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On Extremal Polynomials on a System of Curves and Arcs

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Abstract: Strong asymptotic is given for a sequence of extremal polynomials with respect to a Szegö measure supported on a system of a rectifiable Jordan curves and arcs and perturbed by an infinite Blaschke sequence of point masses.

Key words: Asymptotic behavior, orthogonal polynomials, capacity logarithmic, green function

INTRODUCTION

Hereafter in this study, let σ be a finite positive Borel measure with an infinite compact support in the complex plane. Denote by $\{T_n(z)\}$ the monic polynomial of degree n orthogonal with respect to the measure σ i.e.

$$T_n(z) = z^n + \ldots,$$
$$\int T_n(z)\overline{z}^{k} d\alpha = 0; \quad k = 0,1,\ldots n-1$$

Let P_n be the set of polynomials of degree n. It is well known that $T_n(z)$ satisfies the extremal properties:

$$\|T_n(z)\|^2_{L_2(\sigma)} = \min_{Q \in P_{n-1}} \|z^n + Q\|^2_{L_2(\sigma)} = m_n(\sigma) \tag{1.1}$$

where as usual

$$\|f\|_{L_2(\sigma)} = \left\{\int |f(\xi)|^2 d\sigma(\xi)\right\}^{1/2}$$

One of the major areas of research in the study of extremal polynomials is to investigate the strong asymptotics behavior of $T_n(z)$ as $n\to\infty$. Other commonly used names are power asymptotic, Szegö asymptotic, or full exterior asymptotic.

In the classical case of a real interval and unit circle the strong asymptotics was investigated by Bernstein and Szegö (1975). In Widom (1969), we find some general results concerning the strong asymptotics for orthogonal polynomials with respect to measure supported by a finite system of arcs and curves and which satisfy Szegö condition. An extension of Widom's results has been given by Kaliaguine and Konorova (2000) for a measure concentrated on a system of arcs and curves and perturbed by a finite Blaschke sequence of point masses. For more details on this subject see (Khaldi, 2004; 2005; 2004; Li and Pan, 1994; Marcellan and Maroni, 1992; Nuttal and Singh, 1977; Peherstorfer and Yuditskii, 2001; Rakhmanov, 1977; Rakhmanov, 1983; Rudin, 1968; Szegö, 1975; Widom, 1969).

In the present study we study the strong asymptotics of orthogonal polynomials associated to a measure of the type

$$\sigma = \alpha + \gamma$$

where α denotes the absolutely continuous part of the measure σ on E i.e.,

$$d\alpha(\xi) = \rho(\xi)d\xi,\ \rho \geq 0, \rho \in L^1(E, |d\xi|) \tag{1.2}$$

and γ is a point measure supported on a denumerable set of points $\{z_k\}_{k=1}^{\infty}$ off the system i.e.,

$$\gamma = \sum_{k=1}^{\infty} A_k \delta_{z_k},\ A_k > 0, \sum_{k=1}^{\infty} A_k < \infty. \tag{1.3}$$

Note that the study of the case where the point measure is supported by an infinite discrete part (Kahldi, 2004, 2005; Peherstorfer and yudistshii, 2001), is completely different from the finite one (Gonchar, 1975; Kaliaguine, 1995; Kaliaguine and Konorova, 2000; Li and Pan, 1994), so we try to bring over some of the foremost ideas of (Kaliaguine and Konorova, 2000) to the infinite case, where the situation turns out to be much more difficult, for this reason, no much is known about extremal polynomials on a system of curves and arcs, although they plays an important role in modern solid state physics see (Gasper and Cyrot, 1973; Heine, 1980), since the densities of states live on several arcs.

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This study is organized as follow, we give some preliminaries and notations to be able to state our results, we recall the definition of the complex Green function, describe the Hardy multi-valued functions. We expose the extremal problems. The main results are proved.

Premilinaries and notations: Let $E = \bigcup_{k=1}^{m} E_k$ be a union of complex rectifiable Jordan curves and arcs of class C^{2+} with $E_i \bigcap E_j = \phi, \forall i \neq j$. By E^1 we denote the union of curves and E^2 the union of arcs. Denote by Ω the connected component of C\E and we suppose that ∞∈Ω.

