Journal of Engineering and Applied Sciences 2 (3): 619-624, 2007

Contribution to the Study of Jeffery-Hamel Flow

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Abtract: The stability of a perfect fluid flow was the subject of several studies. It is well known that in very many idustrial applications, the improvement of output of any machine crossed by a fluid requires a good knowledge of stability and instability zones of this fluid. The aim of this research is to contribute to the study of Jeffery-Hamel flow stability. Flow of an incompressible viscous fluid from a line source or sink at the intersection of the two rigid planes.

Key words: Flow, Navier-Stokes, Jeffery-Hamel, perturbation, stability, intersection

INTRODUCTION

Today it is well known that in many applications of industry (aerospace, chemical, civil and mechanical engineering) the output improvement of any machines crossed by a fluid requires a good knowledge of the stability and instability zones of this fluid in the machines pipes. Jeffery-Hamel flows are a family of exact solutions of the Navier-Stokes equations for steady two-dimensional flows of an incompressible viscous fluid from a line source or sink at the intersection of the two rigid planes. They were discovered by Jeffery (1915) and independently by Hamel (1916) and have extendively studied and discussed by several authors, e.g., Sternberg and Kolter (1958), Fraenkel (1962,1963), Lugt and Schwiderski (1965), Batchelor (1967), Allemen and Eagles (1984), Georgiou and Eagles (1985), Sobey and Drazin (1986), Banks *et al.* (1988), McAlpine and Drazin (1998), etc.

For our part, one tried to contribute to the study of the Jeffery-Hamel flow stability in accordance with the small perturbances. The (Fig. 1) represents the geometry of this flow; the movement is uniform along the Z axis and it is natural to suppose that it is purely radial, i.e.:

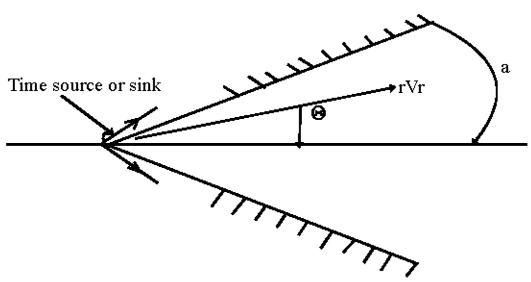

Fig. 1: The geometrical configuration of Jeffery-Hamel flows

$$\vec{v}\begin{cases} v_r = v(r,\theta) \\ v_\theta = 0. \\ v_z = 0. \end{cases} \quad (1)$$

Basic flow: Consider flow of an incompressible viscous fluid. Flow is governed by the cauchy equations:

$$\rho.\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla}).\vec{v}\right) = -\text{grad}\vec{P} + \mu.\Delta\vec{v} + (\lambda + 3\mu).\text{gra}\vec{d}.\text{div}\vec{v} \quad (2)$$

For Jeffery-Hamel flow, two-dimensional flow between two rigid planes, Eq. 2 is reduced to :

$$\left(\vec{v}.\vec{\nabla}\right).\vec{v} = -\frac{1}{\rho}\text{grad}\vec{P} + \nu.\Delta\vec{v} \quad (3)$$

ν : Kinematic viscosity of the fluid.
The radial velocity for J-H flow,Vr , is given by:

$$v_r = \frac{f(\theta)}{r}$$

where:
r: radial coordinate.
θ : azimuthal angle ($-\alpha < \theta < +\alpha$).

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The mass flux between the walls:

$$Q = \int_{-\alpha}^{+\alpha} r.v_r.d\theta = C^{te} \Rightarrow Q = Re.v.\int_{-1}^{+1} F(\theta^*,\alpha).d\theta^* \qquad (4)$$

$$\text{Where}: -1 \le \theta^* \le +1, \text{ with } \theta^* = \frac{\theta}{\alpha}$$

Reynolds number is defined by:

$$Re = Q / \nu \qquad (5)$$

We consider the dimensionless parameters:

$$F(\theta) = \frac{v_r}{v_{max}} = \frac{f(\theta)}{f(0)} \quad \text{and} \quad (\theta^*,\alpha) = \frac{f(\theta^*,\alpha)}{f(0,\alpha)}$$

