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Another Aspect of Duality in Linear Systems

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Abstract: In state space context, linear systems' duality is usually expressed in a particular form. Here, we showed that this concept of duality in a linear system transcends mere representation of the system in its dual form; rather, it is also manifested in some of the system's properties. Consequently, we examined 2 salient system's properties and established the manifestation of the duality concept between them.

Key words: State space, duality, controllability, observability, transition matrix, range space and null space

INTRODUCTION

Duality is concept whereby one true statement can be obtained from the other, by merely interchanging two words; though there may sometimes be the need to modify the language of the dual statement in order to make it clearer. Examples of some areas of mathematics where the duality concept has been applied are linear programming, projective geometry, symbolic logic, set theory, etc.

For instance, in projective geometry, it is an established fact that the class of all the points of a plane and the class of all the lines of the same plane are symmetrically related to each other. Thus, to every property of lines in the geometry of points, there corresponds a property of points in the geometry of lines. As a result, having proved a theorem in the geometry of points, we can immediately write down the corresponding theorem about lines by simply changing the word 'points' to 'lines' and vice versa, in the geometry of lines (Semple and Kneeborne, 1952).

In linear programming, for every linear programming problem there corresponds another linear programming problem associated with it which is called its dual. The optimal solutions to the problem and its dual are equivalent, but they are derived through alternative procedures. Thus, the relationship between the problems, once understood, automatically enables us to write down the solutions to both problems, if we have just solution to one of the problems (Bunday, 1984).

Generally, the duality concept is a mathematical trick to use a stone to kill two birds. Usually, what the mathematicians do is to solve the simpler of the problem and its dual while the duality concept is used to obtain the solution to the tougher one.

STATE SPACE SYSTEMS

As we know, state space linear systems are represented in the form below:

$$\left.\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned}\right\} \quad (1)$$

Where matrices A(t), B(t), C(t), D(t) are $n\times n$, $n\times l$, $m\times n$, $m\times l$ continuous functions of time, respectively. Also, x(t) denotes the state of the system, u(t) denotes the input (control) to the system and it is sectionally continuous in $[t_0, t_1]$ while y(t) denotes the output to the system at any point in time. It is important to note that the dynamics of the system above changes with time. Moreover, the first equation in (1) above is referred to as the state equation while the second equation in (1) is called the output equation (Rosenbrock, 1970; Rosenbrock and Storey, 1970).

However, duality in linear systems, as presented in many textbooks and research works, is often expressed as captured in the following statement:
A linear system represented as

$$\left.\begin{aligned}\dot{x}(t) &= Ax(t) + Cu(t)\\ y(t) &= Bx(t) + Du(t)\end{aligned}\right\} \quad (2)$$

is said to be controllable if and only if its dual system

$$\left.\begin{aligned}\dot{x}(t) &= A^T x(t) + C^T u(t)\\ y(t) &= B^T x(t) + D^T u(t)\end{aligned}\right\} \quad (3)$$

is observable and vice versa (Kailath, 1980; Towers, 1998).

As can be seen from above, the system representation in (1) seems to be more general than that in (2), since the former is time varying while the latter is time invariant. Obviously, findings from (1) will be more valuable and pragmatic than those in (2) because most of the systems in real life are time varying coupled with the fact that the dynamics of a system is a lot dependent on the relationship between the coefficients of the system's state and its control (i.e., x(t) and u(t)). Moreover, these findings are always valid for systems' represented in (2).

Nevertheless, there is still the need to show that the duality between system's controllability and system's observability is much deeper than this mere representation in the dual form in (2) and (3). This we shall establish in the sequel.

RESULTS AND DISCUSSION

The two important system's properties that will be considered in this study are system's controllability and observability. These properties are important because they play significant role in the determination of system's stability. As a reminder, we will give brief definitions of the two properties.

