

Asymptotic Integration of A Class of Nonlinear Parabolic Equations

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Abstract: The objective of this research is to give the explicit computation for the normal form and the normalization mapping of a class of the nonlinear parabolic equations developed in.

Key words: Nonlinear parabolic equations, normal form, normalization mapping, asymptotic expansions, asymptotic integration

INTRODUCTION

Let us consider the class of nonlinear parabolic equations:

$$\begin{cases} u_t + Au + f(u) = 0 \\ u(0) = u_0 \end{cases} \quad (1.1)$$

where A is an unbounded linear operator of the domain D(A) and f is a polynomial such that:

$$f(u) = \sum_{n=2}^{2p+1} a_n u^n, \quad p \geq 1, \quad a_{2p+1} > 0 \quad (1.2)$$

In a previous work^[1], we constructed an asymptotic expansion and a normal form for the Eq. (1.1). In a non resonant case, this normal form is a linear parabolic equation

$$U_t + AU = 0 \quad (1.3)$$

In a general case, the normalization mapping denoted by W builded in^[1] associate to a function u_0 belonging to a natural space of initial data an element $W(u_0)$ of a Frechet Space \mathfrak{S}_A such that for every solution $u(t)$ of (1.1), the function $v(t) = W(u(t))$ satisfies the following equation in \mathfrak{S}_A .

$$v_t + Av + F(v) = 0 \quad (1.4)$$

where the mapping F(v) (defined in section 1) is a generally nonlinear operator which involves only terms corresponding to resonances in the spectrum of A.

Moreover (1.4) is equivalent to an infinite system of non homogeneous linear ordinary differential equations, which can be (elementarily) integrated. More precisely, the components of F can be expressed in terms of a sequence of polynomials

$\{P_j\}_{j=1}^{\infty}$ and the operator f.

The objective of this research is to show uniqueness of the polynomial P_j and to give a way to compute them by induction, although the algorithm which gives the P_j is some what complicated.

Finally, we obtained that the normal form of (1.4) is canonical and we characterize the expansion in terms of its coordinates.

The application of the method to other nonlinear equation has been made in.

NOTATIONS AND PRELIMINARIES

Let V and H be two separable Hilbert spaces such that

$$V \subset H \text{ with compact injection,} \quad (2.1)$$

$$V \text{ is dense in } H. \quad (2.2)$$

We denote by $\|\cdot\|$ and $|\cdot|$ the corresponding norms.

Consider the unbounded operator A with a rang in H

$$D(A) = \{u \in V, Au \in H\}. \quad (2.3)$$

Supplying D(A) with the graph norm, A is then an isomorphism of D(A) in H, so there exists a sequence

$$\{\lambda_i\}_{i=1}^{\infty}$$

of eigenvalues of A

$$0 < \lambda_1 < \lambda_2 < \dots, \quad (2.4)$$

each of a finite multiplicity and also a sequence

$$\{\omega_i\}_{i=1}^{\infty}$$

orthonormal basis (in H and V) of the associated eigenvectors.

$$A\omega_i = \lambda_i\omega_i, i = 1, 2, \dots \quad (2.5)$$

The orthogonal projection in H of the linear span($\omega_1, \dots, \omega_m$) will be denoted by P_m and R_j denotes the orthogonal projection in the eigenspace of λ_j

$$R_i\omega = \sum_{\lambda_k} (\omega, \omega_k)_H \omega_k \quad (2.6)$$

Then we have

$$R_j R_k = 0 \text{ si } i \neq j, R_1 \oplus R_2 \oplus \dots = 1 \quad (2.7)$$

We shall also consider the Frechet space containing H

$$\mathfrak{S}_A = R_1 H \oplus R_2 H \oplus \dots, \quad (2.8)$$

equipped with the topology of convergence of components, the operator A and the semi-group, e^{-At} generated by A , extend to \mathfrak{S}_A .