Also, we can write:

$$\left.\begin{aligned} F'(\theta) &= \frac{1}{\alpha}.F'(\theta^*,\alpha.) \\ .F'(\theta) &= \frac{1}{\alpha}.F'(\theta^*,\alpha). \\ F'''(\theta) &= \frac{1}{\alpha^3}.F'''(\theta^*,\alpha) \end{aligned}\right\} \qquad (6)$$

Finally, by considering (5) and (6), we obtain:

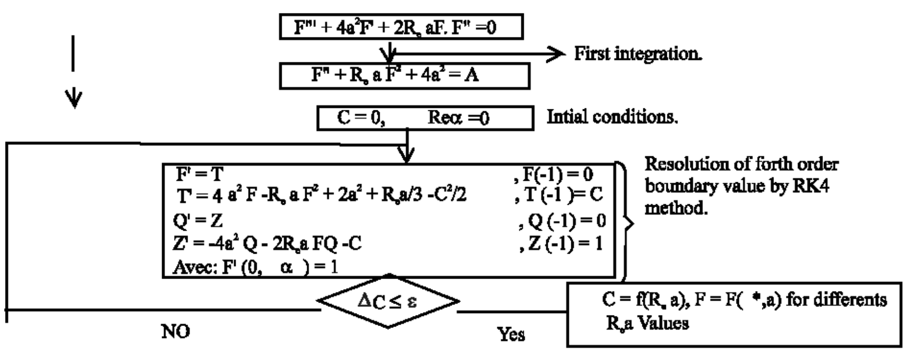
$$F'''(\theta^*,\alpha) + 4\,\alpha^2\,F'(\theta^*,\alpha) + 2\,R_e\,\alpha\,F(\theta^*,\alpha).F''(\theta^*,\alpha) = 0 \qquad (7)$$

With boundary conditions:

$F(\pm 1, \alpha) = 0$on $\theta^* = \pm 1$,

$F(0, \alpha) = 1$ and $F'(0, \alpha) = 0$at channel center ($\theta^* = 0$).

Thus we have a fourth order boundary-value problem. The resolution procedure of this problem is summarized in organigram 1.



Organigram 1-Numerical Evaluation of basic flow.

The C constant, which represents fluid-wall friction, is evaluated by:

$$\Delta C = C_{i+1} - C_i = \frac{(1 - F_0^i).\dfrac{\partial F_0^i}{\partial C} - T_0^i.\dfrac{\partial T_0^i}{\partial C}}{\left(\dfrac{\partial F_0^i}{\partial C}\right)^2 + \left(\dfrac{\partial T_0^i}{\partial C}\right)^2}$$

For the case of the convergent flow $(Q\langle 0)$, the function f(θ) being everywhere negative and varies from 0 for $\theta = \pm\alpha$ at the value - f(0) (f(0) >0) for θ=0. The convergent flow is symmetrical compared toθ = 0 (i.e.$f(0) = f(-\theta)$) and possible for any angle$\alpha\langle\pi$ and any Reynolds number (Fig. 2 and 3).

In the study of the divergent flow, (Q⟨0) the function f(θ) being everywhere positive and varies from 0 for $\theta = \pm\,\alpha$ at the value +f(0) (f(0) >0) for The divergent flow, everywhere symmetrical compared to is not possible, for the given angle, only for a Reynolds number, Re, not higher than a given limit. This is called return point or separation point (Fig. 4 and 5).In this point, flow changes direction. When the Reynolds number, Re, becomes large, the solution of the symmetrical divergent flow is not more legitimate and, it appears other symmetrical solutions with max and min velocities. These velocities increase as Re increases, which leads to a mode of turbulent flow.

