Journal of Engineering and Applied Sciences 3 (3): 246-260, 2008
ISSN: 1816-949X

Generalized Stochastic Perturbation Technique in Engineering Computations

Marcin Kamiñski
Chair of Mechanics of Materials, Technical University of Lódź,
Al. Politechniki 6, 93-590 Lódź, Poland

Abstract: The main aim of the study is to provide the generalized stochastic perturbation technique based on the classical Taylor expansion with a single random variable. The main problem discussed below is an application of this expansion to the solution of various partial differential equations with random coefficients by the fundamental numerical methods, i.e., Boundary Element Method, Finite Element Method as well as the Finite Difference Method. Since, nth order expansion is employed for this purpose, the probabilistic moments of the solution can be determined with *a priori* given accuracy. Contrary to the second order techniques used before, a perturbation parameter ε is also included in the relevant approximations, so the overall solution convergence can be speeded up by some modification of its value. Application of computational methodologies presented in transient problems (dynamics or heat transfer) are also commented in the study together with stochastic processes modelling by the double Taylor expansion.

Key words: Stochastic perturbation technique, Boundary Element Method, Finite Element Method, Finite Difference Method

INTRODUCTION

Stochastic computational techniques have been implemented until now using various theoretical and computational methods, i.e., stochastic spectral approaches (Ghanem and Spanos, 1991; Honda, 2005), some Monte-Carlo simulation techniques (Hurtado and Barbat, 1998) as well as many numerical realizations of the perturbation technique (Kamiñski, 1999, 2001, 2005, 2006; Kleiber and Hien, 1992; Liu *et al*., 1986). As it is known (Kleiber and Hien, 1992; Oden *et al*., 2005; Babuška *et al*., 2005), the usage, precision and computational implementation algorithms strongly depend on the input random fields types, their correlations, interrelations between the first few probabilistic moments as well as numerical technique used to solve the basic deterministic problem. Some of those issues are discussed here in terms of Gaussian input included in the boundary value problem with random parameters solved by the stochastic perturbation method, which is based on nth order Taylor expansion with the single random variable (Nayfeh, 1973). It should be noticed that the assumption that the random input has normal distribution is not justified by the experimental statistics on most of engineering design and material parameters (Bendat and Piersol, 1971). Whereas, elasticity modulus, heat conductivity (Manolis and Shaw, 1996) or geometrical parameters (Papadopoulos and Papadrakakis, 2005) usually exhibits Gaussian character limited to the non-negative values only, then Poisson ratio is restricted to quite narrow interval around 0 value; strength parameters are approximated frequently using Weibull or the other non-symmetric probability density functions; the other non-Gaussian random variables are employed in non-linear stochastic dynamics, for instance (Falsone, 2005). Quite separate problem is correlation or its lack in material and/or geometrical parameters on higher order random response of the structure, which was also studied before by the Stochastic Finite Element Method (SFEM), Stochastic Boundary Element Method (SBEM) and Stochastic Finite Difference Method (SFDM) implemented according to the second order expansions (Kamiñski, 1999, 2001; Kleiber and Hien, 1992).

The main motivation behind an application of the generalized perturbation technique is to eliminate the restriction on the input second probabilistic moments to be smaller than 0.15 (Kleiber and Hien, 1992; Liu *et al*., 1986) and impossibility of reliable computations of higher than the second probabilistic moments for the output. Since, mathematical derivation and computational implementation for the basic numerical methods is quite similar, fundamental necessary equations for the generalized versions of the SBEM, SFEM and SFDM are compared below. Let us note here that the convergence of the generalized version of the SFEM was discussed using symbolic computations of up to 10th order approximations for the expected values and standard deviations with deterministic solution for some elementary flow and statics problems with linear 2-noded 1D stochastic finite elements (Kamiñski, 2006; Kamiñski and Carey, 2005).

