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Effect of Lateral Conduction on the Efficiency of Rectangular Fins

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Abstract: A finite element numerical study has been conducted of two-dimensional heat conduction in a rectangular fin with a variable heat transfer coefficient (h_x) in the flow direction. The results show that the predicted fin efficiency with variable local heat transfer coefficient can be significantly lower than when a uniform h_x value is assumed. The fin efficiency is lower because of the spreading resistance associated with the lateral conduction of heat toward the leading edge. The effect of variable h_x was found to increase as the fin Length-to-Width ratio (L/W) decreased. For laminar boundary layer flow at L/W = 1/10, the fin efficiency with variable h_x was as much as 8.7% lower than with a uniform heat transfer coefficient. For a turbulent flow conditions, the effect of variable h_x was found to be less significant.

Key words: Fins, extended surfaces, variable heat transfer coefficient, two-dimensional, finite element method, laminar boundary layer

INTRODUCTION

A wide variety of extended surface problems have been solved with the assumption of constant heat transfer coefficient over the entire fin surface. However, it has been shown experimentally that for many practical applications this assumption can lead to a significant error in the predicted fin efficiency (Stachiewicz, 1969). In practice, the heat transfer coefficient usually varies from a low value near the fin base to a higher value toward the fin tip. The heat transfer coefficient also varies in the direction of fluid flow because of the developing thermal boundary layer. For forced convection, the heat transfer coefficient has a high value near the leading edge of the fin and decreases significantly in the direction of flow. The current study focuses on the effect that the variation in heat transfer coefficient in the flow direction has on fin efficiency.

The effect of the variable heat transfer coefficient along the length of the fin (base-to-tip) has received substantial attention in the literature. Kraus (1988) has extensively reviewed the experimental and numerical research on extended surfaces up to the late nineteen eighties, including a thorough review of the effects of variable heat transfer coefficient. Analytical solutions to one-dimensional steady conduction for various fin profiles have been obtain for several different classes of functions. For example, analytical solutions have been published for the heat transfer coefficient varying along the fin length as: (i) A power function (Han and Lefkowitz 1960; Barrow, 1986; Agwu Nnanna *et al*., 2002), (ii) An exponential (Chen and Zyskowski, 1963) and (iii) a polynomial (Evenko, 2002). All of these analytical solutions neglected the temperature variation across the thickness of the fin and the heat transfer coefficient was assumed to be temperature independent.

Several studies have also included the effect of fin thickness, along with a variable heat transfer coefficient from base to tip. Ma *et al*. (1991) solved the problem of two-dimensional heat conduction in a thick fin with variable heat transfer coefficient along its length using a Fourier series approach. More recently (Chu and Chang, 2002) used a numerical method to solve two-dimensional transient conduction in a thick round pin fin with variable heat transfer coefficient on the lateral and fin tip surfaces.

Numerous studies have also been conducted that considered a temperature dependent variable heat transfer coefficient. This problem is of particular importance for applications where the heat transfer mode is by free convection or boiling. Sparrow and Acharya (1981) have obtained a numerical solutions for conjugate free convection from a vertical rectangular fin. Also, analytical solutions have been obtained for one-dimensional conduction in various types of fins with the heat transfer coefficient varying as a power function of the local fin-to-ambient fluid temperature difference (Unal, 1985; Sen and Trinh, 1981; Dul'kin and Garas'ko, 2002a,b). Recently, (Mokheimer, 2002; 2003) considered one-dimensional heat transfer from a variety of fin profiles with a temperature dependent heat transfer coefficient using a finite difference method. The local heat transfer coefficient variation was obtained from correlations for natural convection.