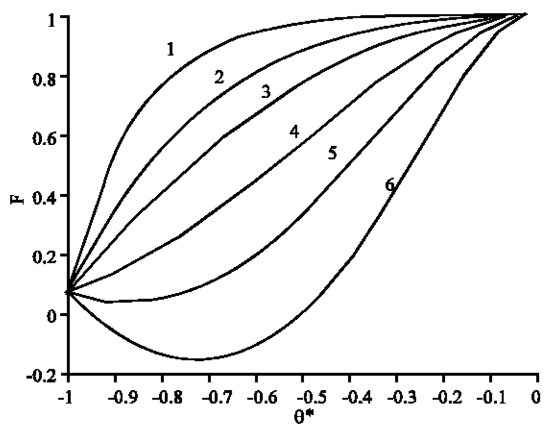


Fig. 2: Dimensionless profiles of velocity (α = 0.001) for different values of Reα. 1. Re α = -28, 2. Re α = -7, 3. Re α = 0, 4. Re α = +7, 5. Re α = +14, 6. Re α = +28.

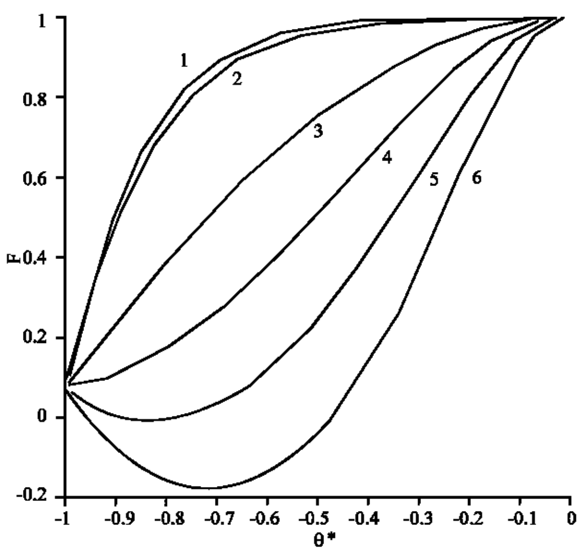


Fig. 3: Dimensionless profiles of velocity (α =0.001) for different values of Reα

Stability analysis:

The perturbation equation: The stability analysis is done by superimposing to the stationary solution, $v(r, \theta)$a time dependant perturbation, u (r, θ, t) such as the resulting movement is:

$$\begin{cases} v_r + u_r \\ v_\theta + u_\theta \\ p + \overline{p} \end{cases} \quad (8)$$

Where:

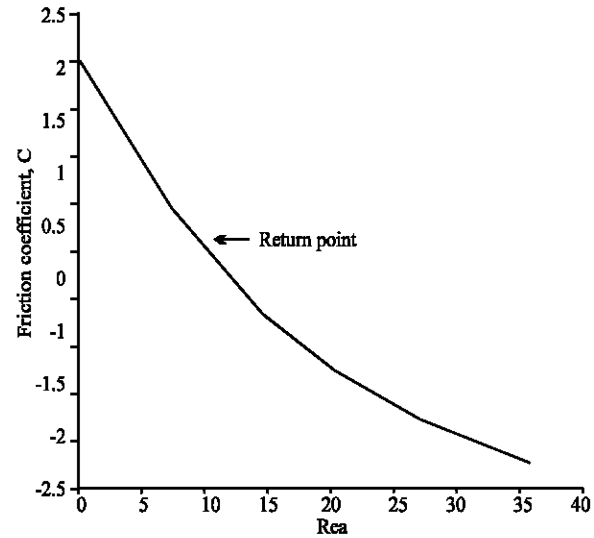


Fig. 4: Friction coefficient in divergent (=0.001) for different values of Re α channel (case of α = 0.001). $\alpha = \frac{\pi}{6}$ for different values of Re α, channel (case of $\alpha = \frac{\pi}{6}$) 1. Re α = -28, 2. Re α = -7, 3. Re α = 0, 4. Re α = +7, 5. Re α = +14, 6. Re α = +28. 79

v^r vθ and: Satisfied basic flow.
u_r, u_θ : Radial and azimuthal velocity perturbation and: $\overline{P}$ the pressure perturbation.
Substituting (8) in (3), we obtain:

$$\frac{\partial^2 u_r}{\partial\theta.\partial t} - r.\frac{\partial^2 u_\theta}{\partial r.\partial t} - \frac{\partial u_\theta}{\partial t} + v_r.\frac{\partial^2 u_r}{\partial\theta.\partial r} + \frac{\partial v_r}{\partial\theta}.\frac{\partial u_r}{\partial r}$$
$$+u_r.\frac{\partial^2 v_r}{\partial\theta.\partial r} + \frac{\partial u_r}{\partial\theta}.\frac{\partial v_r}{\partial r} + \frac{1}{r}.u_\theta.\frac{\partial^2 v_r}{\partial\theta^2} + \frac{1}{r}.\frac{\partial u_\theta}{\partial\theta}.\frac{\partial v_r}{\partial\theta} - r.v_r.\frac{\partial^2 u_\theta}{\partial r^2}$$
$$-r.\frac{\partial v_r}{\partial r}.\frac{\partial u_\theta}{\partial r} - 2.v_r.\frac{\partial u_\theta}{\partial r} - u_\theta.\frac{\partial v_r}{\partial r} = \nu.[\frac{\partial^3 u_r}{\partial\theta.\partial r^2} + \frac{1}{r}.\frac{\partial^3 u_r}{\partial\theta^3}$$
$$+\frac{1}{r}.\frac{\partial^2 u_r}{\partial\theta.\partial r} - \frac{2}{r^2}.\frac{\partial^2 u_\theta}{\partial\theta^2} - \frac{1}{r^2}.\frac{\partial u_r}{\partial\theta} - r.\frac{\partial^3 u_\theta}{\partial r^3} - 2.\frac{\partial^2 u_\theta}{\partial r^2} - \frac{1}{r}.\frac{\partial^3 u_\theta}{\partial r.\partial\theta^2}$$
$$+\frac{1}{r^2}.\frac{\partial^2 u_\theta}{\partial\theta^2} - \frac{2}{r}.\frac{\partial^2 u_r}{\partial r.\partial\theta} + \frac{2}{r^2}.\frac{\partial u_r}{\partial\theta} + \frac{1}{r}.\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}] \quad (9)$$

With boundary conditions: $u(r, \theta, t) = 0$, at $\theta = \pm\alpha$.

The radial and azimuthal velocities, u_r and u_θ, may be written in terms of generalized streamfunction Ψ as:

$$u_\theta = \frac{\partial\psi}{\partial r} \quad (10)$$

$$u_r = \frac{1}{r}.\frac{\partial\psi}{\partial\theta} \quad (11)$$

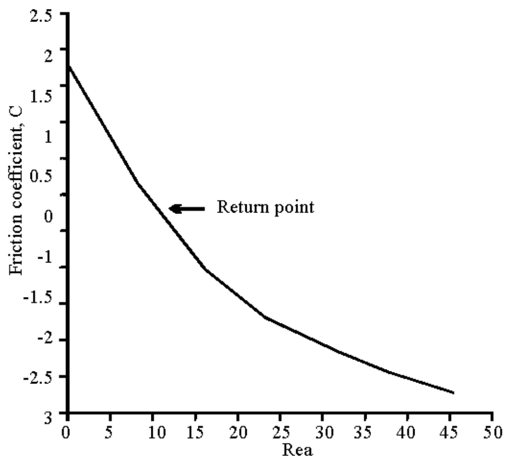


Fig. 5: Friction coefficient in divergent channel (case of $(\alpha = \frac{\pi}{6})$

Consider the new dimensionless variables:

$$\begin{cases} r^* = r/r2 \\ \psi^* = \psi/f(o) \end{cases} \qquad (12)$$

and knowing that $\theta^* = \theta/\alpha$, $f(\theta^*) = f(\theta^*,\alpha)/f(o, \alpha)$, $Re = f(o).\alpha/\nu$, by using the operator∇, we have:

$$\left.\begin{aligned} \psi^*{}_{r^*} &= \frac{\partial \psi^*}{\partial r^*}, \qquad (\nabla^2\psi^*)_{t^*} = \frac{\partial}{\partial r^*}(\nabla^2\psi^*) \\ \psi^*{}_{\theta} &= \frac{\partial \psi^*}{\partial \theta^*}, \qquad (\nabla^2\psi^*)_{r^*} = \frac{\partial}{\partial r^*}(\nabla^2\psi^*) \end{aligned}\right\} \qquad (13)$$