We denote by

$$0 < \Lambda_1 < \Lambda_2 < \dots < \Lambda_j < \dots, \quad (2.9)$$

The sequence of eigenvalues of multiplicity m_k and by $\{S(t)\}_{t \geq 0}$ the nonlinear semigroup defined by

$$S(t): V \rightarrow V, u_0 \rightarrow S(t)u_0 \quad (2.10)$$

Finally, we call resonance in the spectrum of A , the relation:

$$a_1 \Lambda_1 + a_2 \Lambda_2 + \dots + a_k \Lambda_{k+1}, a_i \in \mathbb{N}, i = 1, k \quad (2.11)$$

We shall denote by

$$0 < \mu_1 = \Lambda < \mu_2 < \dots \quad (2.12)$$

The elements of the additive semi-group \mathfrak{S} generated by Λ_k 's and $k_j = \text{Max}\{k, \Lambda_k \leq \mu_j\}$.

On the other hand, we define the power A^α of the operator A , for $\alpha \in \mathfrak{R}$ and we denote by

$$V_\alpha = D(A^{\frac{\alpha}{2}}) \text{ where } V_0 = H, V_1 = V \quad (2.13)$$

Supplying V_α with the following norm

$$\|u\|_\alpha = \|A^{\frac{\alpha}{2}} u\|_0 \quad (2.14)$$

then V_α is a Hilbert space.

Finally, we introduce a sequence

$$\{E_m\}_{m=1}^{\infty}$$

of Hilbert space such that

$$E_{m+1} \subset E_m \forall m \text{ with continuous injection} \quad (2.15)$$

$$\forall m, V_m \text{ is close subspace of } E_m \quad (2.16)$$

The following theorem showed some of the main results of^[1].

Theorem 2.1:

i) There exists a one to one analytic mapping

$$W: V \rightarrow \mathfrak{S}_A$$

satisfying $W'(0) = 1$, such that for every regular solution $u(t)$ of (1.1), $v(t) = W(u(t))$ satisfies the equation

$$V_t(t) + Av(t) + F(v(t)) = 0 \quad (2.17)$$

where

$$F_k(v) = R_k F(v) = \sum_{\mu_{q_1} + \dots + \mu_{q_j} = \Lambda_k} R_k \left[P_{q_1}(v_1, \dots, v_{k_{q_1}}) \dots P_{q_j}(v_1, \dots, v_{k_{q_j}}) \right] \quad (2.18)$$

and

$$V = v_1 \oplus v_2 \oplus \dots \in \mathfrak{S}_A \quad (2.19)$$

ii) In (2.18), for $j = 1, 2, \dots, P_j$ are $E_m \cap D(A)$ value polynomials defined on

$$R_1 H \oplus R_2 H \oplus \dots \oplus R_k H$$

depending on the spectrum of and. Such that

$$W_{\mu_j}(0, u_0) = P_j(W_1(u_0), \dots, W_{k_j}(u_0))$$

where

$$W_1(u_0) = R_1 W_{\Lambda_1}(0, u_0)$$

Moreover, if

$$M(X_1, \dots, X_{k_j})$$

is a monomial in

$$P_j(X_1, \dots, X_{k_j})$$

of degree

$$m_1, \dots, m_{k_j}$$

in

$$X_1, \dots, X_{k_j}$$

respectively, then

$$m_1 \Lambda_1 + \dots + m_{k_j} \Lambda_{k_j} = \mu_j.$$

Furthermore if μ_j is eigenvalue Λ_k i.e.,

$$\mu_j = \Lambda_{k_j}$$

then

$$P_j(X_1, \dots, X_{k_j}) = X_{k_j} + \text{higher order terms in } X_1, \dots, X_{k_j-1}$$

We recall that, the mapping W was constructed in^[1] from expansion of every regular solution of (1.1), the following theorem gives the properties of the expansion which will be used in the present research.

Theorem 2.2: For each $n \in \mathbb{N}$, the solution $u(t)$ of (1.1) admits the following expansion in H

$$u(t) = W\mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t} + \dots + \mu_N e^{-\mu_N t} + V_N(t) \quad (2.20)$$

where $W\mu_j(t)$ is a $E_m \cap D(A)$ -valued polynomial in t and $V_N(t) \in ([0, \infty), V) \cap C^\infty((t_1, \infty), E_m \cap D(A))$, $t_1 > 0$, $M \geq 0$. This expansion satisfies the following properties:

- $|V_N(t)|_m = O(e^{-(\mu_N + \epsilon_N)t})$, $\forall \epsilon_N > 0$, $m \geq 0$.
- $d_j^0 = \deg W\mu_j(t) \leq j - 1$, $j = 1, 2, \dots, N$.
- if $\Lambda_j \leq \mu_N$, $\Lambda_j \in \sigma(A)$ a non resonant eigenvalue, then W_{μ_j} is a constant in t and $R_j W_{\mu_j} = W_{\Lambda_j}$.
- if $\mu_j \leq \mu_N$ is not a nonresonant eigenvalue, W_{μ_j} satisfies the equation

$$\frac{d}{dt} W_{\mu_j} + (A - \mu_j) W_{\mu_j}(t) + \sum_{\mu_{\alpha_1} + \dots + \mu_{\alpha_n} = \mu_j} W_{\mu_{\alpha_1}} \bullet \dots \bullet W_{\mu_{\alpha_n}} \quad (2.21)$$

If Λ_j is a resonant eigenvalue we have

$$\deg W_{\mu_j}(t) \leq 1 + \text{Max}_{\mu_{\alpha_1} + \dots + \mu_{\alpha_n} = \mu_j} (d^0 \alpha_1 + \dots + d^0 \alpha_n) \quad (2.22)$$

Moreover,

$$R_k W_{\mu_j}(t)$$

for $k \neq j$ and the order of the coefficient will be obtained from

$$R_{\Lambda_1} W_{\Lambda_1}(0), \dots, R_{\Lambda_{j-1}} W_{\Lambda_{j-1}}(0)$$

- For any $\mu_j \in \mathfrak{S}$

$$W_{\mu_j}(0, u_0) = P_j(W_1(u_0), \dots, W_{k_j}(u_0)) \quad (2.23)$$

where P_j are the polynomial defined in theorem 1.1 and $W_k(u_0) = R_{\Lambda_k} W_{\Lambda_k}(0, u_0)$ are the components of the normalizing mapping W .

- Let $\prod_n: \mathfrak{S}_A \rightarrow R_1 H \oplus R_2 H \oplus \dots \oplus R_N H$ be the canonical projection $N = 1, 2, \dots$ then the range of $\prod_n W$ contains a ball centred at 0.

Remark 2.1:

We will henceforth use the notation

$$S(t, u_0) \approx \sum_1^\infty W_{\mu_j}(t, u_0) e^{-\mu_j t} \quad (2.24)$$

for the asymptotic expansion (2.19).

THE ALGEBRA OF THE ASYMPTOTIC EXPANSION

Proposition 3.1: For every $u \in V$, the asymptotic expansion (2.19) can be rewritten as follows

$$S(t)u \approx \sum_{j=1}^\infty P_j(W_1(S(t)u), \dots, W_{k_j}(S(t)u)) \quad (3.1)$$

Proof: We denote $k_j = \{k, \Lambda_k \leq \mu_j\}$;

Let, according to (2.23), has the asymptotic expansion

$$S(t)u \sim \sum_{j=1}^\infty e^{-\mu_j t} W_{\mu_j}(t, u) \quad (3.2)$$

It results from theorem 2.2

$$W_{\mu_j}(0, u) = P_j(W_1(u), \dots, W_{k_j}(u)) \quad (3.3)$$

We also have for any $t, t \geq 0$

$$S(t + t_0, u) \approx \sum_{j=1}^{\infty} e^{\mu_j t} W_{\mu_j}(t + t_0, u) \quad (3.4)$$

Hence that

$$S(t + t_0, u) \sim \sum_{j=1}^{\infty} e^{\mu_j t} W_{\mu_j}(t, S(t_0)u) \quad (3.5)$$

By the uniqueness of the expansion (2.19), it follows that

$$W_{\mu_j}(t, S(t_0)u) = e^{-\mu_j t_0} W_{\mu_j}(t + t_0, u) \quad (3.6)$$

Which for $t = 0$, we obtain

$$W_{\mu_j}(0, S(t_0)u) = e^{-\mu_j t_0} W_{\mu_j}(t, u) \quad (3.7)$$

So from theorem 2.1, we have

$$W_{\mu_j}(t, u) = e^{-\mu_j t} P_j(W_1(S(t)u), \dots, W_{k_j}(S(t)u)) \quad (3.8)$$

We deduce then from (2.23) and (3.8), the formula

$$S(t)u \sim \sum_{j=1}^{\infty} e^{-\mu_j t} W_{\mu_j}(t, u) = \sum_{j=1}^{\infty} P_j \left(\begin{matrix} W_1(S(t)u) \\ \dots, \\ W_{k_j}(S(t)u) \end{matrix} \right) \quad (3.9)$$

The next proposition gives further properties of the P_j .