The second reason to contrast those methods is that Finite, Boundary Element as well as Finite Difference Methods have quite different areas of application in computational engineering. The Finite Difference Method (Chou, 1991; Kunz and Luebbers, 1993; Mitchell and Griffiths, 1980; Taflove, 1998; Wang and Anderson, 1995) and the SFDM (Kamiñski, 2001; Chatterjee and Poggie; Grzywiñski and Sluzalec, 2000) are mainly applied in electromagnetics, electrodynamics, heat transfer and groundwater flow modelling as well as in geophysics. Since, the computational domain is discretized by deterministic regular or irregular grids, therefore SFDM is the most efficient in modelling of randomness in physical properties and natural/essential boundary conditions. The Boundary Element Method (Beer, 2001; Berger and Tewary, 1997; Brebbia and Dominguez, 1996) and the SBEM (Honda, 2005; Kamiñski, 1999; Manolis and Shaw, 1996; Burczyñski, 1995; Cheng and Lafe, 1991) are used most frequently in mechanical and geotechnical problems solutions but, contrary to the remaining methods, are extremely efficient in numerical analysis of cracks as well as of randomness in boundary shape and/or its variations. The FEM and the SFEM (Babuška *et al*., 2005; Kamiñski and Carey, 2005; Kleiber and Hien, 1992; Keese and Matthies, 2005; Liu *et al*., 1986; Zienkiewicz, 1971) were traditionally widely employed in computational mechanics from elastostatics to nonlinear dynamics. Considering probabilistic computations aspects the SFEM is recommended in modelling of random material properties as well as in the external both static and dynamic structural loads.

It must be underlined that an application of the stochastic perturbation technique in BEM, FEM and FDM are demonstrated for simplicity here in terms of static equilibrium for a homogeneous domain. However, it can be shown that its implementation in elastodynamics (Kleiber and Hien, 1992) or transient heat transfer problems proceed quite similarly (Kamiñski and Carey, 2005; Minkowycz *et al*., 1998). Finally, a common stochastic extension of the fundamental numerical methods is made taking into account computational issues. Since, up to nth order systems of linear equations are solved then both parallelisation procedures (Keese and Matthies, 2005) and also the other solution and accuracy topics (Papadrakakis, 1993) may appear quite analogous.

The presentation starts here from a general description of the theoretical aspects of the generalized nth order stochastic perturbation technique based on the Taylor series representation with random coefficients. Immediately after this part, the application of this technique in the conjunction with the traditional Finite Element Method (FEM) is shown. The utilized variational principles of the nth order together with the algebraic equations adequate for the Stochastic Finite Element Method are formulated and discussed. Although, they are valid for the simple elliptic boundary value problems, one can relatively easy use this strategy to modify deterministic transient or nonlinear problems for their stochastic expansions. Further, the main topic is a usage of the Boundary Element Method (BEM) for a solution of boundary value problems with random coefficients. This methodology known as the Stochastic Boundary Element Method (SBEM) is provided here to demonstrate, analogously to the previous section, a solution by the discrete numerical technique for some engineering problems with a single random parameter where complete knowledge on their fundamental probabilistic moments and coefficients is supported. Although, the SBEM fits perfectly the needs of such problems with random geometry, those issues are not raised in this presentation. The second last section contains an implementation of the Finite Difference Method in the context of the Taylor expansion with random parameters. It should be underlined that the generalized stochastic perturbation technique is still under initial development so that its only the few illustrative applications with the FEM or BEM have been published until now without wider comparison studies against the other methods. The generalized Stochastic Finite Difference Method is still not explored very well and, therefore, the comparison of all of those methodologies may steer the next discoveries in the field of stochastic computational mechanics.

THEORETICAL ASPECTS OF THE GENERALIZED STOCHASTIC PERTURBATION TECHNIQUE

Let us introduce the random field b(x) and its probability density function as p(b). Then, the first 2 probabilistic moments of this field are defined as (Bendat and Piersol, 1971; Feller, 1967; Vanmarcke, 1983):

$$E[b] \equiv b^0 = \int_{-\infty}^{+\infty} b\, p(b)\, db, \qquad (1)$$

and

$$\mathrm{Cov}\left(b(x_r), b(x_s)\right) \equiv$$

$$S_{rs} = \int_{-\infty}^{+\infty} \left[b(x_r) - b^0(x_r)\right]\left[b(x_s) - b^0(x_s)\right] p(b)\, db \qquad (2)$$

The basic idea of the stochastic perturbation approach is to expand all the input variables and all the state functions of the given problem via Taylor series about their spatial expectations using some small parameter ε>0. In case of random quantity b = e, the following expression is employed (Kamiñski, 2006; Kamiñski and Carey, 2005)

$$e = e^0 + \sum_{n=1}^{\infty} \tfrac{1}{n!}\varepsilon^n \frac{\partial^n e}{\partial b^n}(\Delta b)^n, \quad (3)$$