While considering (10), (11), (12) and (13), we obtain finally:

$$\begin{aligned} &(\nabla^2\psi^*)_{t^*} + \frac{Re.F(\theta^*,\alpha)}{\alpha.r^*}(\nabla^2\psi^*)_{r^*} - \\ &\frac{2.Re.F'(\theta^*,\alpha)}{\alpha^2 r^{*4}}\psi^*{}_{\theta} - \frac{Re.F''(\theta^*,\alpha)}{\alpha^3 r^{*3}}\psi^*{}_{r^*} = \nabla^4\psi^*) \end{aligned} \qquad (14)$$

with boundary conditions:

$$\psi^* = 0, \quad \begin{cases} \psi^*{}_{\theta} = 0 \\ \text{at}\, \theta = \pm\alpha \\ \psi^*{}_{r^*} = 0 \end{cases} \qquad (15)$$

The general solution of the linear differential Eq. 14 of the perturbation can be represented by the sum of the particular solutions where Ψ_* depends on time by the factor: $e^{\grave{u}.t^*}$, such as:

$$\Psi^*(t^*, r^*, \theta, \alpha, Re) = e^{\grave{u}.t^*}.\ \chi(r^*, \theta, \alpha, Re) \qquad (16)$$

Substituting (16) in (14), we obtain:

$$\begin{aligned} &\omega.\nabla^2\chi + \frac{Re.F(\theta^*,\alpha)}{\alpha.r^*}(\nabla^2\chi)r^* - \frac{2.Re.F'(\theta^*,\alpha)}{\alpha^2 r^{*4}} \\ &\chi\theta - \frac{Re.F''(\theta^*,\alpha)}{\alpha^3 r^{*3}}\chi r^* = \nabla^4\chi \end{aligned} \qquad (17)$$

with boundary conditions:

$$\chi = 0,\ \chi_\theta = 0 \text{ à } \theta = \pm\alpha \qquad (18)$$

(17) represents the reduced fourth order linear differential equation of the perturbation.

The case of an arbitrary reynolds number:

Very small kinematic viscosity $\nu << 1$: In this study, we consider the new dimensionless variables:

$$\hat{r} = r^* (-)½ \qquad (19)$$

Substituting (19) in (17), we obtain:

$$\begin{aligned} &\nabla^2\chi - \frac{Re.F(\theta^*,\alpha)}{\alpha.\hat{r}}(\nabla^2\chi)_{\hat{r}} + \frac{2.Re.F'(\theta^*,\alpha)}{\alpha^2 \hat{r}^4} \\ &\chi\theta + \frac{Re.F''(\theta^*,\alpha)}{\alpha^3 \hat{r}^3}\chi_{\hat{r}} = -\nabla^4\chi \end{aligned} \qquad (20)$$

The solution of the Eq. 20 may be written in the form:

$$\chi(\hat{r},\theta) = \sum_{i=0} \hat{r}^{(i+\lambda)}\Phi i(\theta) \qquad (21)$$

Where λ is an arbitrary constant.

Finally, it comes:

$$\begin{aligned} &\sum_i[(\Phi''_i + (i+\lambda)^2\Phi_i)_{i+2}(\frac{Re.F(\theta^*,\alpha)}{\alpha}. \\ &(i+\lambda)) + \Phi'_{i+2}.\frac{2.Re.F'(\theta^*,\alpha)}{\alpha^2} + \\ &\Phi_{i+2}.(i+2+\lambda).(\frac{Re.F''(\theta^*,\alpha)}{\alpha^3} - \frac{Re.F(\theta^*,\alpha)}{\alpha}. \\ &(i+\lambda).(i+2+\lambda))] = -\sum_i[\Phi^{(4)}{}_{i+2} + 2.\Phi''{}_{i+2}. \\ &(2+(i+\lambda)..(i+2+\lambda)) + \Phi_{i+2}(i+\lambda)^2\ (i+2+)^2]. \end{aligned} \qquad (22)$$

with boundary conditions,

$$\Phi(\theta) = 0,\ \Phi'(\theta) = 0 \text{ à } \theta = \pm\alpha \qquad (23)$$