Proposition 3.2:

- If $\mu_j \in \mathbb{S}$ is the eigenvalue Λ_{k_j} , then

$$P_j(X_1, X_2, \dots, X_{k_j}) = X_{k_j} + Q_{k_j}(X_1, X_2, \dots, X_{k_{j-1}}) \quad (3.10)$$

where

$$d^0 Q_{k_j} \geq 2 \text{ and } R_k Q_{k_j}(X_1, X_2, \dots, X_{k_{j-1}}) = 0 \quad (3.11)$$

- For $u \in V$, we denote $X_k(t) = W_k(u(t))$, then for every j

$$\sum_{m=2}^{k_j} (D_m P_j) F_m = (A - \mu_j) P_j + \sum_{\mu_{\alpha_1} + \dots + \mu_{\alpha_0} = \mu_j} P_{\alpha_1} \dots P_{\alpha_0} \quad (3.12)$$

where D_m is the derivative with respect to the m^{th} component.

Remark 3.1: In (3.12) the polynomials are evaluated at

$$X_1(t), \dots, X_{k_j}(t),$$

but the formula is valid for arbitrary

$$X_1, \dots, X_{k_j}$$

since $W(0) = 1$.

Proof: Part i) results from the theorem 1.1 and ^[1]. Part ii) from (2.18) and (3.9), we derive for $\mu_j \in \mathbb{S}$, the Eq

$$\begin{aligned} & \frac{d}{dt} \left(e^{\mu_j t} P_j \left(\begin{matrix} X_1, X_2, \\ \dots, X_{k_j} \end{matrix} \right) \right) + (A - \mu_j) e^{\mu_j t} P_j \left(\begin{matrix} X_1, X_2, \\ \dots, X_{k_j} \end{matrix} \right) \\ & + \sum_{\mu_{\alpha_1} + \dots + \mu_{\alpha_0} = \mu_j} e^{\mu_j t} P_{\alpha_1}(X_1, \dots, X_{\alpha_1}) \dots P_{\alpha_0}(X_1, \dots, X_{\alpha_0}) \end{aligned} \quad (3.13)$$

From theorem 2.1, the Eq. (3.13) can be rewritten as

$$\begin{aligned} & \frac{d}{dt} P_j \left(\begin{matrix} e^{\Lambda_1 t} X_1, \dots \\ e^{\Lambda_{k_j} t} X_{k_j} \end{matrix} \right) + (A - \mu_j) P_j \left(\begin{matrix} e^{\Lambda_1 t} X_1, \\ \dots, e^{\Lambda_{k_j} t} X_{k_j} \end{matrix} \right) \\ & + \sum_{\mu_{\alpha_1} + \dots + \mu_{\alpha_0} = \mu_j} P_{\alpha_1} \left(\begin{matrix} e^{\Lambda_1 t} X_1, \\ \dots, e^{\Lambda_{\alpha_1} t} X_{\alpha_1} \end{matrix} \right) \dots P_{\alpha_0} \left(\begin{matrix} e^{\Lambda_1 t} X_1, \\ \dots, e^{\Lambda_{\alpha_0} t} X_{\alpha_0} \end{matrix} \right) \end{aligned} \quad (3.14)$$

Differentiation, $e^{\Lambda_k t} X_k(t)$, $k = 1, 2, \dots$, with respect to t , we get

$$\frac{d}{dt} (e^{\Lambda_k t} X_k(t)) = \Lambda_k e^{\Lambda_k t} X_k(t) + e^{\Lambda_k t} X_k(t) \quad k = 1, 2, \dots \quad (3.15)$$

and from (2.17) we have

$$\frac{dX_k}{dt} = \dot{X}_k(t) = -\Lambda_k X_k - F_k(X(t)) \quad (3.16)$$

then

$$\frac{d}{dt} (e^{\Lambda_k t} X_k(t)) = -\dot{X}_k(t) = -e^{\Lambda_k t} F_k \quad (3.17)$$

Coming back to (3.14), we deduce then

$$\sum_{m=1}^{k_j} (D_m P_j) F_m = (A - \mu_j) P_j + \sum_{\mu_{\alpha_1} + \dots + \mu_{\alpha_0} = \mu_j} P_{\alpha_1} \dots P_{\alpha_0} \quad (3.18)$$

where the summation starts at $m = 2$ since $F_1 = 0$.

