

On the Existence and Uniqueness of a Problem Arising from the Theory of Combustion

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Abstract: We consider the species and energy equation and we impose some initial and boundary conditions on the equations. It is of great important to establish that a new combustion problem has a unique solution. We re-affirm that the solution exist and it has physical implications. The existence and uniqueness theorem stated can be used for other similar problems in fluid dynamics.

Key words: Existence, uniqueness, problem arising, theory of combustion

INTRODUCTION

The interest of scientists after modeling a problem or designing any model, are the questions: Does a solution exist? How many solutions does the problem have? Is the solution unique? The uniqueness of solution is very important in modeling physical problem or designing a model.

Okoya (1989) gave some conditions for existence and uniqueness of some particular problems. He also gave criteria for multi-solution of another problem.

Olanrewaju (2002, 2005) determines some criteria for a design or model to have only one solution and the effect of certain parameters on these solutions.

Njoku (2000) examined an existence and uniqueness result for a system of non-linear volterra equations and established forward modifications to Urysohn integral operators. In this study, we consider a particular problem from the thesis.

Olanrewaju (2002) and impose conditions so that we have only one solution.

MATHEMATICAL FORMULATION

We consider the species and energy equations,

$$\rho \left(\frac{\partial y}{\partial t} + u \frac{\partial y}{\partial x} \right) = \frac{D \partial^2 y}{\partial x^2} - A y e^{-E/RT} \quad (1)$$

$$\rho \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} \right) = \frac{k \partial^2 T}{\partial x^2} + q A y e^{-E/RT} \quad (2)$$

Where $a=1$ (i.e the order of reaction is 1) Let the initial and boundary conditions be $y(x,0) = 0$.

$$\begin{aligned} y(L,t) &= y_0, t > 0 \\ y(L,t) &= 0 \text{ or } y(L,0) = 0 \\ T(x,0) &= T_0 \end{aligned}$$

$$\begin{aligned} T(0,t) &= T_0, t > 0 \\ T(L,t) &= T_1, t > 0, T_1 > T_0 \\ \text{Let } \rho u &= a_1 \text{ (constant } t) \end{aligned}$$

And

Where

- y = Pre-mixed reactants
- ρ = Density of the reactants
- c = Specific heat
- T = Temperature
- q = Heat released/unit mass
- D = Diffusion coefficient
- k = Thermal conductivity
- R = Universal gas constant
- t = Time
- E = Activation energy

Steady case of (1) and (2) becomes

$$u \frac{\partial y}{\partial x} = D \frac{\partial^2 y}{\partial x^2} - A y e^{-E/RT} \quad (3)$$

$$\rho c u \frac{\partial T}{\partial x} = \frac{k \partial^2 T}{\partial x^2} + q A y e^{-E/RT} \quad (4)$$

We can write (3) and (4) as

$$q a_1 c \frac{\partial y}{\partial x} = q D \frac{\partial^2 y}{\partial x^2} - q A y e^{-E/RT} e^{-E/RT_0} e^{-E/RT_0} \quad (5)$$

$$a_1 c \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial x^2} - q A y e^{-E/RT} e^{-E/RT_0} e^{-E/RT_0} \quad (6)$$

Let

$$\varepsilon = \frac{RT_0}{E}$$

Then

$$T = \theta \epsilon T_0 + T_0 \text{ or } T = T_0 (\theta \epsilon + 1)$$

Non-dimensionalizing process

Let

$$y' = \frac{y}{y_0}, \quad u' = \frac{u}{U}, \quad x' = \frac{x}{L}, \quad t = \frac{t}{L}$$

Where

- y_0 = References length along y-axis
- U = Characteristic velocity component along x-axis
- L = References length along x-axis
- t = Time

Non-dimensionalizing (5), (6) and after dropping the “we have

$$\frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial x^2} + \left[\frac{1}{2} a \left(\frac{e^{x/b} - 1}{e^{x/b-1}} \right) - \theta \right] e^{1+\epsilon \theta} \quad (7)$$

$$\theta(0) = 0, \theta(1) = 0.5$$

Where

- d = Modified dimensionless thermal conductivity,
- b = Modified dimensionless thermal conductivity,
- a = Frank-Kamenetskii parameter

METHOD OF SOLUTION

We utilize shooting method in solving (7)

$$\text{Let } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x \\ \theta \\ \theta' \end{pmatrix} \quad (8)$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 \\ x_3 \\ \frac{1}{d} \left[x_3 - \frac{1}{2} a \left(\frac{e^{x_1/b} - 1}{e^{x_1/b-1}} \right) - x_2 \right] e^{1+\epsilon x_2} \end{pmatrix} \quad (9)$$

Satisfying

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} \quad (10)$$

THE EXISTENCE AND UNIQUENESS OF THE SOLUTION

For Eq. 8 which satisfies (9)

(HI): $S > 0, 0 = x_1 = 1, 0 = x_2 = L, a > 0$ and $-m = x_3 = M$, where L, m and M are positive constants.

Theorem 1: Let (HI) hold. Then problem (8) has a unique solution which satisfies (9).

Proof:

From (9), we have

$$\begin{aligned} \left| \frac{\partial f_1}{\partial x_1} \right| &= \left| \frac{\partial f_1}{\partial x_2} \right| = \left| \frac{\partial f_1}{\partial x_3} \right| = \left| \frac{\partial f_2}{\partial x_1} \right| = \left| \frac{\partial f_2}{\partial x_2} \right| = 0, \left| \frac{\partial f_2}{\partial x_3} \right| = \left| \frac{\partial f_3}{\partial x_3} \right| = 1 \\ \left| \frac{\partial f_3}{\partial x_1} \right| &= \left| \frac{a}{2bd} \right| = \left| e^{x/b} - 1 \right| e^{x/b} = \lambda_1 < \infty \\ \left| \frac{\partial f_3}{\partial x_2} \right| &= \left| \frac{L}{e^{1+\epsilon L}} \right| \left| 1 + \frac{L}{(1+\epsilon L)^2} \right| = \lambda_2 < \infty \end{aligned}$$

f_1, f_2, f_3 , are given by (9) clearly

$$\frac{\partial f_i}{\partial x_i}$$

is bounded for $i, j = 1, 2, 3$.

Thus $f_i, i = 1, 2, 3$ are Lipschitz continuous. Hence there exists a solution of (9) which satisfies (10).

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