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Diffusion Processes with Reflecting Boundaries and Random Initial States

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Abstract: Kolmogorov forward equation for one-dimensional time-homogeneous diffusion processes X(t) is considered. This equation is solved explicitly for the Wiener and the Ornstein-Uhlenbeck processes, in particular, in the case when there is a (time-dependent) reflecting boundary. Moreover, the initial state X(0) is a random variable.

Key words: Brownian motion, method of separation of variables, special functions

INTRODUCTION

A one-dimensional diffusion process X(t) is characterized by its transition density function

$$p(x,t;x_0,t_0) := \frac{P[X(t)\in(x,x+dx)\mid X(t_0)=x_0]}{dx}$$

This function satisfies the Kolmogorov forward equation (also called Fokker-Planck equation)

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\{v(x,t)p(x,t;x_0,t_0)\} - \frac{\partial}{\partial x}\{m(x,t)p(x,t;x_0,t_0)\} = \frac{\partial}{\partial t}p(x,t;x_0,t_0)$$

(Cox and Miller[1], for instance), where m(x,t) and v(x,t) are respectively the drift and the dispersion (or, in finance, volatility) of the process. The functions m(x,t) and v(x,t) must be such that, for all s > 0, we have (see Lamberton and Lapeyre[2]):

$$\int_0^s |m(x,t)|\,dt < \infty \quad \text{and} \quad \int_0^s v(x,t)\,dt < \infty$$

Next, the probability density function f(x,t) of X(t) is defined by

$$f(x,t) = \frac{P[X(t)\in(x,x+dx)]}{dx}$$

and can also be expressed as follows:

$$f(x,t) = \int_{-\infty}^{\infty} p(x,t;x_0,t_0)f(x_0,t_0)\,dx_0$$

for $t > t_0$. Therefore, it satisfies the same partial differential equation as $p(x,t;x_0,t_0)$:

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\{v(x,t)f(x,t)\} - \frac{\partial}{\partial x}\{m(x,t)f(x,t)\} = \frac{\partial}{\partial t}f(x,t).$$

The most important diffusion processes used in the applications are time-homogeneous, which implies that m(x,t) = m(x) and v(x,t) = v(x). In particular, the standard Brownian motion is such that m(x) = 0 and v(x) = 1, so that

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}f(x,t) = \frac{\partial}{\partial t}f(x,t).$$

When the starting value $X(t_0)$ (= x_0) of the stochastic process is deterministic, say $X(t_0) = x_{00}$, so that

$$f(x_0,t_0) = \delta(x_0 - x_{00})$$

where δ(.) is the Dirac delta function, we find that the solution of the above partial differential equation (which is the heat equation) is

$$f(x,t) \equiv p(x,t;x_{00},t_0) = \frac{1}{\sqrt{2\pi(t-t_0)}}\exp\left\{-\frac{(x-x_{00})^2}{2(t-t_0)}\right\}$$

for $x \in \Re$ and $t > t_0$. That is, $X(t) \mid \{X(t_0) = x_{00}\}$ has a Gaussian distribution with mean x_{00} and variance $t-t_0$.

The problem that we consider in this note is that of finding *probabilistic* solutions to the partial differential equation

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\{v(x)f(x,t)\} - \frac{\partial}{\partial x}\{m(x)f(x,t)\} = \frac{\partial}{\partial t}f(x,t) \quad (1)$$

subject to the appropriate conditions, that can be obtained by the method of separation of variables.

The case when X(t) is a standard Brownian motion, an Ornstein-Uhlenbeck process and a geometric Brownian motion.

RESULTS AND DISCUSSION

If we assume that

$$f(x,t) = g(x)h(t),$$

then Eq. (1) becomes

$$\frac{1}{2}\frac{d^2}{dx^2}\{v(x)g(x)\}h(t) - \frac{d}{dx}\{m(x)g(x)\}h(t) = g(x)h'(t).$$

Hence, we deduce that

$$h(t) = c_0 e^{ct},$$

where c_0 and c are constants. It follows that the function g(x) must satisfy the ordinary differential equation

$$\frac{1}{2}\frac{d^2}{dx^2}\{v(x)g(x)\} - \frac{d}{dx}\{m(x)g(x)\} = cg(x).$$

Standard brownian motion: When m(x) = 0 and v(x) = 1, we get the simple ordinary differential equation

$$\frac{1}{2}\frac{d^2}{dx^2}g(x) = cg(x),$$

whose general solution is

$$g(x) = c_1 \exp\left\{\sqrt{2c}\,x\right\} + c_2 \exp\left\{-\sqrt{2c}\,x\right\}.$$