CONSTRUCTION OF THE P_j

We denote by

$$R_{\mu_j} = R_k$$

if $k = k_j$ and

$$R_{\mu_j} = 0$$

otherwise.

In the other hand we set

$$Q_{\mu_j} = I - R_{\mu_j}$$

and setting

$${}^0X_1 \in R_1H, {}^0X_2 \in R_2H, \dots$$

The following theorem gives the main result concerning the construction of the P_j.

Theorem 4.1: We define a sequence of polynomials q_j as follows

$$q_1(t) = {}^0X_1 \tag{4.1}$$

and by induction

$$\left\{ \begin{aligned} q_j(t) &= -\sum_{n \geq 0} (1-t)^n \left[(A - \mu_j) \right]^{-n-1} \left(\frac{d}{dt} \right)^n \\ &\quad \sum_{\mu_{\alpha_1} + \dots + \mu_{\alpha_n} = \mu_j} q_{\alpha_1} \dots q_{\alpha_n} \\ &\quad \text{if } \mu_j \notin \sigma(A) \end{aligned} \right. \tag{4.2}$$

or,

$$\left\{ \begin{aligned} q_j(t) &= {}^0X_{k_j} + \int_0^t \sum_{\mu_{\alpha_1} + \dots + \mu_{\alpha_n} = \mu_j} q_{\alpha_1} \dots q_{\alpha_n} R_{k_j} \left[\begin{matrix} q_{\alpha_1}(\tau) + \\ \dots + q_{\alpha_n}(\tau) \end{matrix} \right] \\ &+ \sum_{n \geq 0} (-1)^n (A - \mu_j) (I - R_{k_j})^{-n-1} \left(\frac{d}{dt} \right)^n \times \\ &\quad \sum_{\mu_{\alpha_1} + \dots + \mu_{\alpha_n} = \mu_j} (I - R_{k_j}) (q_{\alpha_1} \dots q_{\alpha_n}) \quad \text{if } \mu_j = \mu_{k_j} \end{aligned} \right. \tag{4.3}$$

then

$$q_j(t) = P_{j,0} \left(\begin{matrix} 0 \\ X_1, \dots, X_{k_j} \end{matrix} \right) + t P_{j,1} \left(\begin{matrix} 0 \\ X_1, \dots, X_{k_j} \end{matrix} \right) + t^{j-1} P_{j,j-1} \left(\begin{matrix} 0 \\ X_1, \dots, X_{k_j} \end{matrix} \right) \tag{4.4}$$

where the P_{j,i} are polynomials maps in

$$\begin{matrix} 0 \\ X_1, \dots, X_{k_j} \end{matrix}$$

and

$$P_j \left(\begin{matrix} 0 \\ X_1, \dots, X_{k_j} \end{matrix} \right) = P_{j,0} \left(\begin{matrix} 0 \\ X_1, \dots, X_{k_j} \end{matrix} \right)$$

Proof: To prove theorem 4.1, we have need the following algebraic lemma.

Lemma 4.1: Let L an inversible (not necessarily bounded) linear operator in some Hilbert space H. Then for any H-valued polynomial r(t). The equation

$$q' = Lq + r \tag{4.5}$$

has a unique polynomial solution given by

$$q(t) = \sum_{k \geq 0} (-1)^k L^{-k} r^{(k)}(t) \tag{4.6}$$

Proof of lemma 4.1: We deduce after an immediate computation that (4.6) is a solution of the Eq. (4.5).

To proof the uniqueness of q, we consider q₁ and q₂ two polynomial solutions of (4.5), the difference q = q₁ - q₂ is solution of equation

$$q' = Lq \tag{4.7}$$

Since q is a polynomial, then q⁽ⁿ⁾ = 0 for n > d⁰q which with (4.7) yields 0 = Lq⁽ⁿ⁻¹⁾ and since L is inversible, we obtain q⁽ⁿ⁻¹⁾ = 0.

