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The Effects of Linearly Varying Distributed Moving Loads on Beams

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Abstract: The dynamic behaviour of a Bernoulli-Euler Beam traversed by a linearly varying distributed moving load is investigated. Using a series solution for the dynamic deflection in terms of normal modes, the equation governing the model is reduced to a set of ordinary differential equations whose solution is obtained in form of a Duhamel integral. Several numerical results are presented to show the effects of linearly varying distributed moving load on the dynamic behaviour of the beam. Important conclusions are drawn for structural design purposes.

Key words: Linearly varying distributed, beam, dynamic behaviour, structural design, moving load

INTRODUCTION

Several investigations have been carried out on the problems of the dynamic response of structures to moving loads (Fryba, 1972; Kopmaz and Telli, 2002; Gbadeyan and Dada, 2001, 2006; Sadiku and Leipholz, 1987; Mohmoud and Abouzaid, 2002; Esmailzadeh and Gborashi, 1995; Dugush and Eisenberger, 2002; Adetunde *et al.*, 2007; Michactsos and Kounads, 2001; Michaltsos, 2002). Such studies are of importance in the field of transportation and in designing space station facilities and machine parts. In fact, a moving load induces larger deflections and stresses on the structure on which it moves than does an equivalent static load. As there are several different structures on which loads move so are many types of load. Consequently, moving load problems did and still continues to draw attention of researchers (Fryba, 1972; Kopmaz and Telli, 2002; Gbadeyan and Dada, 2001, 2006; Sadiku and Leipholz, 1987; Mohmoud and Abouzaid, 2002; Esmailzadeh and Gborashi, 1995; Dugush and Eisenberger, 2002; Adetunde *et al.*, 2007; Michactsos and Kounads, 2001; Michaltsos, 2002). in the field of engineering, applied mathematics and physics.

For simplicity, moving loads can be assumed or approximated to be concentrated otherwise they are distributed. The problems that involved the former, that is concentrated moving load problems, have been the most common subject of investigation among researchers. This is perhaps due to the fact that it is a simplified formulation for moving load problems. On the other hand, a more realistic approach is to assume that the load is distributed over a length or contact area as it moves. In the case of a distributed load, a load that is distributed with constant magnitude is referred to as uniformly distributed while load's distribution of the form $C_1 + C_2.X$, where C_1 and C_2 are constants and X is a variable, is said to be linearly distributed. While there are a very limited number of publications on beams with distributed loads, most of these few publications focused on uniformly distributed problems (Gbadeyan and Dada, 2001, 2006; Esmailzadeh and Gborashi, 1995; Adetunde *et al.*, 2007). The idea behind assuming a uniform distribution is encouraging in that it results in a considerable simplification of moving load's distribution problems. However, in the practical sense, in the area of road transports, designing of machine parts and aerospace engineering, uniform distribution is just a specific case and the simplest. Hence, it is more practically useful to consider the load as linearly distributed as opposed to a uniformly distributed.

In this context, this study focuses on the effects of a linearly varying distributed moving load on the deflection of a beam. Numerical example involving a simply supported beam is presented.

MATHEMATICAL FORMATION AND SIMPLICATION OF THE GOVERING EQUATION

The vibration of a beam as described by Bernoulli-Euler's differential equation, based on the assumption that the theory of small deformations, Hooke's law, Navier's hypothesis and saint-venants's principle, is being applied. Further assumptions are as follows: The beam is of constant cross-section and constant mass per unit length, the moving mass moves at constant speed from left to right and the beam damping is proportional to the velocity of the vibration.

Under the above assumptions, the governing equation of motion is described by the following equation (Fryba, 1972).