Where,

$$\varepsilon \Delta b = \varepsilon\left(b - b^0\right) \quad (4)$$

is the first variation of the variable b about its expected value and with

$$\varepsilon^2 (\Delta b)^2 = \varepsilon^2 \left(b - b^0\right)^2 \quad (5)$$

denoting the second variation of b about b^0. Symbol $(.)^0$ represents the function value (.) taken at the expectation b^0, while up to nth order partial derivatives with respect to b are evaluated at b^0. Let us analyse further the expected values of any state function f(b) defined analogously to the formula (3) by its expansion via Taylor series as follows:

$$\begin{aligned} E\left[f(b);b\right] &= \int_{-\infty}^{+\infty} f(b)\, p(b)\, db \\ &= \int_{-\infty}^{+\infty} \left(f^0 + \sum_{n=1}^{\infty} \tfrac{1}{n!}\varepsilon^n f^{(n)} \Delta b^n\right) p(b)\, db \end{aligned} \quad (6)$$

Let us remind that this power expansion is valid only if the state function is analytic in ε and the series converge. Therefore, any criteria of convergence should include the magnitude of the perturbation parameter; perturbation parameter is taken as equal to 1 in numerous practical computations (Kamiñski and Carey, 2005; Kleiber and Hien, 1992). Contrary to the previous analyses in that area, now the quantity ε is treated as the parameter in further analysis, so that it is included explicitly in all further derivations demanding analytical expressions.

From the numerical point of view, the expansion provided by the formula (6) is carried out for the summation over the finite number of components. Considering various probability distributions, the essential difference is noticed between symmetric distribution functions, where,

$$E\left[f(b);b\right] = f^0 + \int_{-\infty}^{+\infty} \left(\sum_{n=1}^{2M} \tfrac{1}{(2n)!}\varepsilon^{2n} \frac{\partial^{2n} f}{\partial b^{2n}} \Delta b^{2n} \right) p(b)\, db, \quad (7)$$

and non-symmetric probability functions

$$E\left[f(b);b\right] = f^0 + \int_{-\infty}^{+\infty} \left(\sum_{n=1}^{N} \tfrac{1}{(n)!}\varepsilon^{n} \frac{\partial^{n} f}{\partial b^{n}} \Delta b^{n} \right) p(b)\, db \quad (8)$$

In both cases, the natural quantities N, M must guarantee a satisfactory accuracy of approximation for the additional probabilistic moments. It can be done for the expected values and the variances introducing the following statistical error measures:

- For the expectations

$$\forall_{\delta_1 \in \Re^+} \exists_{N_1 \in N} \left| E\left[f_{N_1}(b)\right] - E\left[\tilde{f}(b)\right] \right| \le \delta_1 \quad (9)$$

- For the variances

$$\forall_{\delta_2 \in \Re^+} \exists_{N_2 \in N} \left| Var\left(f_{N_2}(b)\right) - Var\left(\tilde{f}(b)\right) \right| \le \delta_2 \quad (10)$$

The real positive numbers δ_1 and δ_2 denote the admissible errors for determination of expectations and the variances. Natural quantities N_1 and N_2 correspond to the orders of perturbation resulting in a desired accuracy; maximum of these 2 numbers fulfils satisfactory accuracy conditions. Of course, the exact solutions in both above equations are extract from statistical approximation (Bendat and Piersol, 1971) as:

$$E\left[\tilde{f}(b)\right] = \lim_{L\to\infty} \frac{1}{L} \sum_{i=1}^{L} \tilde{f}_i(b), \quad (11)$$

together with

$$Var\left(\tilde{f}(b)\right) = \lim_{L\to\infty} \frac{1}{L-1} \sum_{i=1}^{L} \left(\tilde{f}_i(b) - E\left[\tilde{f}(b)\right]\right)^2, \quad (12)$$

where, L is the total number of random trials used to compute the estimators for the random function f(b). As it is seen, sufficiently accurate modelling of the moments by the perturbation technique needs initial Monte-Carlo simulation to determine the necessary order of the perturbation for a given problem.