It is the fourth order differential equation of small perturbation of the flow for any Reynolds number, Re, When $\nu\_ < < 1$, the second member of the Eq. 22 disappears and it comes:

$$\sum_i [(\Phi''_i + (i+\lambda)^2 \Phi i) - \Phi''_{1+2} (\frac{Re.F(\theta^*,\alpha)}{\alpha}. (i+\lambda)) + \Phi'_{1+2} . \frac{2.Re.F'(\theta^*,\alpha)}{\alpha^2} + \Phi_{1+2}.(i+2+\lambda)..(\frac{Re.F''(\theta^*,\alpha)}{\alpha^3} - \frac{Re.F(\theta^*,\alpha)}{\alpha}. (i+\lambda)(i+2+\lambda))] = 0. \quad (24)$$

So that the Eq. 24 adapts to the numerical processing, it is necessary to standardize, the function $\Phi(\theta)$, like its first and second derivative. The Eq. 24 becomes:

$$\Phi''i(\theta^*,\alpha) + \alpha^2 (i+\lambda)^2 \Phi i (\theta^*, \alpha) = \Phi''_{i+2}(\theta^*, \alpha) . \frac{Re.F(\theta^*,\alpha)}{\alpha}(i+\lambda) - \Phi_{i+2}(\theta^*,\alpha). \frac{2.Re.F'(\theta^*,\alpha)}{\alpha} - \Phi_{i+2}(\theta^*,\alpha)(i+2+\lambda). \frac{Re.F(\theta^*,\alpha)}{\alpha} - ReF(\theta^*,\alpha)\,\alpha(i+\lambda)(i+2+\lambda)) \quad (25)$$

$$\Phi(\theta^*, \alpha) = 0, \Phi'(\theta^*, \alpha) = 0 \text{ at } \theta^* = \pm \alpha \quad (26)$$

Consequently, it is necessary to solve this second order differential equation analytically by seeking its general solution.

The perturbation model: The combination between (10), (11), (16) and (21), with the dimensionless variable =r. $(-\omega)^{½}$, leads to:

$$\frac{u_\theta}{u_r} = -\frac{\sum_i (i+\lambda).\bar{r}^{(i+\lambda-1)}.\Phi_i(\theta^*,\alpha)}{\sum_i \bar{r}^{(i+\lambda-1)}.\frac{\Phi'_i(\theta^*,\alpha)}{\alpha}} \quad (27)$$

We choose, now, a model such as in point M (r=1, $\theta\_* = -\beta\_$) (divergent or convergent flow), the components $u_r = u_\theta$ (Fig. 6).

At point M, Eq. 27 may be expressed in the one dimensional form:

$$\sum_i (-\omega)^{½}[(i+\lambda)\,\Phi_i(\theta^*, \alpha) + \frac{\Phi'i(\theta^*,\alpha)}{\alpha}] = 0 \quad (28)$$

with, $(-\omega) = z^2$, we obtain finally:

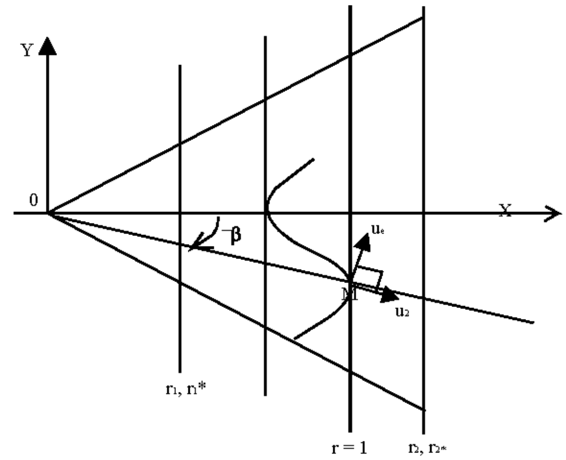


Fig. 6: Perturbation model

$$\sum_i z^i.[(i+\lambda)\Phi_i(\theta^*,\alpha) + \frac{\Phi'i(\theta^*,\alpha)}{\alpha}] = 0 \quad (29)$$