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$$El\frac{\partial^4 w(x,t)}{\partial x^4}+\mu\frac{\partial^2 w(x,t)}{\partial t^2}+2\mu\omega_b\frac{\partial w(x,t)}{\partial t}=q(x,t) \quad (1)$$

Where
x is the length coordinate with the origin at the left-hand end of the beam,
t is the time coordinate with the origin at the instant of the force arriving on the beam,
w (x,t) is the beam deflection at point x and time t, measured from the equilibrium position when the beam is loaded with its own weight,
E is young's modulus of the beam,
I is the constant moment of intertia of the beam cross section,
μ is the constant mass per unit length of the beam,
ω_b is the circular frequency of the damping of the beam
q(x, t) is the applied force

The linearly varying distributed moving applied force is described by the expression (Pilkey and Pilkey, 1974)

$$q(x,t)=\{w_1<x-\alpha_1>^0+\frac{w_2-w_1}{d}<x-\alpha_1> \\ -w_2<x-\alpha_2>^0-\frac{w_2-w_1}{d}<x-\alpha_2>\} \quad (2)$$

Where $w_1 = M_1g$ and $w_2 = M_2g$ are the forces produced by masses M_1 and M_2, respectively at the end points of the load as shown in Fig. 1, $d = a_2 - a_1$ is the length of the load, $a_1 = Vt\text{-}d/_2$,$a_2 = Vt+d/_2$, v is the velocity of load, g is the acceleration due to gravity and the Macaulay notation is defined as

$$<x-\alpha>^n=\begin{cases}0 & , x\geq\alpha_2\\(x-\alpha)^n & , x\geq\alpha_2\end{cases} \quad (3)$$

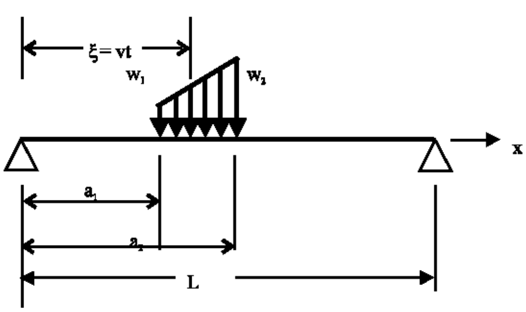

Fig. 1: A beam of span L under a linearly varying distributed load

A series solution of Eq. 1 in terms of the normal modes will be sought in the form $w(x,t)=\sum_{n=1}^{N}x_n(X)t_n(t)$ and in the absence of damping (Neglecting the damping term), Eq. 1 becomes

$$El\sum_{n=1}^{N}X_n^{1111}(x)t_n(x)+\mu\sum_{n=1}^{N}X_n(x)\ddot{T}n(t)=w_1<x-\alpha_1>^0 \\ +\frac{w_2-w_1}{d}<x-\alpha_1>-w_2<x-\alpha_2>0\frac{w_2-w_1}{d}<x-\alpha_2> \quad (4)$$

The n-th normal mode of vibration of a uniform beam satisfies

$$X_n(x)=A_n,Sin\frac{\lambda_n x}{L}+B_nCos\frac{\lambda_n x}{L}C_nSinh\frac{\lambda_n x}{L}+ \\ D_nCosh\frac{\lambda_n x}{L} \quad (5)$$

where λ_n, A_n, B_n, C_n and D_n are constants whose values are obtained by applyingthe boundary conditions of the beam.

The natural modes defined by Eq. 5 satisfy the homogeneous differential equation

$$El\ X_n^{1111}(x)-\mu\omega_n^2X_n(x)=0 \quad (6)$$

Where the natural circular frequencies $\omega_n^2=\frac{\lambda_n^4 El}{\mu L^4}$

Equation 6 may be expressed as

$$ElX_n^{1111}(x)T_n(t)=\mu\omega^2X_n(x)T_n(t) \quad (7)$$

Multiplying both sides of Eq. 4 by X_k (x), making use of Eq. 7 and then integrating it along the entire length of the beam, we have.

$$\ddot{T}_n(t)+\omega_n^2T_n(t)=\frac{1}{\alpha\mu}\left[w1\int_{\alpha_1}^{L}X_k(x)dx+\frac{w_2-w_1}{d}\right. \\ \int_{\alpha_1}^{L}X_k(x)(x-\alpha_1)dx-w_2\int_{\alpha_1}^{L}X_k(x)dx-\frac{w_2-w_1}{d} \\ \int_{\alpha_1}^{L}X_k(x)dx-\frac{w_2-w_1}{d}\int_{\alpha_1}^{L}X_k(x)(x-\alpha_1)dx \quad (8)$$

Where

$\int_{\alpha_1}^{L}X_n^{(x)}X_n^{(x)}dx=\begin{cases}0 & ,n\neq k\\ \alpha & ,n=k\end{cases}$ and α is a constant.