Definition 1: The state $x^{(0)}$at $t = t_0$ is said to be controllable at t_0 if there exists some $\tau > t_0$ and some control u(t) defined on $[t_0, \tau]$ such that the solution of

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{4}$$

which passes through $x^{(0)}$ at time t_0 later passes through the zero state ($x(t) = 0$ at $t = \tau$) for arbitrary $x^{(0)}$.

Definition 2: The system

$$y(t) = C(t)x(t) + D(t)u(t) \tag{5}$$

is said to be observable on the $[t_0, t_1]$, whenever we can obtain the initial state of the system $x^{(0)}$ given that we already know the u(t) and y(t) ; $\forall t \in [t_0, t_1]$ and arbitrary $x^{(0)}$.

Now, we will examine systems' controllability. To do this, let us consider a simple dynamical system of the form

$$\dot{z}(t) = B(t)u(t) \tag{6}$$

We can solve Eq. 6 in place of (4) since both has same solution based on the transformation $z(t) = \Phi(t_0, t)$ x(t) Then, when will the system (6) be controllable at t_0? In answering this question, let us integrate (6) to give

$$z(t) = z^{(0)} + \int_{t_0}^{t_1} B(\tau)u(\tau)d\tau \tag{7}$$

Suppose $z^{(0)}$ is controllable at t_0 then u(y) must be such that

$$\underline{0} = z^{(0)} + \int_{t_0}^{t_1} B(\tau)u(\tau)d\tau \tag{8}$$

for some $t_1 > t_0$

Due to the difficulty that may be encountered in solving the integral Eq. 8, we will interpret (8) in the vector space setting. Suppose the linear map $L : u(t) \rightarrow z$ is defined by

$$z = L(u) \equiv -\int_{t_0}^{t_1} B(\tau)u(\tau)d\tau \tag{9}$$

is a linear mapping of vector spaces. i.e., $L : C_S [t_0, t_1]^l \rightarrow R^n$ is linear.

In this context, denoting the range space of L by R(L), we observe that the statement (8) is equivalent to the statement that $z^{(0)} \in R(L)$. Therefore, $z^{(0)}$ is controllable at t_0 for the system (6) iff $z^{(0)} \in R(L)$ for some $t_1 > t_0$ Hence, the set of states that are controllable at t_0 on the time interval $[t_0, t_1]$ is exactly R(L). Thus, controllability deals with studying the range space of a linear map.

Now, we will consider system's observability in a similar way. We know from definition of observability that it has to do with finding the initial state $x^{(0)}$. However, the essential problem to be solved in determining $x^{(0)}$ from the given information is illustrated below:

To determine initial state $x^{(0)}$, let us adopt the variation of parameter formula in Eq. 5, we will have

$$y(t) = C(t)\Phi(t, t_0)x^{(0)} + \int_{t_0}^{t} C(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t) \tag{10}$$

Obviously, we know u(t) and the last two terms on the RHS of the above equation; if we incorporate these with the known term y(t) on the LHS, without loss of generality, the problem is reduced to determining $x^{(0)}$ from the equation

$$y(t) = C(t)\Phi(t, t_0)x^{(0)} \tag{11}$$

where y(t), C(t), $\Phi(t, t_0)$ are known. Therefore, the problem of determining the system's observability is reduced to solving Eq. 11.

Considering Eq. 11, it is obvious that it is only C(t) and A(t) (note that A(t) determines the transition matrix $\Phi(t, t_0)$) that contribute directly to the equation while the control terms play no role therein.

Consequently, for system's observability, we shall use the system equations of the form

$$\left.\begin{aligned}\dot{x}(t) &= A(t)x(t)\\ y(t) &= C(t)x(t)\end{aligned}\right\} \tag{12}$$

We will, then, progress the study of observability or equivalently the study of (11) by introducing the linear transform $L : R^n \to C[t_0, t_1]^m$ defined by

$$L(x^{(0)}) = H(t)x^{(0)} \equiv C(t)\Phi(t,t_0)x^{(0)} \tag{13}$$

Now, system (1) will be observable iff Eq. 13 determines a unique $x^{(0)}$ for the given y(t) and this will only be so if the map (13) is injective (i.e., 1-1). The condition for such map to be injective is that *Kernel of* L = $\{\underline{0}\}$. Hence, system's observability leads to the study of null space of a linear mapping.