Complex green function: Let g(z,a) be the real Green function of Ω with singularity at a, denote by ğ(z,a) it's harmonic conjugate, then the function G(z,a) = g(z,a)+ iğ(z,a) is called complex Green function for Ω with the pole at a. If a=8 we denote by g(z) and G(z) the real and complex Green function with singularity at infinity. The function Φ(z) = exp[G(z)] is locally analytic in Ω, has no zero with a pole at infinity and |Φ(ξ)| = 1, ξ∈∂Ω. The logarithmic capacity of E denoted by C(E) is the positive number C(E) = exp(-γ), where γ is the Robin's constant of Ω:

$$\gamma = \lim_{z\to\infty}\left[g(z) - \log|z|\right]$$

If the weight function ρ (which defines the absolutely part of the measure σ satisfies the Szegö condition:

$$\int_E \log(\rho(\xi))|\Phi'(\xi)||d\xi| > -\infty$$

then, there exists the real function h harmonic in Ω with the boundary condition on E: h(ξ) = log(ρ(ξ)), ξ∈E. The function $R(z) = \exp\{h(z) + i\tilde{h}(z)\}$ is locally analytic in Ω, has a non-tangential limit value on E and |R(z)| = ρ(ξ), ξ∈E. The function $D(z) = \exp\{(1/2)(h(z) + i\tilde{h}(z))\}$ is called Szegö function associated with the weight function ρ.

The Hardy spaces of multi-valued functions $H^2(\Omega,\rho)$ One says that a function f locally analytic in Ω with single valued modulus and multi-valued argument is from $H^2(\Omega,\rho)$ space if the function $|f(z)^2R(z)|$ has a harmonic majorant in Ω. Each function f from $H^2(\Omega,\rho)$ has limit values a.a. on E and

$$\|f\|^2_{H^2(\Omega,\rho)} = \oint_E |f(\xi)|^2 \rho(\xi)|d\xi| < \infty,$$

here we have use the following notation

$$\oint_E f(\xi)|d\xi|. := \int_{\partial\Omega} f(\xi)|d\xi|,$$

where the boundary ∂Ω of the region Ω is essentially the set *E* where any arc of *E* is taken twice.

Extremal problems: We define μ(α) as the extremal value of the following problem:

$$\mu(\alpha) = \inf\left\{\|\varphi\|^2_{H^2(\Omega,\rho)}, \varphi \in H^2(\Omega,\rho), \varphi(\infty) = 1\right\} \quad (3.1)$$

It is proved in (Widom, 1969) that the extremal function of the problem (3.1) is unique up to the complex constant factor of modulus 1. We denote the extremal function of the problem (3.1) by φ.

Lemma 3.1: The extremal function of the following problem

$$\mu(\sigma) = \inf\left\{\|\varphi\|^2_{H^2(\Omega,\rho)}, \varphi \in H^2(\Omega,\rho), \varphi(\infty) = 1, \varphi(z_k) = 0, k = 1, 2, \ldots\right\} \quad (3.2)$$

is given by ψ = φB, in addition

$$\mu^\infty(\sigma) = \mu(\alpha)\prod_{k=1}^{\infty}|\Phi(z_k)|^2$$

where the constant μ(α) and the function φ* are defined by the problem (3.1) and the function *B* is the product:

$$B(z) = \prod_{k=1}^{\infty}\frac{\Phi(\infty, z)}{\Phi(z, z_k)}$$

Proof: The proof is the same as given in (Khaldi, 2004).

Remark 3.1: In the case when Ω is simply connected that is E is a curve or an arc (m = 1) the function B is

$$B(z) = \prod_{k=1}^{\infty}\frac{\Phi(z) - \Phi(z_k)}{\Phi(z_k)\overline{\Phi(z_k)} - 1}\frac{\left|\Phi(z_k)^2\right|}{\Phi(z_k)}$$

Statement of the main results

Definition 4.1: A measure σ = α+γ is said to belong to a class A, if the absolutely continuous part α and the discrete part γ satisfy the conditions (1.2), (1.3) and the Blaschke's condition, i.e.,

$$\sum_{k=1}^{\infty}\left(\Phi(z_k) - 1\right) < \infty \quad (4.1)$$

Remark 4.2: The condition (4.1) is natural and it guarantees the convergence of the Blaschke product

$$\prod_{k=1}^{\infty}\left|\Phi(z_k)\right|^2$$

Definition 4.3: A rectifiable curve or an arc E is said to be of class C^{2+} if in the canonical parameterization z(t) of E, the second derivative of the function z(s) satisfies a Lipschitz condition with some positive exponent.

