Journal of Engineering and Applied Sciences 2 (6): 1054-1057, 2007

Coefficient Inequalities and Convolution Properties Associated with Certain Classes of Analytic Functions

Oladipo Abiodun Tinuoye
Department of Pure and Aplied Mathematics,
Ladoke Akintola University of Technology, P.M.B. 4000, Ogomoso, Nigeria

Abstract: In the present study we defined two classes of analytic functions with negative coefficients using Salagean derivative operator. Coefficient inequalities and convolution properties associated with these classes are investigated

Key words: Coefficient inegualities, convolution properties, certain classes, analytic function

INTRODUCTION

Let $T(\rho)$ denote the class of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=\rho}^{\infty} a_\rho z^\rho, \ (a_k \geq 0, \rho \in N \setminus \{1\} = \{2,3,...\}) \quad (1)$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$

A function $f(z) \in T(\rho)$ is said to be in the class $T_{n,\rho}(\lambda,\beta)$ if it satisfies the condition

$$\text{Re}\left\{\frac{D^{n+1}f(z)}{\lambda D^{n+1}f(z) + (1-\lambda)D^n f(z)}\right\} > \beta \quad (2)$$

For some $0 \leq \beta < 1$, $0 \leq \lambda < 1$, $n = 0,1,2,...$, $z \in E$ and D is the same as Salagean derivative operator defined as

$$D^0 f(z) = f(z), D^1 f(z) = zf'(z), ...$$
$$D^n f(z) = z(D^{n-1}f(z))' \text{ (Abdulhalim, 1992)} \quad (3)$$

Also, let $C_{n,\rho}(\lambda,\beta)$ denote the subclass of $T(\rho)$ of all functions f(z) satisfying the following inequality

$$\text{Re}\left\{\frac{D^{n+2}f(z)}{(1-\lambda)D^{n+1}f(z) + \lambda D^{n+2}f(z)}\right\} > \beta \quad (4)$$

for some $0 \leq \beta < 1, 0 \leq \lambda < 1, n = 0,1,2,...$, $z \in E$ and D is the same as defined in (3).

In particular

$$C_{0,2}(\lambda,\beta) \equiv C_2(\lambda,\beta) \text{ and } T_{o,2}(\lambda,\beta) \equiv T_2(\lambda,\beta)$$

were studied by Altintas and Owa (1988). Also, putting $n = 0$, $\lambda = 0$ we obtain the classes $T_\rho(0,\beta) \equiv T_\rho(\beta)$ and $C_\rho(0,\beta) \equiv C_\rho(\beta)$ which were investigated by Choi and Kim (1996). They are subclasses of order β and convex of order β, respectively see (Srivastava and Owa, 1992; Duren, 1983)

Let $f_j(z) \in T(\rho)$ $(j = 1,2,...,m)$ be given by

$$f_j(z) = z - \sum_{k=\rho}^{\infty} a_{k,j} z^k \quad (5)$$

then the convolution (or Hadamard product) $f_j(z)$ of is defined by

$$\prod_{j=1}^{m} f_j(z) = (f_1 * ... * f_m)(z) = z - \sum_{k=p}^{\infty}\left(\prod_{j=1}^{m} a_{k,j}\right)z^k \quad (6)$$

Owo and Srivastava (2003), Saitoh and Owa (2001).

Here we define $D^n f(z)$ by

$$D^n f(z) = z - \sum_{k=\rho}^{\infty} k^n a_k z^k \quad (7)$$

COEFFICIENT INEQUALITIES

Lemma 1: A function f(z) defined by (1) is in $T_{n,\rho}(\lambda,\beta)$ if and only if

$$\sum\left[k^{n+1} - \beta(\lambda k^n(k-1) + k^n)\right]a_k \leq 1 - \beta \quad (8)$$

for some $0 \leq \beta < 1, 0 \leq \lambda < 1, n = 0,1,2,...,$.

Proof: Since $f(z) \in T(\rho)$. Then by using (7) in (2) with some simple transformation the result follows.

Lemma 2: A function f(z) defined by (1) is in $C_{n,\rho}(\lambda,\beta)$ if and only if

$$\sum\left[k^{n+2} - \beta(\lambda k^{n+1}(k-1) + k^{n+1})\right]a_k \le 1-\beta \qquad (9)$$

for some $0 \le \beta < 1, 0 \le \lambda < 1, n = 0,1,2,...,$

Proof: Since $f(z) \in T(\rho)$. Then, using (7) in (4) with some simple transformation the result follows.

Lemma 1 and 2 shall be used in obtaining our next results.