At this juncture, we remark that for simply supported boundary conditions, we have w(0, t) = w(L, t) = $w^3(0, t)$ = w^3 (L, t) = 0, while the initial conditions are w(x,0) = w(x,0) = 0

Applying these conditions, we have

$$X_n(x) = \text{Sin}\frac{n\pi x}{L}, \alpha = \frac{1}{2} \text{ and } \omega_n^2 = \frac{n^4\pi^4 El}{\mu L^4} \quad (9)$$

Equation 8 can be evaluated by substituting the set of Eq. 9 into it and we obtained

$$\ddot{T}_n(t) + \omega_n^2(t) = P(t)$$

Where

$$P(t) = \frac{2}{\mu L}\left[(w_2 - w_1)\frac{L}{n\pi}(-1)^n + \frac{L}{n\pi}\right.$$
$$\left\{w_1 \text{Cos}\frac{n\pi}{L}(vt - d/2) - w_1 \text{Cos}\frac{n\pi}{L}(vt + d/2)\right\} +$$
$$\left[\frac{w_2 - w_1}{n\pi}\right]L(-1)^n + \left[\frac{w_2 - w_1}{d}\right] \quad (10)$$
$$\left.\frac{L^2}{n^2\pi^2}\left\{\text{Sin}\frac{n\pi}{L}(vt + d/2) - \text{Sin}\frac{n\pi}{L}(vt - d/2)\right\}\right]$$

The solution of Eq. 10 are given in terms of Duhamel's integral as

$$T_n(t) = \frac{1}{\omega_n}\int_0^1 Pn(\tau)\sin\omega n(t-\tau)d\tau \quad (11)$$

Equation 11 may be written as

$$T_n(t) = \frac{2}{\omega_n \mu L}\left[(w_2 - w_1)\frac{L}{n\pi}(-1)^n \int_0 \text{Sin}\omega_n(t-\tau)\right.$$
$$+\frac{L}{n\tau}\left\{\begin{matrix} w_1 \int_0 \text{Cos}\frac{n\pi}{L}(v\tau - d/2)\text{Sin}\omega_n(t-\tau)d\tau - w_2 \\ \int_0 \text{Cos}\frac{n\pi}{L}(v\tau + d/2)\text{Sin}\omega_n(t-\tau)d\tau \end{matrix}\right\}$$
$$+\left(\frac{w_1 - w_2}{n\tau}\right)L(-1)^n \int_0 \text{Sin}\omega(t-\tau)d\tau + \left(\frac{w_2 - w_1}{d}\right) \quad (12)$$
$$\frac{L^2}{n^2\pi^2}\left\{\int_0 \text{Sin}\frac{n\tau}{L}(v\tau + d/2)\text{Sin}\omega_n(t-\tau)\right.$$
$$\left.\left. - \int_0 \text{Sin}\frac{n\pi}{L}(v\tau - d/2)\text{Sin}\omega_n(t-\tau)d\tau\right\}\right]$$