Let us focus now on analytical derivation of basic probabilistic moments for some structural response function. According to Eq. (6) it yields for the input random variable with symmetric probability density function in the second order perturbation approach:

$$
\begin{aligned}
E\left[f(b);b\right] &= \int_{-\infty}^{+\infty} f(b)\,p(b)\,db \\
&= \int_{-\infty}^{+\infty} \left(f^{0} + f^{,b}\Delta b + \tfrac{1}{2} f^{,bb}\Delta b\Delta b\right) p(b)\,db \\
&= f^{0}(b) + 0\times \varepsilon f^{,b}(b) + \tfrac{1}{2}\varepsilon^{2} f^{,bb}(b) S_{bb}
\end{aligned}
\tag{13}
$$

This expected value can be calculated or symbolically computed only if it is given as some analytical function of the random input parameter b. Many existing models in various branches of engineering can be adopted to achieve this goal. Computational implementation of the symbolic calculus programs, combined with powerful visualization of probabilistic output moments, assures the fastest solution of those problems.

If higher order terms are necessary (because of a greater random deviation of the input random variable about its expected value), then the following extension can be proposed

$$
\begin{aligned}
&E\left[f(b);b\right] = f^{0}(b) + \tfrac{1}{2}\varepsilon^{2} f^{,bb}(b)\mu_{2}(b) + \\
&\tfrac{1}{4!}\varepsilon^{4} f^{,bbbb}(b)\mu_{4}(b) + \tfrac{1}{6!}\varepsilon^{6} f^{,bbbbbb}(b)\mu_{6}(b) + \ldots
\end{aligned}
\tag{14}
$$

where, $\mu_n(b)$ denotes nth order central probabilistic moment of the quantity b and where all terms with the odd orders are equal to 0 for the Gaussian random deviates; higher than the 6th order terms are neglected here. Thanks to such an extension of the random output, any desired efficiency of the expected values as well as higher probabilistic moments can be achieved by an appropriate choice of the parameters m and ε corresponding to the input probability density function (PDF) type, relations between the probabilistic moments, acceptable error of the computations etc. This choice can be made by comparative studies with long (almost infinite) series Monte-Carlo simulations or theoretical results obtained from the direct symbolic integration. Similar considerations lead to the 6th order expressions for a variance (Kamiñski, 2006; Kamiñski and Carey, 2005). There holds

$$
\begin{aligned}
\mathrm{Var}(f) = \int_{-\infty}^{+\infty} \Big(& f^{0} + \Delta b f^{,b} + \tfrac{1}{2}(\Delta b)^{2} f^{,bb} + \tfrac{1}{3!}(\Delta b)^{3} \\
& f^{,bbb} + \tfrac{1}{4!}(\Delta b)^{4} f^{,bbbb} + \tfrac{1}{5!}(\Delta b)^{5} f^{,bbbbb} - E[f]\Big)^{2} \\
& p(f(b))db
\end{aligned}
\tag{15}
$$

Hence,

$$
\begin{aligned}
\mathrm{Var}(f) \cong & \int_{-\infty}^{+\infty} (\Delta b)^{2} \left(f^{,b}\right)^{2} p(f(b))db + \\
& \int_{-\infty}^{+\infty} \tfrac{1}{4}(\Delta b)^{4} \left(f^{,bb}\right)^{2} p(f(b))db + \\
& \int_{-\infty}^{+\infty} \Delta b f^{,b}\, \tfrac{1}{3!}(\Delta b)^{3} f^{,bbb} p(f(b))db + \\
& \int_{-\infty}^{+\infty} \Delta b f^{,b}\, \tfrac{1}{3!}(\Delta b)^{3} f^{,bbb} p(f(b))db + \\
& \int_{-\infty}^{+\infty} \tfrac{1}{3!}(\Delta b)^{3} f^{,bbb}\, \tfrac{1}{3!}(\Delta b)^{3} f^{,bbb} p(f(b))db + \\
& \int_{-\infty}^{+\infty} \tfrac{1}{4!}(\Delta b)^{4} f^{,bbbb}\, \tfrac{1}{2}(\Delta b)^{2} f^{,bb} p(f(b))db + \\
& \int_{-\infty}^{+\infty} \tfrac{1}{4!}(\Delta b)^{4} f^{,bbbb}\, \tfrac{1}{2}(\Delta b)^{2} f^{,bb} p(f(b))db + \\
& \int_{-\infty}^{+\infty} \tfrac{1}{5!}(\Delta b)^{5} f^{,bbbbb} \Delta b f^{,b} p(f(b))db + \\
& \int_{-\infty}^{+\infty} \tfrac{1}{5!}(\Delta b)^{5} f^{,bbbbb} \Delta b f^{,b} p(f(b))db
\end{aligned}
\tag{16}
$$