As we can see from above, system's controllability relates to the range space of a linear map while system's observability deals with the null space of a linear map. Without doubt, it is an established fact that the null space is the dual of the range space. Consequently, the above thus shows the manifestation of the duality concept between system's controllability and system's observability.

Additional results: We need to characterize the range and null space of the respective linear maps. This will be established in the following lemmas:

Lemma 1:

$z \in R(L)$ iff $z \in R(W(t_0, t_1))$ where $W(t_0, t_1)$ is a linear map given by

$$W(t_0,t_1) = \int_{t_0}^{t_1} B(\tau)B^T(\tau)d\tau \tag{14}$$

Proof : If $z \in R(W(t_0, t_1))$ then there exists a vector $\eta \in \Re^n$ such that $W(t_0, t_1)\,\eta = z$.
Using a particular control $u_1(t) = -B^T\eta$, then

$$\begin{aligned}L(u_1) &= -\int_{t_0}^{t_1} B(\tau)u_1(\tau)d\tau\\ &= \int_{t_0}^{t_1} B(\tau)B^T(\tau)\eta d\tau\\ &= \{\int_{t_0}^{t_1} B(\tau)B^T(\tau)d\tau\}\eta\\ &= W(t_0,t_1)\eta\\ &= z\end{aligned}$$

Thus, $z \in R(L)$ as required.

For the converse, suppose that $z \notin R(W(t_0, t_1))$, then we must show that $z \notin R(L)$ Obviously, $W(t_0, t_1)$ is symmetric based on its definition. Then, using the dimension theorem for the linear mapping, it follows that every vector $z \in \Re^n$ has a unique decomposition of the form

$$z = z_1+z_2 \tag{15}$$

with $z_1 \in \ker(W(t_0, t_1))$ and $z_2 \in R(W(t_0, t_1))$

It follows that $W(t_0, t_1)\, z_1 = 0$ and that $\exists\ \eta \in \Re^n$ such that $z_2 = W(t_0, t_1)\eta$ Hence,

$$\begin{aligned}{z_1}^T z_2 &= {z_1}^T W(t_0,t_1)\eta\\ &= (W(t_0,t_1)z_1)^T\eta = 0\end{aligned} \tag{16}$$

Now, if $z \notin R(W(t_0, t_1))$ as assumed earlier, then in (15) we must have $z_1\ (\neq \underline{0}) \in \Re^n$. Thus, it follows from (16) that

$$\begin{aligned}{z_1}^T z &= {z_1}^T z_1 + {z_1}^T z_2\\ &= {z_1}^T z_1\\ &= \|z_1\|^2 \neq 0.\end{aligned} \tag{17}$$

Therefore, $z \in R(W(t_0, t_1))$ implies that there exists $z_1\ (\neq \underline{0}) \in \Re^n$. such that ${z_1}^T z \neq 0$ Consequently, there exists a control $u_1(t)$ defined on the $[t_0, t_1]$ such that

$$z = -\int_{t_0}^{t_1} B(\tau)u_1(\tau)d\tau$$

and it follows from (17) that

$$\begin{aligned}-\int_{t_0}^{t_1} (B^T(\tau)z_1)^T u_1(\tau)d\tau &= -\int_{t_0}^{t_1} {z_1}^T B(\tau)u_1(\tau)d\tau\\ &= -{z_1}^T\int_{t_0}^{t_1} B(\tau)u_1(\tau)d\tau\\ &= {z_1}^T z \neq 0.\end{aligned} \tag{18}$$

However, $z_1 \in \ker(W(t_0, t_1))$ and so ${z_1}^T W(t_0, t_1)\, z_1 = 0$.