Simplification of Eq. 12 gives

$$T_n(t) = \frac{2}{\omega_n \mu L}\left[\frac{L}{2n\pi\left(\omega_n + \frac{n\pi v}{L}\right)}\left\{w_1\left\{\text{Cos}\frac{n\pi}{L}(vt + d/2) - \text{Cos}\left(\omega_n t - \frac{n\pi d}{2L}\right)\right\}\right.\right.$$
$$-w_2\left\{\text{Cos}\frac{n\pi}{L}(vt - d/2) - \text{Cos}\left(\omega_n t + \frac{n\pi t}{2L}\right)\right\} + \left(\frac{w_1 - w_2}{d}\right)\frac{L}{n\pi}\left\{\text{Sin}\frac{n\pi}{L}(vt + d/2)\right.$$
$$\left.\left. - \text{Sin}\left(\frac{n\pi d}{2L} - \omega_n t\right) - \text{Sin}\frac{n\pi}{L}(vt - d/2) - \text{Sin}\left(-\frac{n\pi d}{2L} - \omega_n t\right)\right\}\right\} \quad (13)$$
$$+\frac{L}{2n\pi\left(\frac{n\pi v}{L} - \omega_n\right)}\left\{w_1\left\{\text{Cos}\left(\omega_n t + \frac{n\pi d}{2L}\right) - \text{Cos}\frac{n\pi}{L}(vt + d/2)\right\}\right.$$
$$-w_2\left\{\text{Cos}\left(\omega_n t - \frac{n\pi d}{2L}\right) - \text{Cos}\frac{n\pi}{L}(vt - d/2)\right\} + \left(\frac{w_2 - w_1}{d}\right)\frac{L}{n\pi}$$
$$\left.\left.\left\{\text{Sin}\left(\frac{n\pi d}{2L} + \omega_n t\right) - \text{Sin}\frac{n\pi}{L}(vt + d/2) + \text{Sin}\left(-\frac{n\pi d}{2L} + \omega_n t\right) - \text{Sin}\frac{n\pi}{L}(vt - d/2)\right\}\right\}\right]$$

Using the non-dimensional quantities

$$\bar{t}\ \frac{t}{t_1},\ \bar{d} = \frac{d}{L},\ \bar{x} = \frac{x}{L},\ \bar{v} = \frac{ct_1}{L},\ \bar{g} = \frac{gt_1^2}{L},$$

$$\bar{w}_1 = \frac{w_1}{\left(\mu L / t_1^2\right)},\ \bar{w}_2 = \frac{w_2}{\left(\mu L / t_1^2\right)} \text{ and } \omega_n = \omega_n t_1$$

Where

$$t_1 = L^2\left(\frac{\mu}{EI}\right)^{\frac{1}{2}}$$

The non-dimensional displacement is $\bar{w}(\bar{x}, \bar{t}) = \frac{X(x)T(t)}{(gt_1)^2}$

$$= \frac{Sin(n\pi\bar{x})}{\pi n\bar{\omega}_n}\left[\frac{1}{(\bar{\omega}_n + n\pi\bar{v})}\right]\left\{ \bar{w}_1\left\{\{Cos\, n\pi(\bar{v}\bar{t}+\frac{\bar{d}}{2}) - Cos\left(\bar{\omega}_n t - \frac{n\pi\bar{d}}{2L}\right)\right\}\right.$$
$$-\bar{\omega}_2\left\{Cos\frac{n\pi}{L}(\bar{v}\bar{t} - d/2) - Cos\left(\bar{\omega}_n t + \frac{n\pi\bar{d}}{L}\right)\right\} + \left(\frac{\bar{w}_2 - \bar{w}_1}{n\pi\bar{d}}\right)\left\{Sin\, n\pi(\bar{v}\bar{t} + \frac{\bar{d}}{2})\right.$$
$$\left.\left. - Sin\left(\frac{n\pi\bar{d}}{2L} - \bar{\omega}_n\bar{t}\right) - Sin\, n\pi(\bar{v}\bar{t} - \bar{d}/2) - Sin\left([-\frac{n\pi\bar{d}}{2L} - \bar{\omega}_n t\right)\right\}\right\} + \frac{1}{(n\pi\bar{v} - \bar{\omega})} \tag{14}$$
$$\left\{\bar{w}_1 Cos\left(\bar{\omega}_n\bar{t} + \frac{n\pi\bar{d}}{2}\right) - Cos\, n\pi(\bar{v}\bar{t} - \bar{d}/2)\right\} - \bar{w}_2\left\{Cos\left(\bar{\omega}_n\bar{t} - \frac{n\pi\bar{d}}{2}\right) - Cos\, n\pi(\bar{v}\bar{t} - d/2)\right\}$$
$$\left.\left. + \left(\frac{\bar{w}_2 - \bar{w}_1}{n\pi d}\right)\left\{Sin\left(\frac{n\pi\bar{d}}{2} + \bar{\omega}_n\bar{t}\right) - Sin\, n\pi(\bar{v}\bar{t} + \bar{d}/2) + Sin\left(\frac{-n\pi\bar{d}}{2} + \bar{\omega}_n\bar{t}\right) - Sin\, n\pi(\bar{v}\bar{t} - \bar{d}/2)\right\}\right\}\right]$$