The previous study most closely related to the current research was by Huang and Shah. In this detailed review study, Huang and Shah make a critical assessment of the error caused by many of the usual idealizations adopted in classical one-dimensional fin analysis, including the effect of variable h along the length of the fin. In addition to a thorough review of the literature, Huang and Shah also present new results of a numerical study on the effect of lateral conduction along the fin in the flow direction caused by a variable ambient temperature. Using a constant heat transfer coefficient, they solved the two-dimensional temperature field in the fin, accounting for the increase in the bulk fluid temperature from the inlet to the outlet of the fin array. They state in their conclusions that the longitudinal heat conduction effect on fin efficiency (η) is less than 1 % for η>10% and hence can be neglected. While true within the context of their study, this statement is potentially misleading. It will be shown in the current study that the issue of lateral conduction cannot be addressed without considering the effect of the variable heat transfer coefficient in the flow direction.

The purpose of the current study is to determine the effect of lateral conduction on the fin efficiency of rectangular fins subject to a variable heat transfer coefficient in the flow direction. The high convective heat transfer rate near the leading edge of the fin produces a lateral heat flow, resulting in lower fin efficiencies than when a constant heat transfer coefficient is assumed. The primary aim is to determine the range of conditions for which this span-wise conduction effect is significant.

PROBLEM DEFINITION

Figure 1 shows a sketch of the problem geometry. This study considers a rectangular fin of constant cross section with length L and width W. The fin surface is exposed to a flow in the x-direction, which has a constant ambient Temperature (T∞). Heat conduction in the rectangular fin is assumed to be steady and the thermal conductivity of the fin material (k_s) is assumed to be constant. It is further assumed that the Biot number based on the fin thickness is much less than one, such that the temperature gradients across the fin (z-direction) can be neglected. The variable heat transfer coefficient (h_x) is assumed to be only a function of the primary flow direction (x-coordinate) and identical on both sides of the fin. With these assumptions, the governing equation for the temperature field in the fin can be expressed as:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{2h_x}{k_s t}(T-T_\infty) \quad (1)$$

where k_s is the thermal conductivity of the fin. The following dimensionless variables are introduced:

$$\theta=\frac{T-T_\infty}{T_b-T_\infty},\quad X=\frac{x}{W},\quad Y=\frac{y}{W} \quad (2)$$

where T_b is the base temperature of the fin (at Y=0). In terms of these new variables, the governing equation becomes:

$$\frac{\partial^2 \theta}{\partial X^2}+\frac{\partial^2 \theta}{\partial Y^2}=\frac{2W^2 h_x}{k_s t}\theta \quad (3)$$

Defining the local Nusselt number as:

$$Nu_x=\frac{h_x x}{k_f} \quad (4)$$

where k_f is the conductivity of the fluid, Eq. 3 can be written as:

$$\frac{\partial^2 \theta}{\partial X^2}+\frac{\partial^2 \theta}{\partial Y^2}=\frac{2Nu_x}{k_R t^*}\frac{\theta}{X} \quad (5)$$

where k_R is the solid-to-fluid conductivity ratio ($k_R = k_s/k_f$) and t* is the dimensionless fin thickness (t* = t/W). From Eq. 5, it can be seen that the temperature field will be a function of the parameter $Nu_x/k_R t^*$, as well as the aspect ratio of the fin, L/W.

The temperature field depends upon the details of the variation of the local convection coefficient distribution in the flow direction. In the current study, external forced convection has been assumed. The local Nusselt number variation has been assumed to be independent of temperature and have a power law variation, given by:

