a non-negative definite solution, then the state feedback control law Eq. 25 provides the optimal solution to the linear quadratic optimal control problem in the stabilizing class of admissible controls.

Proof: Using the non-negative solution P_0, P_1, P_2 to the Riccati equations (Obinabo, 2008; Burns, 1989; Gibson 1983; Payne and Silverman, 1973):

$$A_o' p_o + p_o A_o + p_1'(0) + p_1(0) + Q_o - p_o BR^{-1}B'p_o = 0 \quad (27)$$

$$-\frac{\partial}{\partial\beta}p_1(\beta) + A_o' p_1(\beta) + p_2(0,\beta) + Q_1(\beta) - p_o BR^{-1}B'p_1(\beta) = 0 \quad (28)$$

and the trajectory x of the system Eq. 21 for the admissible control u, we defined the quadratic form:

$$(x(t), x_t)'\{p_0, p_1, p_2\}(x(t), x_t)$$

Differentiating with respect to t, substituting Eq. 21 and 27, 28, re-arranging terms and integrating both sides with respect to t over the interval [0, T], we obtain:

$$\begin{aligned} &(x(T), x_T)'\{p_0, p_1, p_2\}(x(T), x_T) + \\ &\int_0^T \begin{bmatrix} (x(t), x_t)'\{Q_0, Q_1, Q_2\} \\ (x(t), x_t) + u'(t)Ru(t) \end{bmatrix} dt \\ &= (z,\phi)'\{p_o, p_1, p_2\}(z,\phi) + \\ &\int_0^T \left\| u(t) + R^{-1}B\left\{p_o v(t) + \int_{-h}^0 p_1(\beta)x(t+\beta)d\beta\right\}\right\|_{R^2} dt \end{aligned} \quad (29)$$

Where $\|y\|R^2 = yRy$ for y in R. If u is a stabilizing control, the corresponding trajectory x is an element in R^2 (0, ∞); R^n so that the system is exponentially stable and $(x(T, x_{t)} \to 0$ as $T \to \infty$. Further, J (u)<∞ for the stabilizing control u, since $\{Q_0, Q_1, Q_2\}$ defines a bounded operator on and M^2 is a constant matrix. Therefore, for the stabilizing control u, letting T go to infinity on both sides of Eq. 29, we have:

$$\begin{aligned} &J(u) = (z,\phi)\{p_0, p_1, p_2\}(z,\phi) + \\ &\int_0^\infty \left\| u(t) + R^{-1}B'\begin{Bmatrix} p_o x(t) + \\ \int_h^0 p_1(\beta)x(t+\beta)db \end{Bmatrix} \right\|_{R^2} dt \end{aligned} \quad (30)$$

This identity implies that the control law Eq. 25 attains the minimum of the cost provided that the control Eq. 25 is stabilizable. This is shown, first by considering from Eq. 29 the functional:

$$\begin{aligned} &(x(T), x_T)'\{p_0, p_1, p_2\}(x(T), x_T) + \\ &\int_0^T \begin{bmatrix} (x(t), x_t)'\{Q_0, Q_1, Q_2\} \\ (x(t), x_t) + u'(t)Ru(t) \end{bmatrix} dt \\ &= (z,\phi)'\{p_o, p_1, p_2\}(z,\phi) \end{aligned} \quad (31)$$

If, however, system Eq. 2 is not stable given the control law Eq. 25, then the non-negative definite triplet $\{P_0, P_0, P_0\}$ is not positive definite and a contradiction will result if the triplet is positive definite, with the notion that for the emerging unstable trajectory x of system Eq. 21 with control Eq. 25, the left hand side of Eq. 31 goes to infinity or becomes indefinite as $T \to \infty$ while the right hand side remains constant. Therefore, there exists a non-zero element (z, ϕ) in M^2 such that:

$$(z,\phi)'\{p_o, p_1, p_2\}(z,\phi) = 0 \quad (32)$$

Now, since the first and second terms of this equation are both non-negative, we have from Eq. 31

$$\int_0^t \begin{bmatrix} (x(t), x_t)'\{Q_0, Q_1, Q_2\} \\ (x(t), x_t) + u'(t)Ru(t) \end{bmatrix} dt = 0 \quad (33)$$

Since $\{Q_0, Q_1, Q_2\}$ is non-negative definite and R is positive definite, the identity Eq. 33 implies that the control u given by Eq. 25 should be identically zero. The concept of complete controllability of the system Eq. 21 was of central importance in the establishment of the optimal regulator and was tacitly assumed in obtaining the results of the previous section. The state x∊X is controllable if and only if there exists ω∊Ω such that:

$$z^n.x + \bar{G}_{\Sigma}(\omega)=0,\quad \partial^0\omega < n \tag{34}$$

In other words, x is controllable if it can be transferred to the zero state by applying some input sequence.

Here, we denote the set of all controllable states by X_c. If X = X_c, Σ will be completely controllable. A subset Y of X will be controllable in bounded time if there exists an integer N≥0 so that for all , x∊y, ∃ω∊Ω such that:

$$z^n.x + \bar{G}_{\Sigma}(\omega)=0,\quad \partial^0\omega < n \tag{35}$$

In other words, every state in Y can be transferred to zero in at most N steps.