Inductively, we have q = 0, so q₁ = q₂ and we have the uniqueness of q.

Let us take back the proof of theorem 4.1 and one assumes that $\mu_j \notin \sigma(A)$.

Let us consider the equation when q_j is a polynomial of t.

$$q'_j(t) + (A - \mu_j)q_j(t) + \sum_{\mu_{\alpha_1} + \dots + \mu_{\alpha_n} = \mu_j} q_{\alpha_1}(t) \dots q_{\alpha_n}(t) = 0 \tag{4.8}$$

$$q_j(0) = P_j \left(\begin{matrix} 0 \\ X_1, \dots, X_{k_j} \end{matrix} \right)$$

We remark that, the Eq. (4.8) is exactly of type (4.5) where $-L = A - \mu_j$ and the second member

$$b = \sum_{\mu_{\alpha_1} + \dots + \mu_{\alpha_0} = \mu_j} q_{\alpha_1} \dots q_{\alpha_0}.$$

It follows from the lemma 4.1 that

$$q_j(t) = - \sum_{n \geq 0} (1-t)^n \left[(A - \mu_j) \right]^{n-1} \frac{d^n}{dt^n} \sum_{\mu_{\alpha_1} + \dots + \mu_{\alpha_0} = \mu_j} q_{\alpha_1} \dots q_{\alpha_0} \quad (4.9)$$

and of course

$$q_j(t) = P_{j0} \left(\begin{matrix} 0 \\ X_1, \dots, X_{k_j} \end{matrix} \right) + t P_{j,1} \left(\begin{matrix} 0 \\ X_1, \dots, X_{k_j} \end{matrix} \right) + t^{j-1} P_{j,j-1} \left(\begin{matrix} 0 \\ X_1, \dots, X_{k_j} \end{matrix} \right) \quad (4.10)$$

To prove (4.3), we assume that

$$\mu_j = \Lambda_{k_j} \in \sigma(A)$$

and we set

$$q_j(t) = \beta_j(t) + Q_{\Lambda_{k_j}} q_j(t) \quad (4.11)$$

where

$$\beta_j(t) = R_{\Lambda_{k_j}} q_j(t) \in R_{\Lambda_{k_j}} H$$

and

$$Q_{\Lambda_{k_j}} = (I - R_{\Lambda_{k_j}}).$$

Coming back to (4.8) and using (4.11), we obtain

$$\beta_j'(t) + \sum_{\mu_{\alpha_1} + \dots + \mu_{\alpha_0} = \mu_j} R_{k_j} [q_{\alpha_1} \dots q_{\alpha_0}] = 0 \quad (4.12)$$

and

$$Q_{\Lambda_{k_j}} q_j(t) + (A - \mu_j) (Q_{\Lambda_{k_j}} q_j) + \sum Q_{\Lambda_{k_j}} [q_{\alpha_1} \dots q_{\alpha_0}] = 0 \quad (4.13)$$

The eq. (4.12) will be integrated

$$\beta_j(t) = \beta_j(0) + \int_0^t \sum_{\mu_{\alpha_1} + \dots + \mu_{\alpha_0} = \mu_j} R_{\Lambda_{k_j}} [q_{\alpha_1}(\tau) \dots q_{\alpha_0}(\tau)] d\tau \quad (4.14)$$

and it follows that

$$R_{\Lambda_{k_j}} q_j(t) = \overset{0}{X}_{k_j} + \int_0^t \sum_{\mu_{\alpha_1} + \dots + \mu_{\alpha_0} = \mu_j} R_{\Lambda_{k_j}} [q_{\alpha_1}(\tau) \dots q_{\alpha_0}(\tau)] d\tau \quad (4.15)$$

The Eq. (4.13) is exactly of type (4.5), or its results from lemma 4.1 that

$$(I - R_{\Lambda_{k_j}}) q_j = - \sum_{n \geq 0} (1-t)^n \left[(I - R_{\Lambda_{k_j}}) \right]^{n-1} \left(\frac{d}{dt} \right)^n \times \sum_{\mu_{\alpha_1} + \dots + \mu_{\alpha_0} = \mu_j} (I - R_{\Lambda_{k_j}}) (q_{\alpha_1} \dots q_{\alpha_0}) \quad (4.16)$$

Finally, we deduce (4.2) and (4.3) from (4.15) and (4.16).