The applied force for the concentrated moving load is described by

$$q_i(x, t) = Mg\delta(x - a_1)$$

Where

$\delta(x - a_1)$ is the Dirac-delta function.

Following the same procedure, the non-dimensional displacement forconcentrated moving force is $w_i(\bar{x}, \bar{t})$

$$w_i(\bar{x}, \bar{t}) = Sin(n\pi\bar{x}\left\{\frac{sin(n\pi\bar{v}\bar{t}) + sin\bar{\omega}_n t}{(n\pi\bar{v} + \bar{\omega}_n)} + \frac{sin\bar{\omega}_n t - Sin(n\pi\bar{v}\bar{t})}{(n\pi\bar{v} - \bar{\omega}_n)}\right\} \tag{15}$$

Where $\bar{M}$ is the dimensionless mass of the concentrated load.

RESULTS AND DISCUSSION

In this study, numerical results are presented in both graphical and tabular forms. The effects of linearly varying distributed loads moving with constant velocity are discussed. To illustrate these effects, the model considered is a beam of length L = 12 m, mass per unit length μ = 45 kg m^{-1} and EI = 2587 Nm^{-2} which is traversed by a moving load of weight with velocities 3, 4 and 5 m sec^{-1}.

The influence of a moving load's distribution on a beam's deflection for d = 0.1, or $\bar{d}$ = 0.0083 with various dimensionless values of mass $\bar{M}$ = 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4 and 1.6 are shown in Table 1. For various values of distribution's slopes G = 0, 5, 10, 15, 20 and 25 considered, the results show that the amplitude of the deflection $w_m = 1000\ \bar{w}$ increases with an increase in the value of the slope G. It can be noticed from the table that as the mass M increases, the percentage R = 100 $(w_m-w_u)/w_u$ decreases persistently where w_u is the dimensionless deflection for G = 0. The percentage comparisons between a beam's deflection w_u produced by a uniformly distributed mass and the other that is produced by a linearly distribution mass shows that percentage increase R decreases with an increase in dimensionless mass M for the same value of the slope.

Table 1: Percentage comparison of dimensionless mid -span deflections between uniformly distributed and linearly varying loads for various gradients

		G = 0	G = 05	G = 10	G = 15	G = 20	G = 25
$\bar{M}$ = 0.2	W_M	-0.5901	-0.5671	-0.5494	-0.5362	-0.5253	-0.5160
	R%	0	3.9038	6.8955	9.1421	10.9817	12.5694
$\bar{M}$ = 0.4	W_M	-1.1803	-1.1549	-1.1342	-1.1165	-1.0989	-1.0832
	R%	0	2.1480	3.9038	5.3997	6.8955	8.2223
$\bar{M}$ = 0.6	W_M	-1.7704	-1.7450	-1.7197	-1.7013	-1.6836	-1.6660
	R%	0	1.4320	2.8640	3.9038	4.9010	5.8983
$\bar{M}$ = 0.8	W_M	-2.3605	-2.3352	-2.3098	-2.2860	-2.2684	-2.2507
	R%	0	1.0740	2.1480	3.1558	3.9038	4.6517
$\bar{M}$ = 1.0	W_M	-2.9506	-2.9253	-2.8999	-2.8746	-2.8531	-2.8354
	R%	0	0.8592	1.7184	2.5776	3.3054	3.9038
$\bar{M}$ = 1.2	W_M	-3.5408	-3.5154	-3.4901	-3.4647	-3.4394	-3.4202
	R%	0	0.7160	1.4320	2.1480	2.8640	3.4052
$\bar{M}$ = 1.4	W_M	-4.1309	-4.1055	-4.0802	-4.0548	-4.0295	-4.0049
	R%	0	0.6137	1.2274	1.8411	2.4549	3.0490
$\bar{M}$ = 1.6	W_M	-4.7210	-4.6957	-4.6703	-4.6450	-4.6196	-4.5943
	R%	0	0.5370	1.0740	1.6110	2.1480	2.6850