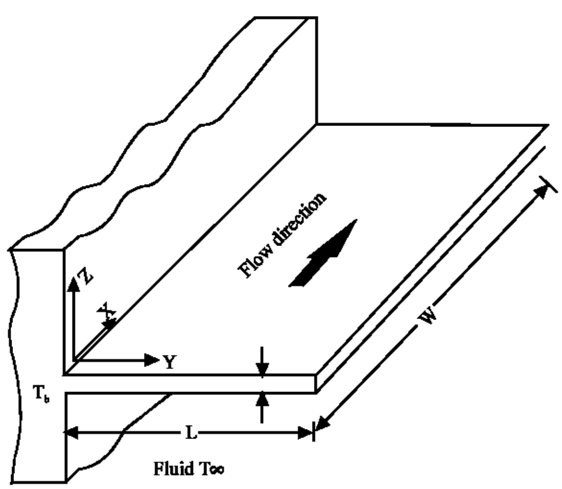

Fig. 1: Fin geometry and coordinate system

$$Nu_x = \frac{q\,x}{k_f(T - T_\infty)} = CRe_x^n Pr^p \qquad (6)$$

where q is the local convective heat flux from the fin surface. For a laminar forced convection boundary layer n = 1/2. For a turbulent forced convection boundary layer n = 4/5. Substituting Eq. 6 into Eq. 5 and noting that $Re_x = XRe_W$ gives:

$$\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2} = \frac{2CRe_W^n Pr^p}{k_R t^*}\frac{\theta}{X^{1-n}} \qquad (7)$$

For normal values of the exponent n associated with this application, the term on the right hand side of Eq. 8 is singular at X = 0. To deal with this numerical problem, the computational domain was split into two parts: Over the first 100th of the fin width ($0 \le \times \le 0.01$), a constant Nusselt number was applied, equal to the average Nusselt number from X = 0 to X = 0.01. Beyond X>0.01, the variable local Nusselt number was applied, as specified by Eq. 6. Tests were conducted to ensure that X = 0.01 was an appropriate division location which did not significantly affect the results.

To solve Eq. 7, boundary conditions are required at the fin edges. The fin base was assumed to be at a uniform Temperature (T_b) and convection from the exposed edges of the fin was neglected. With these assumptions the boundary conditions on Eq. 7 are:

$$\begin{aligned} &\theta = 1 \quad \text{for} \quad Y = 0 \quad 0 \le X \le 1 \\ &\frac{\partial\theta}{\partial X} = 0 \quad \text{for} \quad X = 0,1 \quad 0 \le Y \le L/W \\ &\frac{\partial\theta}{\partial Y} = 0 \quad \text{for} \quad Y = L/W, \quad 0 \le X \le 1 \end{aligned} \qquad (8)$$

Equation 7 has been solved numerically subject to the above boundary conditions using a standard Galerkin finite element method. Three-node triangular elements were used with linear interpolation functions for temperature. Results are presented in terms of fin efficiency. The fin efficiency results have been plotted as a function of the standard fin parameter, mL, defined as:

$$mL = \sqrt{\frac{\bar{h}P}{k_s A_c}}L = \sqrt{\frac{2\bar{h}}{k_s t}}L = \sqrt{\frac{2CRe_W^n Pr^p}{n k_R t^*}}\frac{L}{W} \qquad (9)$$

Validation and grid study: Prior to the study, the numerical procedure was validated using an analytical solution for one-dimensional conduction in fin with a variable heat transfer coefficient along its length. The following power law heat transfer coefficient variation from the base to tip was used:

$$h_y = (\gamma + 1)\bar{h}\left(\frac{y}{L}\right)^\gamma \qquad (10)$$

where $\bar{h}$ is the average heat transfer coefficient. For this power law variation, (Han and Lefkowitz, 1960) have shown that the fin efficiency is:

$$\eta = \left[\frac{(\gamma+2)^\gamma(\gamma+1)}{(mL)^{2\gamma+2}}\right]^{1/(2+\gamma)}\left[\frac{I_{(\gamma+1)/(\gamma+2)}(\beta)}{I_{-(\gamma+1)/(\gamma+2)}(\beta)}\right]\left[\frac{\Gamma((\gamma+1)/(\gamma+2))}{\Gamma((1/(\gamma+2))}\right] \qquad (11)$$

where I is a modified Bessel function of the first kind, G is the Gamma function and β is given by the expression:

$$\beta = \frac{2\sqrt{\gamma+1}}{\gamma+2}mL \qquad (12)$$

Figure 2 shows the comparison of the present numerical solution with this exact analytical solution for exponents of γ= 0, 1, 2, 4. The maximum difference between the exact