As it can be recognized from Eq. (16), the first integral corresponds to the second order perturbation, the next three complete 4th order approximation and the rest needs to be included to achieve full 6th order expansion. After multiple integration and indices transformations, one can show that

$$
\begin{aligned}
\mathrm{Var}(f(b)) = \; & \mu_{2}(b) f^{,b} f^{,b} + \mu_{4}(b) \\
& \left(\tfrac{1}{4} f^{,bb} f^{,bb} + \tfrac{2}{3!} f^{,b} f^{,bbb}\right) + \mu_{6}(b)\times \\
& \left(\left(\tfrac{1}{3!}\right)^{2} f^{,bbb} f^{,bbb}\right) + \mu_{6}(b)\times \\
& \left(\tfrac{1}{4!} f^{,bbbb} f^{,bb} + \tfrac{2}{5!} f^{,bbbbb} f^{,b}\right)
\end{aligned}
\tag{17}
$$

Let us mention that it is necessary to multiply each of these equations by the relevant order probabilistic moments of the input random variables or fields to get the algebraic form convenient for any symbolic computations. Because of a great complexity of such a solution, the second order perturbation approach is usually preferred. Recursive derivation of the particular perturbation order equilibrium equations can be powerful in conjunction with symbolic packages with automatic differentiation tools only; it can extend the area of stochastic perturbation technique applications in computational physics and engineering outside the random processes with small dispersion about their expected values (Kleiber and Hien, 1992; Kunz and Luebbers, 1993).

Quite similar expansion can be proposed as a function of the perturbation parameter ε, the perturbation order as well as the input random variable b:

$$E\left[f(b); b, \varepsilon, m\right] = f^0(b) + \tfrac{1}{2}\varepsilon^2 \frac{\partial^2 f}{\partial b^2}\mu_2(b) + \tfrac{1}{4!}\varepsilon^4 \frac{\partial^4 f}{\partial b^4}\mu_4(b) + \tfrac{1}{6!}\varepsilon^6 \frac{\partial^6 f}{\partial b^6}\mu_6(b) + \ldots + \tfrac{1}{(2m)!}\varepsilon^{2m} \frac{\partial^{2m} f}{\partial b^{2m}}\mu_{2m}(b) \quad (18)$$

for any natural m with μ_{2m} being the ordinary probabilistic moment of 2 mth order. Quite similarly we can proceed to extract the third and higher probabilistic moments of the state functions. Let us see the result for the 10th order stochastic Taylor expansion. There holds for the third probabilistic moment:

$$\mu_3(f(b)) = \int_{-\infty}^{+\infty} \left(\sum_{i=1}^{n} \frac{\varepsilon^i}{i!} \Delta b^i \frac{\partial^i f}{\partial b^i} - E[f(b)] \right)^3 p(b)db$$
$$= \int_{-\infty}^{+\infty} \begin{pmatrix} f^0(b) + f^{,b}\varepsilon\Delta b + f^{,bb}\varepsilon^2 \dfrac{(\Delta b)^2}{2!} \ldots \\ \dfrac{\partial^n f(b)}{\partial b^n}\varepsilon^n \dfrac{(\Delta b)^n}{n!} - E[f(b)] \end{pmatrix}^3 p(b)db$$
$$\cong \int_{-\infty}^{+\infty} \left(f^{,b}\varepsilon\Delta b + f^{,bb}\varepsilon^2 \frac{(\Delta b)^2}{2!} \ldots \frac{\partial^{10} f(b)}{\partial b^{10}}\varepsilon^{10} \frac{(\Delta b)^{10}}{10!} \right)^3 p(b)db \quad (19)$$