In Fig. 2, the dimensionless time history of the load on the beam versus dimensionless mid-span deflections for positive slopes 0,10,and 20 are graphically shown. It can be clearly seen from this figure that the greater the slope of the moving load, the higher the amplitude of the

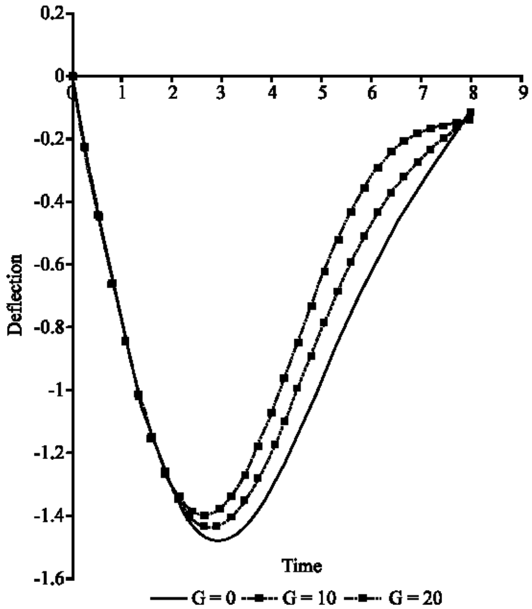

Fig. 2: Dimensionless time history of mid-span dimensionless deflections of the beam for gradients and fixed M = 0.5, $\bar{d}$ = 0.0083 and = $\bar{v}$ 0.1251

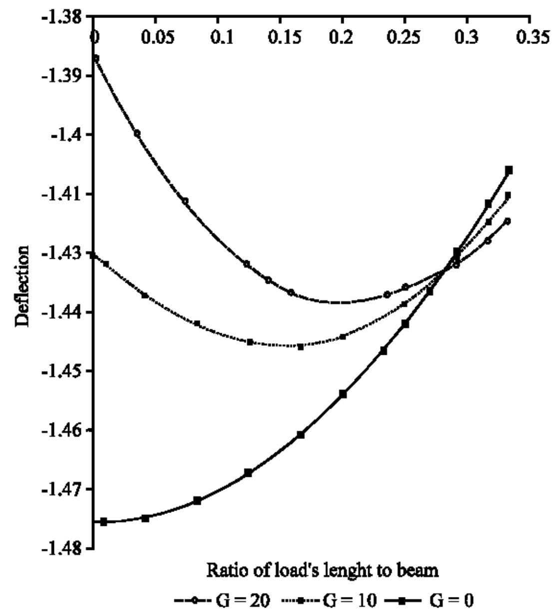

Fig. 4: The effects of ratio $\frac{\bar{d}}{\bar{L}}$ on the dimensionless mid-span deflection for some gradients and fixed M = 0.5 , = $\bar{d}$ 0.0083 and = $\bar{v}$ 0.1251