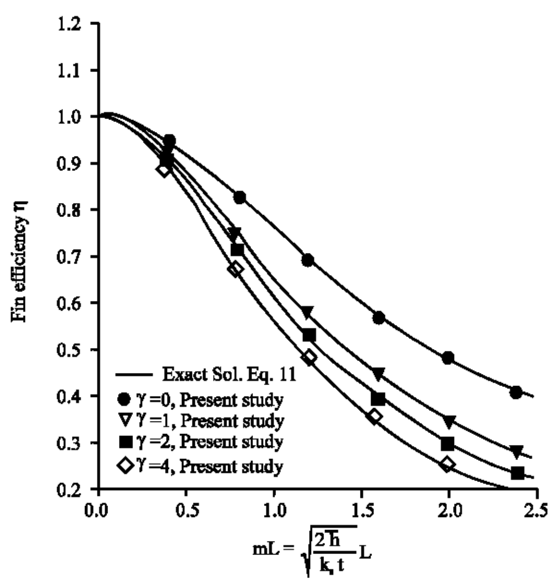

Fig. 2: Comparison of the numerically predicted fin efficiency with the exact solution (Eq. 11) for a fin with a power function variable heattransfer coefficient along its length

solution and the numerically predicted fin efficiency was 0.8%, which occurred at the lowest efficiency studied ($\eta \approx 0.3$).

A solution was also obtained for constant heat transfer coefficient at each fin aspect ratio. For constant heat transfer coefficient, the fin efficiency for a fin with an adiabatic tip can be expressed analytically as:

$$\eta_{\bar{h}} = \frac{\tanh(mL)}{mL} \quad (13)$$

Figure 3 shows the comparison of the numerically predicted fin efficiency and this analytical result. In all cases, the numerically predicted fin efficiency was agreed closely with that predicted by Eq. 13.

Extensive grid studies were done to test the dependence of the results on grid density. Tests were performed at low, mid-range and high fin efficiencies. The result of main interest is the percentage difference between the fin efficiency with constant heat transfer coefficient and the fin efficiency with variable heat transfer coefficient. The results presented in this study were obtained on a non-uniform mesh with approximately 11,500 nodes. With this mesh, the differences in the fin efficiencies with constant and variable heat transfer coefficients are estimated to be grid independent to better than 0.3%.

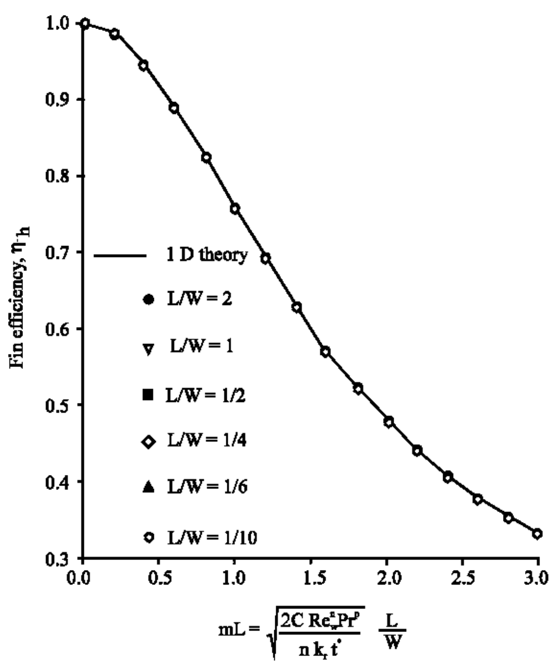

Fig. 3: Comparison of the analytical Eq. 13 and numerically predicted fin efficiency at each fin aspect ratio for constant heat transfer coefficient