Analogously, we can recover the recursive formula for the mth order probabilistic moment as:

$$\mu_m(f(b)) = \int_{-\infty}^{+\infty} \left(\sum_{i=1}^{n} \frac{\varepsilon^i}{i!} \Delta b^i \frac{\partial^i f}{\partial b^i} - E[f(b)] \right)^m p(b)db$$
$$= \int_{-\infty}^{+\infty} \begin{pmatrix} f^0(b) + f^{,b}\varepsilon\Delta b + f^{,bb}\varepsilon^2 \dfrac{(\Delta b)^2}{2!} \ldots \\ \dfrac{\partial^n f(b)}{\partial b^n}\varepsilon^n \dfrac{(\Delta b)^n}{n!} - E[f(b)] \end{pmatrix}^m p(b)db$$
$$\cong \int_{-\infty}^{+\infty} \left(f^{,b}\varepsilon\Delta b + f^{,bb}\varepsilon^2 \frac{(\Delta b)^2}{2!} \ldots \frac{\partial^{10} f(b)}{\partial b^{10}}\varepsilon^{10} \frac{(\Delta b)^{10}}{10!} \right)^m p(b)db \quad (20)$$

It is necessary to point out that this methodology is valid for a single random variable with any probability density function; further simplifications may be obtained by a specification of this PDF. This methodology will be essentially changed in the case of random field as well as of two and more correlated random variables.

VARIATIONAL FORMULATION AND THE STOCHASTIC FINITE ELEMENT METHOD EQUATIONS

Now the goal is to formulate and to implement the so-called the generalised SFEM (GSFEM), where practically any order of the state parameters vector can be determined with an accuracy given a priori. Let us consider a statistically homogeneous and bounded continuum $\Omega \subset \Re$ without free of the initial stresses and strains. Elastic properties and geometry of Ω may be treated as design random parameters and they result in random displacement field u_i (x; ω) and random stress tensor σ_{ij} (x; ω) satisfying the classical boundary-value problem typical for the linear elastostatics. Let us assume that there are non-empty subsets of external boundaries of Ω, namely $\partial\Omega_t$ and $\partial\Omega_u$, where the stress and displacement boundary conditions are defined. The boundary-differential equation system describing this equilibrium problem can be written as follows:

$$\sigma_{ij}(x;\omega) = C_{ijkl}(x;\omega)\, \varepsilon_{kl}(x;\omega), \quad (21)$$

$$\varepsilon_{ij}(x;\omega) = \tfrac{1}{2}\left(u_{i,j}(x;\omega) + u_{j,i}(x;\omega)\right), \quad (22)$$

$$\sigma_{ij,j}(x;\omega) + \rho f_i = 0, \quad (23)$$

$$u_i(x;\omega) = \hat{u}_i(x;\omega);\ x \in \partial\Omega_u, \quad (24)$$

$$\sigma_{ij}(x;\omega)\, n_j = t_i(x;\omega);\ x \in \partial\Omega_t \quad (25)$$

Where,

$$C_{ijkl}(x;\omega) = \delta_{ij}\delta_{kl} \frac{e(x)\nu(x)}{(1+\nu(x))(1-2\nu(x))} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \frac{e(x)}{2(1+\nu(x))} \quad (26)$$

for a = 1, 2,..., n and I, j, k, l = 1, 2. Generally, the equation system posed above is solved using the well-established numerical methods. However, it should be transformed first to the variational formulation. It yields:

$$\int_{\Omega} C_{\alpha\beta\gamma\delta} u_{\alpha,\beta} \delta u_{\gamma,\delta} d\Omega = \int_{\Omega} \rho f_\alpha \delta u_\alpha d\Omega + \int_{\partial\Omega} \tilde{t}_\alpha \delta u_\alpha d(\partial\Omega) \tag{27}$$

where, the left hand side of Eq. (27) corresponds to elastic behaviour of the structure, the first component on the right side includes the body forces effects, while the last one is equivalent to the stress boundary conditions applied. Thus, the stochastic version of the minimum potential energy principle has the following form:

- 0th order equation:

$$\int_{\Omega} \delta u_{i,j} C^0_{ijkl} u^0_{k,l} d\Omega = \int_{\Omega} \rho^0 f^0_i \delta u_i d\Omega + \int_{\partial\Omega_t} \delta u_i \tilde{t}^0_i d(\partial\Omega) \tag{28}$$