We denote as in (1.1) by mn(σ) and mn(σ_N) the extremal values of the following problems:

$$m_n(\sigma)=\int_E\left|T_n(\xi)\right|^2\rho(\xi)\left|d\xi\right|+\sum_{k=1}^{\infty}A_k\left|T_n(z_k)\right|^2$$

$$m_n(\sigma_N)=\int_E\left|T_n^N(\xi)\right|^2\rho(\xi)\left|d\xi\right|+\sum_{k=1}^{\infty}A_k\left|T_n^N(z_k)\right|^2$$

where the measure $\sigma_N=\alpha+\sum_{k=1}^{N}A_k\delta_{z_k}$,

It is easy to see that the extremal property of $T_n^N(z)$ (see (7)) implies that the sequences $\{m_n(\sigma)\}_{k=1}^{\infty}$ is increasing and $m_n(\sigma_N)<m_n(\sigma)$ for every N = 1 and so the following theorem tells us what the limit is.

Theorem 4.4: Assume that the measure $\sigma=\alpha+\sum_{k=1}^{\infty}A_k\delta_z$ satisfies the conditions (1.2) and (1.3), then we have

$$\lim_{N\to\infty}m_n(\sigma_N)=m_n(\sigma)$$

Proof: According to the reproducing property of the kernel polynomial $K_n(\xi,z)$, we have:

$$T_n^N(z_j)=\int_E T_n^N(\xi)\overline{K_{n+1}(\xi,z_j)}\rho(\xi)\left|d\xi\right|$$

using the Schwarz inequality we get

$$\begin{aligned}\left|T_n^N(z_j)\right|^2&\le\int_E\left|T_n^N(\xi)\right|^2\rho(\xi)\left|d\xi\right|\int_E\left|K_{n+1}(\xi,z_j)\right|^2\rho(\xi)\left|d\xi\right|\\&\le m_n(\sigma_N)\sup_{\xi\in E}\left|K_{n+1}(\xi,z_j)\right|^2.\end{aligned}\tag{4.2}$$

From (4.2) and the extremal property of $T_n(z)$ it yields

$$m_n(\sigma)\le m_n(\sigma_N)+\sum_{k=N+1}^{\infty}A_k\left|T_n^N(z_k)\right|^2$$

$$\le m_n(\sigma_N)\left[1+\sup_{\xi\in E,k\ge N+1}\left|K_{n+1}(\xi,z_j)\right|^2,\sum_{k=N+1}^{\infty}A_k\right]$$

Finally, we obtain

$$m_n(\sigma)\le\liminf_{l\to\infty}m_n(\sigma_N)\le\limsup_{l\to\infty}m_n(\sigma_N)\le m_n(\sigma).$$

This achieves the proof of the theorem.

Theorem 4.5: Let E be a system of curves and arcs from the class C^{2+} and the measure σ from the class A. If

$$m_n(\sigma_N)\le\left(\prod_{k=1}^{N}\left|\Phi(z_k)\right|\right)m_n(\alpha),\forall n,\forall N\tag{4.3}$$

then the orthogonal polynomials $T_n(z)$ and the extremal value $m_n(\sigma.)$ have the following asymptotic behavior (n→∞):

(i) $\lim\frac{m_n(\sigma)}{\left[C(E)\right]^{2n}}=\mu_\infty(\sigma)$

(ii) $\lim\int_E\left|\frac{T_n(\xi)}{\left[C(E)\right]^n}-\Psi(\xi)\right|^2\rho(\xi)\left|d\xi\right|=0$

(iii) $T_n(z)=\left[C(E)\Phi(z)\right]^n\left[\Psi(z)+\varepsilon_n(z)\right],$

$\epsilon_n\to0$ uniformly on the compact subsets of Ω.