Example: A linear multivariable system characterized by the general representation Eq. 4 and reducible to an equivalent interconnection of subsystems where each representation for a given system has the same input-output relations is considered. Here, the A and B matrices were obtained experimentally (Obinabo, 2008) as:

$$\begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ b \end{bmatrix}$$

respectively. For improved stability properties the performance criterion of the form:

$$J=\int_0^{\infty}\left(x_1^2 + ru^2\right)dt$$

was defined and the conditions which characterize stability of the interconnection Eq. 4 from the steady state Riccati equation (Obinabo, 2008) were obtained as follows:

$$\dot{x}=-x+u,\qquad x(0)=1$$

$$J=\int_0^{\infty}\left[\{x(t)\}^2 + b\{u(t)\}^2\right]dt,\ b>0$$

The state equations are as follows:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

$$Q=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix};R=b$$

The steady state Riccati equations were computed as follows:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}+\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}-$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\left(\frac{1}{B}\right)[0\ 1]\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}=0$$

$$\begin{bmatrix} -P_1 & -P_2 \\ P_2 & P_3 \end{bmatrix}+\begin{bmatrix} -P_1 & -P_2 \\ -P_2 & P_3 \end{bmatrix}-\begin{bmatrix} P_2^2 & -P_2 \\ P_2P_3 & P_3^2 \end{bmatrix}\frac{1}{b}+\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}=0$$

$$-2P_2 - P_2\frac{2}{b}+1=0$$

$$-P_3 + P_1 - P_2\frac{P_3}{b}=0$$

$$2P_2 = P_3\frac{2}{b}=0$$

$$P_2 = b+\sqrt{b^2+b}$$

$$P_3=\sqrt{-2b^2+2b\sqrt{b^2}}$$

$$U=-R^1B^TPx=-\frac{1}{b}[0\ 1]\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$=-\frac{1}{b}[P_2\ P_3]x$$

$$=\left[\sqrt{1+\frac{1}{b}}-1,\sqrt{-2+\sqrt{1+\frac{1}{b}}}\right]x(t)$$

CONCLUSION

A generalized optimal regulator problem for application to a simple 2×2 interconnection of electric power generating plants was formulated and the stability property of the solution which is desirable from the view point of feedback control designs was clarified.

By generalizing the quadratic form of costs, we were able to bring a pole location method into the optimal regulator design and it was shown that if the generalized cost weighting matrices are chosen appropriately, the generalized optimal regulator problem was reduced to the particular one which can be solved only by computing solutions to finite-dimensional Riccati equations. We have also shown that an improved stability property can be obtained with the performance index of the form:

$$J = \int_{t_0}^{\infty} e^{2\alpha t}\left(x^T Qt + u^T Ru\right)dt$$

REFERENCES

Anderson, B.D.O. and J.B. Moore, 1990. Optimal Control. Prentice-Hall, Englewood, Cliffs, New Jersey.

Burns, R.S., 1989. Application of the Riccati Equation in the Control and Guidance of Marine Vehicles. In: The Riccati Equation in Control Systems and Signals, Bittanti, S. (Ed.). Pitagora Editrice, Bologna, Italy, pp: 18-23.

Cremer, M., 1971. A precompensator of minimal order for decoupling a linear multivariable system. Int. J. Control, 14: 1089-1103.

Elowitz, M.B. and S. Leibler, 2000. A synthetic oscillatory network of transcriptional regulators. Nature, 403: 335-338.

Falb, P.L. and W.A. Wolovich, 1967. Decoupling in the design and synthesis of multivariable control systems. Trans. Automatic Control, 12: 651-659.

Gibson, J.S., 1983. Linear quadratic optimal control of hereditary differential systems: infinite dimensional riccati equations and numerical approximations. SIAM J. Control Optm., 21: 95-139.

Grimble, M.J., 1978. Design of stochastic optimal feedback control systems. Proc. IEE Control Sci., 125: 1275-1284.

Krichman, M., E. Sontag and Y. Wang, 2001. Input-output-to-state stability. Siam J. Control Optm., 39: 1874-1928.

Lehtomaki, N.A., N.R. Jr. Sandell and M. Athans, 1981. Robustness results in linear quadratic gaussian based multivariable control designs. IEEE Trans. Autom. Control, 26: 75-92.

Majumdar, A.K. and A.K. Choudhury, 1973. On the decoupling of linear multivariable systems. Int. J. Control, 17: 225-256.

Morse, A.S. and W.M. Wonham, 1970. Triangular decoupling of linear multivariable systems. IEEE Trans. Autom. Control, 15: 447-449.

Obinabo, E.C., 2008. Optimal control of transient heat flow during interstand cooling of rolled steel in billet mills. Int. J. Signal Syst. Control Eng. Appl., 1: 223-232.

Payne, H.J. and L.M. Silverman, 1973. On the discrete time algebraic riccati equation. IEEE Trans. Autom. Control, 18: 226-234.

Philips, C.L. and R.D. Harbor, 2000. Feedback Control Systems. 4th Edn., Prentice-Hall Inc., Upper Saddle River, N.J..

Riccati, J.P., 1724. Animadversiones in Acquationes Differentiales, Acta Eruditorum Lipsiae. In: Count Riccati and the Early Days of the Riccati Equation, Bittanti, S. (Ed.). Pitagora Editrice, Bologna, Milano.

Safanov, M.J., 1980. Stability and Robustness Feedback Systems. MIT Press, Cambridge, Mass.

Sato, S.M. and P.V. Lopresti, 1971. New results in multivariable decoupling theory. Automatica, 7: 499-508.

Silverman, L.M. and H.J. Payne, 1971. Input-output structure of linear systems with application to the decoupling problem SIAM J. Control, 9: 199-233.

Tempo, R., E.W. Bai and F. Dabbene, 1997. Probabilistic robustness analysis: Explicit bounds for the minimum number of samples. Syst. Control Lett., 30: 237-242.

Wang, S.H., 1970. Design of precompensator for the decoupling problem. Electron. Lett., 6: 739-741.

Wolovich, W.A. and P.L. Falb, 1969. On the structure of multivariable systems. SIAM J. Control, 7: 437-451.