RESULTS

Figure 4 shows the percentage difference between the fin efficiency with constant heat transfer coefficient ($\eta_{\bar{h}}$) and with variable heat transfer coefficient (η_{h_x}) as a function of the fin parameter mL. Calculations were done for fin aspect ratios of L/W = 2, 1, 1/2, 1/4, 1/6 and 1/10 assuming a laminar forced convection boundary layer (n = 1/2). In all cases, the fin efficiency with variable h_x was lower than the efficiency for constant h_x, because of the additional spreading resistance associated with the lateral flow of heat toward the leading edge. Note that in all cases, as the fin efficiency goes to 1.0 (i.e., when $mL \to 0$), the variable h_x has no effect on fin efficiency. This result is expected, since the fin's internal conductance approaches infinity and the heat can flow laterally (in the negative x-direction) with no thermal resistance. It can also be seen in Fig. 4 that the effect of the variable heat transfer coefficient increases with decreasing fin aspect ratio (L/W). As the fin width increases relative to its length, there is a further distance (and hence, more resistance) for heat to flow laterally toward the leading edge.

Figure 4 also shows that, at aspect ratios of L/W = 2 and 1, the effect of variable h_x increases monotonically as mL increases, up to the lowest fin efficiency considered in this study ($h \approx 30\%$). However, for fins with a lower aspect ratio the lateral conduction effect reaches a maximum at approximately mL = 1.5, which corresponds to a nominal fin efficiency of about 60%. For L/W = 1/10, the maximum difference between and was 8.7%.

The reason for the maxima in the curves for low aspect ratio may be explained by examining the

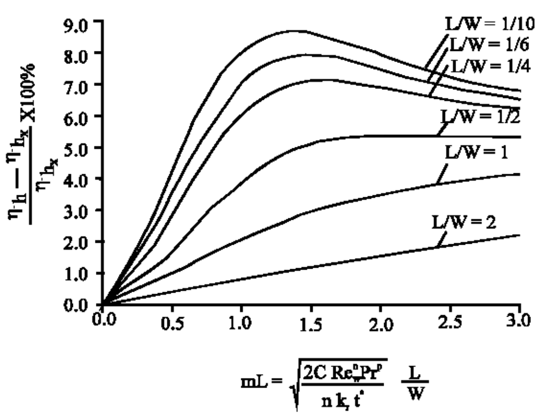

Fig. 4: Percentage difference in the fin efficiency with a constant and variable heat transfer coefficient for laminar boundary layer flow (n = 1/2)

temperature contour plots in Fig. 5. This figure shows the temperature field a fin with an aspect ratio of L/W = 1/4 for various values of mL with variable heat transfer coefficient (n = 1/2). It can be seen that for mL = 0.5, 1.0 and 1.5, the temperature field near the leading edge of the fin is strongly two-dimensional, especially near the fin tip. However, for mL = 3.0 the temperature field appears to be more one-dimensional than these other cases. The reason for the more one-dimensional behavior is low fin efficiency. As efficiency drops, the region of the fin far from the base plays less of a role in total heat transfer rate and the region closest to fin base dominates. The heat flow close to the fin base is almost one-dimensional, in the y-direction. So, for low fin aspect ratios the effect of variable h_x diminishes when the fin efficiency decreases below about 60%.

Figure 6 shows the comparison of effect of lateral conduction for laminar (n = 1/2) and turbulent (n = 4/5) boundary layer flow. Both sets of results are for a fin aspect ratio of L/W=1/4. It can be seen that the effect of spreading resistance is greatly reduced when there is a turbulent boundary layer. In fact, the difference between $\eta_{\bar{h}}$ and η_{h_x} is less than 1 % over the full range of fin efficiency. The effect of lateral heat flow on the fin efficiency is much lower for turbulent flow because the local heat transfer coefficient is much more uniform over the width of the fin. For laminar flow, the region near the leading edge contributes much more to the total heat transfer. For example, for laminar flow on an isothermal surface, the first ten percent of the fin width ($0 \le X \le 0.1$) contributes 32% of the total heat transfer. For turbulent flow, this same area contributes only about 16%. So, there is much less tendency for lateral heat flow within the fin.