Hence the theorem 4.1 is competing.

Corollary 4.1: The polynomials P_j are unique.

CHARACTERIZATION OF THE EXPANSION IN TERMS OF ITS COORDINATES

We recall that, a previous work, we constructed a set of analytic nonlinear manifolds.

$$M_k = \{u \in V, W_1(u) = \dots = W_{k-1}(u) = 0\} \quad (5.1)$$

that are invariant under $S(t)$, of codimension $m_1 + \dots + m_k$ and we have the condition

$$S(t) = O(e^{-\Lambda_k t}) \quad (5.2)$$

Also setting $M_0 = V$, u satisfies

$$u \in M_{k-1} \setminus M_k \Leftrightarrow \begin{cases} |S(t)u| = O(e^{-\Lambda_k t}) \\ \text{and} \\ e^{-\Lambda_k t} = O(S(t)u) \end{cases} \quad (5.3)$$

Consequently, if $u \in M_{k-1}$, then the expansion (3.1) takes the form

$$S(T)u \sim \sum_{k_j \geq k} P_j(0, \dots, 0, W_k(S(t)u), \dots, W_{k_j}(S(t)u)) \quad (5.4)$$

However, for N fixed and $K_N \geq k$, let d_N denoted the degree of the polynomial

$$\sum_{\substack{k_j \geq k \\ j \leq N}} P_j(0, \dots, W_k(v), \dots, W_{k_j}(v)) \quad (5.5)$$

then it follows from ii) of theorem 2.1 that

$$d_N \leq \frac{\mu_N}{\Lambda_k}$$

so if $v \in M_{k-1} \setminus M_k$, then we can write

$$\left| v - \sum_{\substack{k_j \geq k \\ j \leq N}} P_j(0, \dots, W_k(v), \dots, W_{k_j}(v)) \right| = O\left(|v|^{d_N + \frac{\mu_N}{\Lambda_k}}\right) \quad (5.6)$$

where ϵ_N is as theorem 2.2 i). In particular if $v \in M_0 \setminus M_1$ then

$$\left| v - \sum_{j=1}^N P_j(W_k(v), \dots, W_{k_j}(v)) \right| = O\left(|v|^{d_N + \frac{\mu_N}{\Lambda_1}}\right) \quad (5.7)$$

where d_N is the degree of the polynomial

$$\sum_{j=1}^N P_j \text{ in } W_1(v), \dots, W_{k_N}(v)$$

Hence, the expansion (5.4) can be viewed as an asymptotic expansion of $v = S(t)u$ in terms of its coordinates $W_k(v), W_{k+1}(v), \dots$.

Finally, notice that the asymptotic expansion (5.6), (5.7) hold the trajectories but we don't know if they are global or not.

Remark: Present research extends to the equation with second member such that

$$u_t + Au + f(u) = g(x)$$

where $g(x)$ is independent of time, for example:

$$\begin{cases} \frac{d\omega}{dt} - \Delta\omega + 3u_\infty^2\omega + 3u_\infty\omega^2 + \omega^3 = 0 & \Omega \times]0, T[\\ \omega(0) = \omega_0 & \Omega; \quad \omega = 0 \quad \partial\Omega \times]0, T[\end{cases}$$

where $\omega = u - u_\infty$ with u and u_∞ are, respectively solution of equations

$$\begin{cases} u_t - \Delta u + u^3 = g(x) & \Omega \times]0, T[\\ u(0) = u_0 & \Omega; \quad u = 0 \quad \partial\Omega \times]0, T[\end{cases}$$

and

$$\begin{cases} -\Delta u_\infty + u_\infty^3 = g & \Omega \\ u_\infty = 0 & \partial\Omega \end{cases}$$

We obtain the problem (1.1) with:

$$\begin{cases} Au = -\Delta u + 3u_\infty^2 u \\ f(u) = 3u_\infty u^2 + u^3 \end{cases}$$

REFERENCES

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