- 1st order equation:

$$\int_{\Omega} \delta u_{i,j} C^0_{ijkl} u^{,b}_{k,l} d\Omega = \int_{\Omega} \left(\rho^{,b} f^0_i + \rho^0 f^{,b}_i\right) \delta u_i d\Omega + \int_{\partial\Omega_t} \delta u_i \tilde{t}^{,b}_i d(\partial\Omega) - \int_{\Omega} \delta u_{i,j} C^{,b}_{ijkl} u^0_{k,l} d\Omega \tag{29}$$

- 2nd order equation:

$$\int_{\Omega} \delta u_{i,j} C^0_{ijkl} u^{,bb}_{k,l} d\Omega = \int_{\partial\Omega_t} \delta u_i \tilde{t}^{,bb}_i d(\partial\Omega) - \int_{\Omega} \delta u_{i,j} (2C^{,b}_{ijkl} u^{,b}_{k,l} + C^{,bb}_{ijkl} u^0_{k,l}) d\Omega + \int_{\Omega} \left(\rho^{,bb} f^0_i + 2\rho^{,b} f^{,b}_i + \rho^0 f^{,bb}_i\right) \delta u_i d\Omega \tag{30}$$

- Nth order equation

$$\int_{\Omega} \delta u_{i,j} C^0_{ijkl} \frac{\partial^N u_{k,l}}{\partial b^N} d\Omega = \int_{\partial\Omega_t} \delta u_i \frac{\partial^N \tilde{t}_i}{\partial b^N} d(\partial\Omega) - \int_{\Omega} \delta u_{i,j} \sum_{k=1}^{N} \binom{n}{n-k} C^{,k}_{ijkl} u^{,n-k}_{k,l} d\Omega + \int_{\Omega} \sum_{k=0}^{N} \binom{n}{n-k} \rho^{,k}_{ijkl} f^{,n-k}_{k,l} \delta u_i d\Omega \tag{31}$$

If, for instance, Young modulus of Ω is considered as random variable of the problem, then first partial derivatives of elasticity tensor with respect to this variable are derived as:

$$A_{ijkl}(x) = C^{,b}_{ijkl}(x;\omega) = \delta_{ij}\delta_{kl} \frac{\nu(x)}{(1+\nu(x))(1-2\nu(x))} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \frac{1}{2(1+\nu(x))} \tag{32}$$

they are all deterministic quantities. Further, all higher order partial derivatives of this tensor with respect to the same variable are equal to 0. Neglecting the body forces effects and eliminating all partial derivatives of the stress boundary conditions, we can write down the fundamental equations of the problem as:

- 0th order equation:

$$\int_{\Omega} \delta u_{i,j} C^0_{ijkl} u^0_{k,l} d\Omega = \int_{\partial\Omega_t} \delta u_i \tilde{t}^0_i d(\partial\Omega) \tag{33}$$

- 1st order equation:

$$\int_{\Omega} \delta u_{i,j} C^0_{ijkl} u^{,b}_{k,l} d\Omega = -\int_{\Omega} \delta u_{i,j} A_{ijkl} u^0_{k,l} d\Omega \tag{34}$$

- 2nd order equation:

$$\int_{\Omega} \delta u_{i,j} C^0_{ijkl} u^{,bb}_{k,l} d\Omega = -2\int_{\Omega} \delta u_{i,j} A_{ijkl} u^{,b}_{k,l} d\Omega \tag{35}$$

- Nth order equation

$$\int_{\Omega} \delta u_{i,j} C^0_{ijkl} \frac{\partial^N u_{k,l}}{\partial b^N} d\Omega = -(N-1)\int_{\Omega} \delta u_{i,j} A_{ijkl} u^{,N-b}_{k,l} d\Omega \tag{36}$$