Now, the function f(x,t) must be non-negative and such that

$$\int_{\alpha(t)}^{\beta(t)} f(x,t)dx = 1 \quad \forall t \ge t_0 \qquad (2)$$

for appropriate functions α(t) and β(t) that must be determined. Assume that $t_0 = 0$ and choose $c_0 = c = 1$ and $c_1 = 0$, so that

$$f(x,t) = c_2 \exp\left\{-\sqrt{2}\,x\right\} e^t.$$

The condition (2) above is satisfied if we take, in particular, the constant $c_2 = \sqrt{2}$, $\alpha(t) = t/\sqrt{2}$ and $\beta(t) = \infty$

Notice that

$$f(x,0) = \sqrt{2}\exp\left\{-\sqrt{2}\,x\right\} \qquad \text{for } x \in [0,\infty).$$

That is, X(0) is a random variable having an exponential distribution with parameter $\sqrt{2}$.

Thus, X(t) would be a standard Brownian motion taking its values in the interval $[t/\sqrt{2},\infty)$. This is possible if there is a reflecting boundary at $x = t/\sqrt{2}$. The condition for (t-dependent) reflecting barriers is that

$$\frac{\partial}{\partial t}\int_{\alpha(t)}^{\beta(t)} f(x,t)dx = 0.$$

But since the functions α(t) and β(t) where chosen so that the integral is equal to 1, this condition is automatically satisfied.

Ornstein-uhlenbeck process: This important diffusion process is such that its drift is given by m(x) = -αx and its dispersion is v(x) = 1, where α > 0. For simplicity, let us choose α = 1. The ordinary differential equation satisfied by the function g is

$$\tfrac{1}{2}g''(x) + xg'(x) + (c+1)g(x) = 0.$$

Its general solution can be expressed as follows:

$$g(x) = \frac{c_1}{\sqrt{x}}\exp\left\{-x^2/2\right\} M\left(\frac{c+1}{4},\frac{1}{4},x^2\right) + \frac{c_2}{\sqrt{x}}\exp\left\{-x^2/2\right\} W\left(\frac{c+1}{4},\frac{1}{4},x^2\right),$$

where M(.,.,.) and W(.,.,.) are Whittaker functions (Abramowitz and Stegun[3]). In the particular case when c = -1, the solution reduces to

$$g(x) = c_1 + c_2\,\mathrm{erf}(x),$$

where erf(.) is the error function. Choosing ($t_0 = 0$ and) $c_2 = 0$, we obtain that

$$f(x,t) = k\,e^{-t}$$

where k is a positive constant (with respect to x). This time, X(0) has a uniform distribution on an interval of length k, say [0, k].

The condition (2) is fulfilled if we take $\alpha(t) = 0$ and $\beta() = e^t/k$. As in the previous case, we obtain that X(t) has a reflecting barrier at x = 0 and also at $x = e^t/k$. Hence, we can state that the function

$$f(x,t) = k\,e^{-t} \qquad \text{for } x \in [0, e^t/k]$$

is the density function of an Ornstein-Uhlenbeck process whose starting value is a random variable uniformly distributed on the interval [0,k] and evolving between two reflecting boundaries.

Geometric brownian motion: This diffusion process is widely used in mathematical finance. Because it can be expressed as the exponential of a Brownian motion, it has a natural boundary at the origin. We consider the case when the drift is m(x) = 2x and the dispersion is $v(x) = x^2$. The function g(x) is then a solution of

$$\frac{x^2}{2}g''(x) - (c+1)g(x) = 0.$$

We find that

$$g(x) = c_1 x^{\mu} + c_2 x^{\nu},$$

where

$$\mu = \frac{1}{2} + \frac{\sqrt{9+8c}}{2} \quad \text{and} \quad \nu = \frac{1}{2} - \frac{\sqrt{9+8c}}{2}.$$

Choosing ($t_0 = 0$,) c = -1 and $c_2 = 0$, we can write that

$$f(x,t) = k\,x\,e^{-t}.$$

We must have

$$\int_{\alpha(t)}^{\beta(t)} k\,x\,e^{-t}\,dx = 1 \quad \Rightarrow \quad k\frac{\beta^2(t) - \alpha^2(t)}{2}e^{-t} = 1.$$

We can choose k = 2, $\alpha(t) = 0$ and $\beta(t) = e^{t/2}$. Then, we can assert that there is a reflecting boundary at $x = e^{t/2}$.

Finally, the initial distribution of the process is given by

$$f(x,0) = 2x \qquad \text{for } x \in [0,1].$$

CONCLUSION

We have obtained explicit solutions to the Kolmogorov forward equation for very important diffusion processes in the case when these processes evolve in a region bounded by at least one reflecting barrier and their starting values are random.

There are surely other particular solutions that can be obtained by making use of the method of separation of variables for these processes. There are also other important diffusion processes that could be considered, in particular the Bessel process.

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