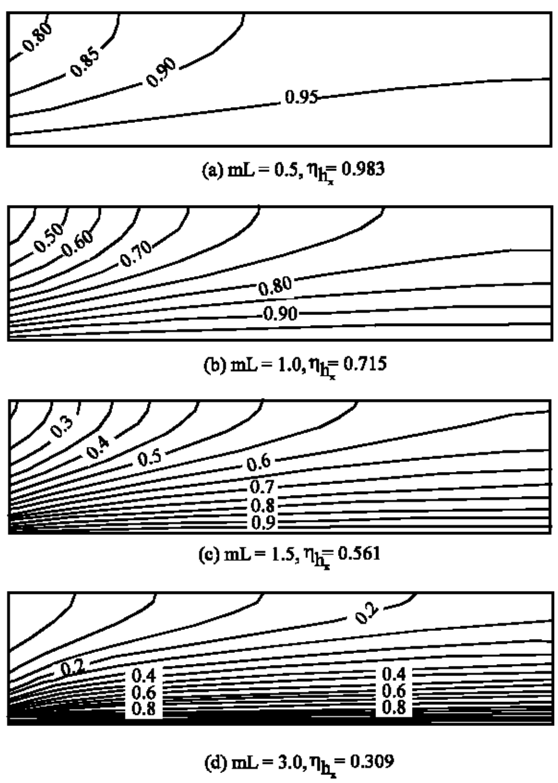

Fig. 5: Temperature contour plots showing the effect of the parameter mL for L/W=1/4, n = 1/2 (temperature contour interval $\Delta\theta$ = 0.05). The direction of flow is from left to right

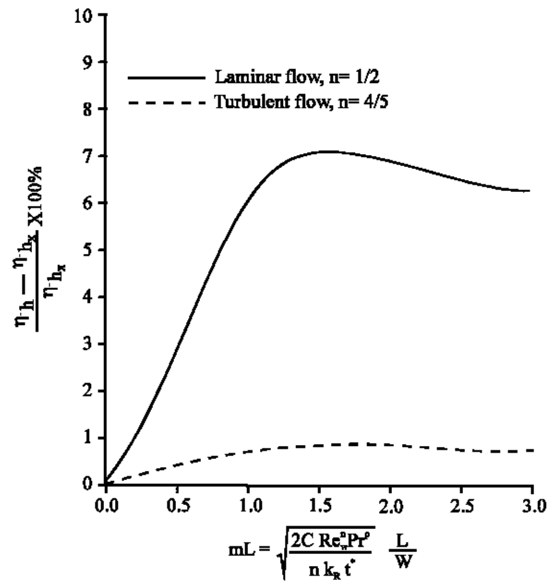

Fig. 6: Percentage difference in the fin efficiency with variable and constant heat transfer coefficient for laminar and turbulent boundary layer flow (L/W = 1/4)

CONCLUSION

A finite element numerical study has been conducted to examine the effect of lateral conduction on a constant cross section rectangular fin with a variable heat transfer coefficient in the direction of fluid flow. The results show that the fin efficiency with variable heat transfer coefficient can be significantly lower than with a uniform h value. The effect of variable h_x was found to increase as the fin Length-to-Width ratio (L/W) decreased, because there was more of a tendency for the lateral heat flow within the fin. For high values of fin aspect ratio L/W, the effect of variable h_x increased monotonically with decreasing fin efficiency. For lower values of aspect ratio, the effect of variable h reached a maximum at about mL = 1.5, which corresponds to a nominal fin efficiency of about 60%. For a local heat transfer coefficient distribution corresponding to an external laminar forced convection boundary layer, the fin efficiency was reduced by as much as 8.7% for L/W = 1/10, compared to when a uniform heat transfer coefficient was assumed. For a turbulent boundary layer, the effect of lateral conduction was found to be much smaller than for laminar flow conditions.

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