CONVOLUTION PROPERTIES

Theorem 1: If $f_j(z) \in T_{n,\rho}(\lambda,\beta), (j = 1,2,...,m)$ then

$$\prod_{j=1}^{m} f_j(z) \in T_{n_j,\rho}(\lambda,\gamma_m)$$

Where

$$\gamma_m = 1 - \frac{(1-\lambda)\prod_{j=1}^{m}(1-\beta_j)}{\prod_{j=1}^{m} 2^{\sum_{j=1}^{m} n_j - n}(2-(1-\lambda)\beta_j) - (1-\lambda)\prod_{j=1}^{m}(1-\beta_j)} \qquad (10)$$

$$\text{and } \sum_{j=1}^{m} n_j - n \ge 0$$

Proof: We need to find the largest γ_m such that

$$\sum_{k=\rho}^{\infty}\left[k^{n+1} - \gamma_m(\lambda k^n(k-1) + k^n)\right]\prod_{j=1}^{m}(a_{k,j}) \le 1-\gamma_m \qquad (11)$$

Note that if m = 1, then $\gamma_1 = \beta_1$. Now suppose m = 2. Then for functions

$f_1(z) \in T_{n_1,\rho}(\lambda,\beta_1)$, and $f_2(z) \in T_{n_2,\rho}(\lambda,\beta_2)$ wehave

$$\sum_{k=\rho}^{\infty}\left[k^{n_1+1} - \beta_1(\lambda k^{n_1}(k-1) + k^{n_1})\right]a_{k,1} \le 1-\beta_1$$

and

$$\sum_{k=\rho}^{\infty}\left[k^{n_2+1} - \beta_2(\lambda k^{n_2}(k-1) + k^{n_2})\right]a_{k,2} \le 1-\beta_2$$

so that

$$\sum_{k=\rho}^{\infty}\frac{[k^{n_1+1} - \beta_1(\lambda k^{n_1}(k-1) + k^{n_1})]}{1-\beta_1}a_{k,1} \le 1$$

and

$$\sum_{k=\rho}^{\infty}\frac{[k^{n_2+1} - \beta_2(\lambda k^{n_2}(k-1) + k^{n_2})]}{1-\beta_2}a_{k,2} \le 1.$$

Hence by Cauchy-Schwarz inequality we have

$$\sum_{k=\rho}^{\infty}\sqrt{\frac{[k^{n_1+1} - \beta_1(\lambda k^{n_1}(k-1) + k^{n_1})][k^{n_2+1} - \beta_2(\lambda k^{n_2}(k-1) + k^{n_2})]}{(1-\beta_1)(1-\beta_2)}(a_{k,1})(a_{k,2})} \le 1 \qquad (12)$$

In order to prove that $(f_1 * f_2)(z) \in T_{n,\rho}(\lambda,\gamma_2)$ it is sufficient to show that

$$\sqrt{(a_{k,1})(a_{k,2})} \le \frac{1-\gamma_2}{k^{n+1} - \gamma_2(\lambda k^n(k-1) + k^n)}\sqrt{\frac{[k^{n_1+1} - \beta_1(\lambda k^{n_1}(k-1) + k^{n_1})][k^{n_2+1} - \beta_2(\lambda k^{n_2}(k-1) + k^{n_2})]}{(1-\beta_1)(1-\beta_2)}}$$

since

$$\sum_{k=\rho}^{\infty}[k^{n+1} - \gamma_2(\lambda k^n(k-1) + k^n)](a_{k,1})(a_{k,2}) \le$$

$$\le 1-\gamma_2\sum_{k=\rho}^{\infty}\sqrt{\frac{[k^{n_1+1} - \beta_1(\lambda k^{n_1}(k-1) + k^{n_1})][k^{n_2+1} - \beta_2(\lambda k^{n_2}(k-1) + k^{n_2}}{(1-\beta_1)(1-\beta_2)}(a_{k,1})(a_{k,2})}$$

$$\leq 1-\gamma_2\left[\left(\frac{\sum_{k=\rho}^{\infty}[k^{n_1+1}-\beta_1(\lambda k^{n_1}(k-1)+k^{n_1})]}{1-\beta_1}(a_{k,1})\right)\left(\frac{\sum_{k=\rho}^{\infty}[k^{n_2+1}-\beta_2(\lambda k^{n_2}(k-1)+k^{n_2})]}{1-\beta_2}(a_{k,2})\right)\right]^{\frac{1}{2}}$$

$$\leq 1-\gamma_2$$

But from 12 we have for all $k=\rho=2,3,...$

$$\sqrt{(a_{k,1})(a_{k,2})}\leq \sqrt{\frac{(1-\beta_1)(1-\beta_2)}{[k^{n_1+1}-\beta_1(\lambda k^{n_1}(k-1)+k^{n_1})][k^{n_2+1}(k-1)+k^{n_2})]}}$$