The constant μ(σ) and the function ψ are defined in Lemma 3.1.and

$$\Psi(\xi)=\Phi^n(\xi)\psi(\xi)\ \text{if}\ \xi\in E^1$$
$$\Psi(\xi)=\Phi_+^n(\xi)\psi_+(\xi)+\Phi_-^n(\xi)\psi_-(\xi)\ \text{if}\ \xi\in E^2$$

Proof: We start with the proof of the upper bound of

$$\limsup_{n\to\infty}\frac{m_n(\sigma)}{\left[C(E)\right]^{2n}}\tag{4.4}$$

Indeed, taking the limit when *N* tends to infinity in (4.3) and using Theorem 4.4, we obtain

$$\frac{m_n(\sigma)}{\left[C(E)\right]^{2n}}\le\left(\prod_{k=1}^{\infty}\left|\Phi(z_k)\right|^2\right)\frac{m_n(\alpha)}{\left[C(E)\right]^{2n}},\tag{4.5}$$

On the other hand it is proved in (Widom, 1969) that

$$\lim_{n\to\infty}\frac{m_n(\alpha)}{\left[C(E)\right]^{2n}}=\mu(\alpha).\tag{4.6}$$

Using (4.5), (4.6) and Lemma 3.1, we get

$$\limsup_{n\to\infty} \frac{m_n(\sigma)}{[C(E)]^{2n}} \le \left(\prod_{k=1}^{\infty} |\Phi(z_k)|^2\right)\mu(\alpha) = \mu(\sigma), \qquad (4.7)$$

Now consider the integral

$$I_n = \int_E \left| \frac{T_n(\xi)}{[C(E)]^n} - \Psi(\xi) \right|^2 \rho(\xi)|d\xi|$$

and transform it in a standard way as the following sum

$$I_n = \int_E \left| \frac{T_n(\xi)}{[C(E)]^n} \right|^2 \rho(\xi)|d\xi| + + \int_E |\Psi(\xi)|^2 \rho(\xi)|d\xi|$$
$$-2\mathrm{Re}\int_E \frac{T_n(\xi)}{[C(E)]^n} \overline{\Psi(\xi)}\rho(\xi)|d\xi|$$
$$= I_n^1 + I_n^2 + I_n^3$$

First step: From the definition of $m_n(\sigma)$ we have

$$I_n^1 = \int_E \left| \frac{T_n(\xi)}{[C(E)]^n} \right|^2 \rho(\xi)|d\xi| \le \frac{m_n(\sigma)}{[C(E)]^{2n}} \qquad (4.8)$$

Second step: Evaluation of the integral I_n^2.
For the curves we get

$$\int_{E^1} |\Psi(\xi)|^2 \rho(\xi)|d\xi| = \oint_{E^1} |\psi(\xi)|^2 \rho(\xi)|d\xi| \qquad (4.9)$$

For the arcs we obtain

$$\int_{E^2} |\Psi(\xi)|^2 \rho(\xi)|d\xi| = \oint_{E^2} |\psi(\xi)|^2 \rho(\xi)|d\xi|$$
$$+2\mathrm{Re}\int_{E^2} \overline{\Phi_+^n(\xi)\psi_+(\xi)\Phi_-^n(\xi)\psi_-(\xi)}\rho(\xi)|d\xi|.$$

since the second integral approaches zero as $n\to\infty$, from Widom's lemma it yields

$$\int_{E^1} |\Psi(\xi)|^2 \rho(\xi)|d\xi| = \mu\ (\sigma) + \beta_n, \quad \beta_n \to 0 \qquad (4.10)$$

So, (4.9) and (4.10) implies

$$\int_{E} |\Psi(\xi)|^2 \rho(\xi)|d\xi| = \mu\ (\sigma) + \beta_n, \quad \beta_n \to 0 \qquad (4.11)$$

Third step: Since $\overline{\Phi_+} = \frac{1}{\Phi_+}$ we have

$$2\mathrm{Re}\int_E \frac{T_n(\xi)}{[C(E)]^n} \overline{\Psi(\xi)}\rho(\xi)|d\xi| =$$

$$2\mathrm{Re}\oint_E \frac{T_n(\xi)}{[C(E)\Phi(\xi)]^n} \overline{\psi(\xi)}\rho(\xi)|d\xi| =$$

then by proceeding as in (Peherestorfer and Yudistskii, 2001) we get

$$I_n^3 = 2\mu(\sigma) + \beta_n, \qquad \beta_n \to 0 \qquad (4.12)$$

Using (4.8), (4.11) and (4.12) we obtain

$$0 \le I_n \le \frac{m_n(\sigma)}{[C(E)]^{2n}} + \mu(\sigma) + \alpha_n - 2\mu(\sigma) - \beta_n \quad (4.13)$$

this implies

$$\liminf_{n\to\infty} \frac{m_n(\alpha)}{[C(E)]^{2n}} \ge \mu(\alpha). \qquad (4.14)$$