$$\begin{aligned}&\Rightarrow \int_{t_0}^{t_1} {z_1}^T B(\tau)B(\tau)^T z_1 d\tau = 0,\\ &\Rightarrow \int_{t_0}^{t_1} \left\|B(\tau)^T z_1\right\|^2 d\tau = 0. \qquad \forall\tau \in [t_0,t_1]\\ &\therefore B(\tau)^T z_1 = 0.\end{aligned} \tag{19}$$

It is clear from the above that (19) contradicts (18). This implies that the assumption that $z \in R(L)$ must be false. Therefore, $z \notin R(L)$ as required.

Now, having proved a lemma relating to controllability, it becomes necessary to establish a similar lemma for observability. Below is the lemma:

Lemma 2: The null space the linear mapping L of (13) coincides with the n×n matrix $M(t_0, t_1)$ given by

$$M(t_0,t_1) = \int H(\tau)^T H(\tau)d\tau \qquad (20)$$

Proof: If $x \in \ker(M(t_0, t_1))$ then

$$0 = x^T M(t_0,t_1)x = \int_{t_0}^{t_1} x^T H(\tau)^T H(\tau)x d\tau$$

$$= \int_{t_0}^{t_1} \|H(\tau)x\|^2 d\tau$$

Hence, $H(t)x = \underline{0}\ \forall t \in [t_0, t_1]$

$\Rightarrow x \in \text{Ker}(L)$.

Conversely, $x \in \text{Ker}(L)$, then

$$H(t)x = \underline{0}\ \forall t \in [t_0,t_1]$$

$$\therefore H(t)^T H(t)x = \underline{0}\ \forall t \in [t_0,t_1]$$

Thus, it follows that

$$\int_{t_0}^{t_1} H(\tau)^T H(\tau)x d\tau = \left(\int_{t_0}^{t_1} H(\tau)^T H(\tau) d\tau\right)x = M(t_0,t_1)x = \underline{0}$$

Therefore, $x \in \ker(M(t_0, t_1))$ as required.

To further characterise the system's controllability, let us consider the below corollary:

Corollary: The system (4) is completely controllable iff, for some $t_1 > t_0$, Rank $W(t_0, t_1) = n$.

Proof: Complete controllability at t_0 requires that every $x^{(0)} \in \Re^n$ lies in $R(W(t_0, t_1))$

$$\Rightarrow \Re^n \subseteq R(W(t_0,t_1))$$

However, $R(W(t_0, t_1))$ is always a subspace of $\Re^n$

$$\Rightarrow R(W(t_0,t_1)) \subseteq \Re^n$$

Hence, $R(W(t_0, t_1)) = \Re^n$

Therefore,

$$\text{RANK}(W(t_0,t_1)) = \dim R(W(t_0,t_1)) = \dim \Re^n = n$$

Similarly, an analogous corollary exists for system's observability.

CONCLUSION

Based on the above established proofs, we have seen that system's controllability deal with the study of range space of a linear map while system's observability deals with the study of null space of a linear map which is a manifestation of duality between these two salient system's properties. Moreover, we also saw that for every statement about system's controllability there exists a corresponding statement about the system observability. Thus, having proved a statement for system's controllability, we can immediately write down the corresponding statement for system's observability by mere replacing the dual-sensitive terms as appropriate in the former statement.

Physically, when a system is controllable, it implies that a controller can be applied on the system such that it achieves the desired output almost immediately irrespective of the time interval. As for system's observability, since the kernel shrinks with time, it shows that the uncertainty surrounding the exact prediction of the initial state of the system reduces as the time interval of system's operation increases. Therefore, the larger the system's operational time interval, the more likely we will be able to predict the system initial state accurately.

Finally, duality in linear system is much deeper than the usual dual representation, rather it is also exhibited in some of the system's properties. In particular, we showed the manifestation of the duality concept in system's controllability and observability.

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