It is a polynomial of degree i (i=0, 1, 2...., N) where the unknown factor $(-z)^2$ which is "the frequency" of the perturbation and coefficients $[(i+\lambda\_).\Phi\_{i}\ (\theta\_{*}\alpha\_) + \frac{\Phi'i(\theta^*,\alpha)}{\alpha}]$ are complex (integral form).

Consequently, the studied flow will be stable or unstable according to whether the real part of $(-z^2)$ will be negative or positive. The resolution of the polynomial (29) is done by the bairstow method intended for resolving the algebraic equations with real coefficients. Once rendering the complex coefficients in real nature, we can apply the bairstow method.

RESULTS AND DISCUSSION

We observe as well in divergent channel as in convergent channel, that the values of frequencies of the amplification of the small perturbation (Table 1), superimposed to steady Jeffery-Hamel flow, have a coherent behavior.

On the one hand, it appears that a symmetrical flow $(Q\langle 0)$ in convergent channel, everywhere stable is possible for any angle $2\alpha\langle\pi$ and for any Reynolds number. So that, which is theoretically confirmed by the solutions of Navier-Stokes equations, with a very high Reynolds number $(Re \rangle\rangle 1)$, the Jeffery-Hamel flow would correspond to a no-viscous potential flow. On the other hand, it appears that a symmetrical flow in divergent channel, everywhere stable $(Q\rangle 0)$ is possible, for a given angle α, only for Reynolds number, Re, not superior at limiting

Table 1: Values of perturbation frequencies

Convergent channel		Divergent channel	
Reα	Frequencies ω	Reα	Frequencies ω
0.0000000E+00	-0.9978040E+00	-0.0000000E+00	-0.9978040E+00
0.3000000E-02	-0.1685628E+01	-0.3000000E-02	0.1686628E+01
0.6000000E-02	-0.1685062E+01	-0.6000000E-02	-0.1687284E+01
0.9000001E-02	-0.1684278E+01	-0.9000001E-02	0.1687869E+01
0.1200000E-01	-0.1683648E+01	-0.1200000E-01	-0.1688542E+01
0.1500000E-01	-0.1682995E+01	-0.1500000E-01	0.1689188E+01
0.1800000E-01	-0.1682365E+01	-0.1800000E-01	-0.1689714E+01
0.2100000E-01	-0.1682055E+01	-0.2100000E-01	0.1690272E+01
0.2400000E-01	-0.1681350E+01	-0.2400000E-01	0.1690915E+01
0.2700000E-01	-0.1680684E+01	-0.2700000E-01	-0.1691659E+01

value, Remax. The point (α, Re_{max}) is indicated like the return point (or inflection point of the flow) like illustrated in (Fig. 4 and 5). The solution in divergent flow thus does not tend, as for the convergent flow, towards the solution of the Euler equations. When the Reynolds number increase, the stationary flow in divergent channel becomes unstable beyond the limits value Re=Remax and it appears a solution for which the velocity has a max and min values. The number of minimum and maximum increases indefinitely in the time. Actually, it is the birth of an instationary flow (turbulent).

CONCLUSION

In this study one tried to contribute to the study of Jeffery-Hamel flow stability. With this intention, on the one hand we have studied dynamically this flow by determining his basic field. On the other hand, we have followed the evolution of the perturbation added to the stationary solution, to give an answer on the stability of Jeffery-Hamel flow. The process of the evaluation of the perturbation was carried out numerically for an arbitrary Reynolds number. In this stage of calculation, several numerical methods such as: fourth order Range Kutta method and Bairstow method were used.

Finally, to improve the obtained results by recording the inconsistency due to the many approximations brought to the methods of numerical processing of the perturbation evolution in the time and to give an answer on the J-H flow stability, it is necessary to solve the general equation of this perturbation in its two-dimensional form.

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