Hence it is sufficient to find the largest γ_2 such that

$$\sqrt{\frac{(1-\beta_2)(1-\beta_2)}{[k^{n_1+1}-\beta_1(\lambda k^{n_1}(k-1)+k^{n_1})][k^{n_2+1}-\beta_2(\lambda k^{n_2}(k-1)+k^{n_2})]}}\leq$$

$$\leq\frac{1-\gamma_2}{[k^{n+1}-\gamma_2(\lambda k^{n}(k-1)+k^{n})}\sqrt{\frac{[k^{n_1+1}-\beta_1(\lambda k^{n_1}(k-1)+k^{n_1})][k^{n_2+1}-\beta_2(\lambda k^{n_2}(k-1)+k^{2})]}{(1-\beta_1)(1-\beta_2)}}$$

That is,

$$1\leq\frac{1-\gamma_2}{k^{n+1}-\gamma_2(\lambda k^{n}(k-1)+k^{n})}\left[\frac{[k^{n_1+1}-\beta_1(\lambda k^{n_1}(k-1)+k^{n_1})][k^{n_2+1}-\beta_2(\lambda k^{n_2}(k-1)+k^{n_2})]}{(1-\beta_1)(1-\beta_2)}\right]$$

$$\gamma_2\leq 1+\frac{(\lambda(k-1)+1-k)\prod_{j=1}^{2}(1-\beta_j)}{\prod_{j=1}^{2}k^{\sum_{j=1}^{2}n_j-n}\left(k-\beta_j(\lambda(k-1)+1)\right)-(1+\lambda)\prod_{j=1}^{2}(1-\beta_j)}\qquad(13)$$

$$\gamma_2=1-\frac{(1-\lambda)\prod_{j=1}^{2}(1-\beta_j)}{\prod_{j=1}^{2}2^{\sum_{j=1}^{2}n_j-n}\left(2-(1+\lambda)\beta_j\right)-(1+\lambda)\prod_{j=1}^{2}(1-\beta_j)}$$

Now suppose that

$$\prod_{j=1}^{m}{}^{*}f_j(z)\in T_{n,\rho}(\lambda,\gamma_m)$$

Where

$$\gamma_m=1-\frac{(1-\lambda)\prod_{j=1}^{m}(1-\beta_j)}{\prod_{j=1}^{m}2^{\sum_{j=1}^{m}n_j-n}\left(2-(1+\lambda)\beta_j\right)-(1+\lambda)\prod_{j=1}^{m}(1-\beta_j)}$$

Then repeating the process above we obtain that

$$\prod_{j=1}^{m+1}{}^{*}f_j(z)\in T_{n,\rho}(\lambda,\gamma_{m+1})$$

and

$$\gamma_{m+1}=1-\frac{(1-\lambda)\prod_{j=1}^{m+1}(1-\beta_j)}{\prod_{j=1}^{m+1}2^{\sum_{j=1}^{m+1}n_j-n}\left(2-(1+\lambda)\beta_j\right)-(1+\lambda)\prod_{j=1}^{m+1}(1-\beta_j)}.$$

Hence the conclusion follow by induction.

Theorem 2: If $f_j(z)\in C_{n,\rho}(\lambda,\beta),\ (j=1,2,...,m)$ then

$$\prod_{j=1}^{m}f_j(z)\in C_{n_j,\rho}(\lambda,\gamma_m)$$

Where

$$\gamma_m=1-\frac{(1-\lambda)\prod_{j=1}^{m}(1-\beta_j)}{\prod_{j=1}^{m}2^{\sum_{j=1}^{m}n_j-n+1}(2-(1-\lambda)\beta_j)-(1-\lambda)\prod_{j=1}^{m}(1-\beta_j)}$$

$$\text{and} \quad \sum_{j=1}^{m} n_j - n + 1 \geq 0$$

Proof: The proof is similar to that of Theorem 1 by using Lemma 2.

Remarks: Putting $\lambda = 0$ and $n = n_j = 0$ in our results we obtain the results in Choi and Kim (1996).

REFERENCES

Altintas, O. and S. Owa, 1988. On subclass of univalent functions with negative coefficients. Pusan Kyougnam Mathe. J., 4: 41-56.

Abduhalim, S., 1992. On a class of Analytic functions involving Salagean derivative operator. Tamkang Journal of Maths, Vol. 23.

Choi, J.H. and Y.C. Kim, 1996. Generalizations of Hadamard products of functions with negative coefficient J. Mathe. Anal. Appl., 199: 495-501.

Duren, P.L., 1983. Univalent Functions. Grundlehrender Mathematschen, Springer-verlag, New York, Berlin, Heidelberg, Tokyo, Vol. 259.

Owa, S. and H.M. Srivastava, 2003. Some generalized convolution properties associated with certain subclasses of analytic functions. J. Inequalities in Pure and Applied Mathe., 42: 1-13.

Srivastava, H.M. and S. Owa, (Ed). 1992. Current Topics in Analytic Functions Theory. World Scientific, Singapore, New Jersey, London, Hong Kong.

Saitoh, S. and S. Owa, 2001. Convolution of certain analytic functions, Alebra Groups Geom., 18: 375-384.