The inequalities (4.7) and (4.14) prove (i) of the Theorem.

On the other hand, we get (ii) of the theorem by passing to the limit when n tends to infinity in (4.13) and taking into account (i) of Theorem.

The proof of (iii) of the Theorem is the same as given in (Kaliagruine and Konorova, 2000). This achieves the proof of the Theorem.

REFERENCES

Aptekarev, A.I., 1986. Asymptotical properties of polynomials orthogonal on a system of contours and periodic motions of Toda lattice. Math. USSR. Sbornik, 53: 233-260.

Bello, M.H. and G.L. Lagomasino, 1998. Ratio and relative asymptotic of polynomials orthogonal on an arc on the unit circle. J. Approx. Theory. 92: 216-244.

Gasper, J.P. and F. Cyrot-Lackman, 1973. Density of state from moment Applications to the impurity band, J. Phys, Solid State Phy., 6: 3077-3096.

Gonchar, A.A., 1975. On convergence of Padé approximation for certain class of meromorphic functions. Mat Sbornik, 97: 4.

Heine, V., 1980. Electronic structure from the point of view of the local atomic environment, Solid State Phy., 34, Academic press, New York.

Kaliaguine, V.A., 1995. A note on the asymptotics of orthogonal polynomials on a complex arc: The case of a measure with a discrete part, J. Approx. Theory, 80: 138-145.

Kaliaguine, V.A. and A.A. Konorova, 2000. Strong asymptotics for polynomials orthogonal on a system of complex arcs and curves: Szego condition on and a mass points off the system, Publication du Laboratoire d'Analyse Numérique et d'Optimisation de Lille I, ANO 410, pp: 1-17.

Khaldi, R., 2004. Strong asymptotics for Lp extremal polynomials off a complex curve, J. Applied Math., 5: 371-378.

Khaldi, R., 2005. Strong asymptotics for Lp extremal polynomials off a circle. Demonstratio Mathematica, pp: 623-632.

Khaldi, R., 2004. Strong Asymptotics of Orthogonal Polynomials on the Segment [-1,1]: the Case of a Measure with Infinite Discrete Part. 5th International Conference on Functional Analysis and Approximation Theory. Acquafredda di Maratea. Potenza, Italy; June pp: 16-23.

Li, X. and K. Pan, 1994. Asymptotic behavior of class of orthogonal polynomials corresponding to measure with discrete part off the unit circle, J. Approx. Theory, 79: 54-71.

Marcellan, F. and P. Maroni, 1992. Sur l'adjonction d'une masse de Dirac à une forme régulière et semi-classique. Ann. Math. Pura Applied, 162: 1-22.

Nuttal, M. and S.R. Singh, 1977. Orthogonal polynomials and Padé" approximations associated with a systems of arcs, J. Approx. Theory 21, pp: 1-42

Peherstorfer, F. and P. Yuditskii, 2001. Asymptotics of orthonormal polynomials in the presence of a denumerable set of mass points, Pro. Amer. Math. Soc., 11: 3213-3220.

Rakhmanov, E.A., 1977. On the asymptotic of ratio of orthogonal polynomials I,, Math. USSRSb, 2: 199-213.

Rakhmanov, E.A., 1983. On the asymptotic of ratio of orthogonal polynomials II, Math. USSRSb, 46: 105-117

Rudin, W., 1968. Real and complex analysis, McGraw-Hill, New York.

Szegö, G., 1975. Orthogonal polynomials, 4th ed. American Mathematical Society, Colloquium Publications, American Mathematical Society, Rhode Island.

Widom, H., 1969. Extremal polynomials associated with a system of curves and arcs in the complex plane, Adv. Math., 3: 127-232.