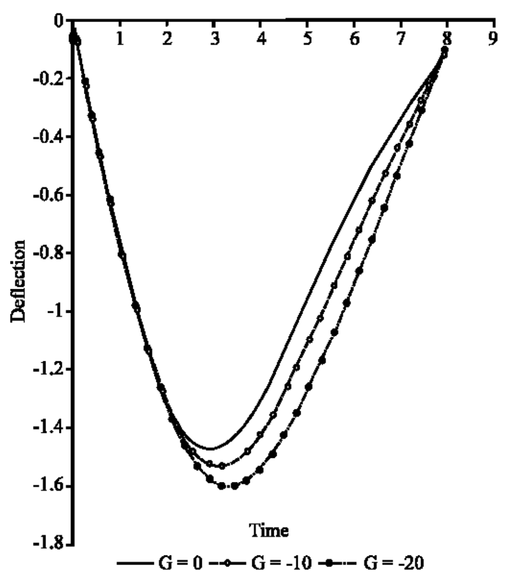

Fig. 3: The dimensionless deflection-time history response at mid-span for some negative gradients and fixed M = 0.5, = $\bar{d}$ 0.0083 and = $\bar{v}$ 0.1251d

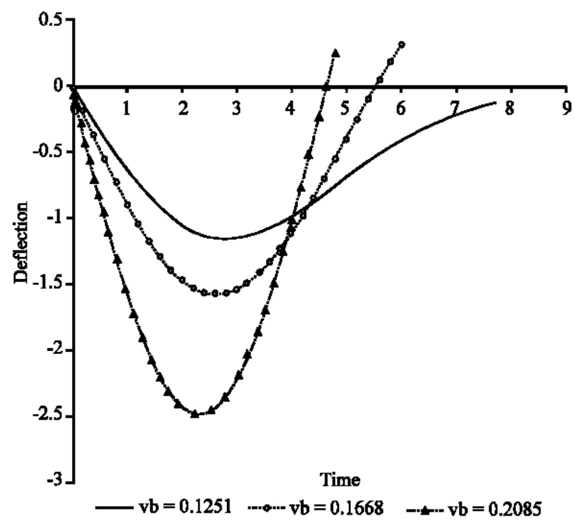

Fig. 5: Dimensionless time history of mid-span dimensionless deflections of the beam for velocities and fixed M = 0.4, = $\bar{d}$ 0.0417 and G = 5

dimensionless deflection w_u. The maximum amplitude of the deflection for slope G = 0, 10 and 20 occurred when the load are linearly distributed on (0.3616≤ $\bar{x}$ ≤0.3699), (0.3283≤ $\bar{x}$ ≤0.3366)and(0.3283≤ $\bar{x}$ ≤0.3366),respectively.

Figure 3 shows the dimensionless deflection-time response at mid-span for negative gradients 0, -10 and -20. The figure shows that amplitude of the dimensionless deflection reduces with the corresponding reduction in the slope.

Figure 4 shows the maximum amplitude of dimensionless deflection for variation of ratio of the load's length to beam's length at different time for fixed $\bar{v}$ = 01251. It is evident from this figure that the amplitude of the deflection is decreasing by increasing the ratio $\frac{\bar{d}}{\bar{L}}$ for all value of G considered.

Figure 5 depicts the time history of the maximum dimensionless deflections of some velocities where other parameters are fixed ($\bar{M}$ = 0.4, d = 0.5 or d = 0.0417; G = 5.) As expected, these results show that the maximum amplitude of dimensionless deflection increases with an increase in velocity.

CONCLUSION

The problem of assessing the dynamic behaviours of simply supported undamped Bernoulli-Euler beams under a linearly varying distributed moving load is considered. The governing equation for the mathematical model is analytically simplified into a set of ordinary differential equations that are solved by using Duhamel integral. Clearly, the linearly varying loads have considerable effects on the dynamic behaviour of the beam.

The result exhibits the following interesting features:

- The effects of the linearly varying load on the dimensionless dynamic mid-span deflection increases with the increase in the slope G of the distribution of the moving load.
- As the ratio of the load's length to beam's length increases, the response amplitude of the dimensionless dynamic deflection reduces.
- The amplitude of the dimensionless deflection increases with an increase in velocities of the moving load.

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