These equations are implemented next using the Finite Element Method to solve some basic elastostatics problem illustrating basic results of the technique involved. Let us introduce for this purpose the following approximation for the displacement field and its nth order partial derivatives with respect to the input random variable using the shape functions $\varphi_{i\alpha}(x)$:

$$u^0_i(x) = \varphi_{i\alpha}(x) \cdot q^0_\alpha, \quad x \in \Omega, \tag{37}$$

$$\frac{\partial^n u_n(x)}{\partial b^n} = \varphi_{i\alpha}(x) \frac{\partial^n q_\alpha}{\partial b^n}, \quad x \in \Omega, \tag{38}$$

i = 1, 2; r, s = 1,..., R; α = 1,..., N_α (N_α is the total number of degrees of freedom introduced in Ω) and n = 1,…, N. The strain tensor components are discretised analogously. It yields:

$$\varepsilon_{ij}^{0}(x)=B_{ij\alpha}(x)\cdot q_{\alpha}^{0};\ x_k\in\Omega \tag{39}$$

$$\frac{\partial^n\varepsilon_{ij}(x)}{\partial b^n}=B_{ij\alpha}(x)\cdot\frac{\partial^n q_\alpha}{\partial b^n};\ x_k\in\Omega \tag{40}$$

where, $B_{ij\alpha}(X_k)$ is the matrix containing the shape functions derivatives

$$B_{ij\alpha}(x)=\tfrac{1}{2}\left[\varphi_{i\alpha,j}(x_k)+\varphi_{j\alpha,i}(x_k)\right],\quad x_k\in\Omega, \tag{41}$$

The FEM approach is obtained as the linear algebraic equations system

$$K_{\alpha\beta}q_\beta=Q_\alpha, \tag{42}$$

where, q_β is the solution vector. When some random quantities are inserted into the matrix $K_{\alpha\beta}$ and the vector Q_α, then Eq. (42) should be rewritten and solved to determine the first consecutive orders of the random structural response. It yields:

- 0th order equations

$$K_{\alpha\beta}^{0}q_\beta^{0}=Q_\alpha^{0}, \tag{43}$$

- 1st order equations

$$K_{\alpha\beta}^{0}q_\beta^{,b}=Q_\alpha^{,b}-K_{\alpha\beta}^{,b}q_\beta^{0}, \tag{44}$$

- 2nd order equations

$$K_{\alpha\beta}^{0}q_\beta^{,bb}=Q_\alpha^{,bb}-2K_{\alpha\beta}^{,b}q_\beta^{,b}-K_{\alpha\beta}^{,bb}q_\beta^{0}, \tag{45}$$

- Nth order equations

$$\sum_{k=0}^{N}\binom{N}{k}K_{\alpha\beta}^{(k)}q_\beta^{(N-k)}=Q_\alpha^{(N)} \tag{46}$$

Let us here recall classical definition of the stiffness matrix in the form of

$$K_{\alpha\beta}=\sum_{e=1}^{E}\int_{\Omega_e}C_{ijkl}^{(e)}B_{ij\alpha}B_{kl\beta}d\Omega \tag{47}$$

where, $C_{ijkl}^{(a)}$ denotes the elasticity tensor components for the finite element 'a' and E stands for the total number of finite elements in Ω. Thus, we can describe the stiffness matrix nth order derivatives with respect to Young modulus 'e' as

$$\frac{\partial K_{\alpha\beta}}{\partial e}=\frac{\partial C_{ijkl}}{\partial e}\sum_{e=1}^{E}\int_{\Omega_e}B_{ij\alpha}B_{kl\beta}d\Omega=A_{ijkl}\sum_{e=1}^{E}\int_{\Omega_e}B_{ij\alpha}B_{kl\beta}d\Omega, \tag{48}$$

and

$$\frac{\partial^n K_{\alpha\beta}}{\partial e^n}=0 \tag{49}$$

for any n≥2. Furthermore, it is seen that all partial derivatives of external load vector are equal to 0. Therefore, it yields

- For 0th order equations

$$K_{\alpha\beta}^{0}q_\beta^{0}=Q_\alpha^{0}, \tag{50}$$

- For Nth order equations

$$K_{\alpha\beta}^{0}\frac{\partial^N q_\beta}{\partial e^N}=-(N-1)K_{\alpha\beta}^{,e}\frac{\partial^{N-1}q_\beta}{\partial e^{N-1}}. \tag{51}$$

Finally, from the first equation of this system zeroth order solution is determined, which, inserted into the next equations, returns first order solution etc. until nth order solution is completed. After all the solution vector components are determined, their expected values, variances and other probabilistic moments can be extracted. It should be underlined that in most of engineering applications, the state function or the state vector are not Gaussian variables. It leads to the necessity of computing higher and especially odd probabilistic moments and can proceed in a similar way.

THE STOCHASTIC BOUNDARY ELEMENT METHOD BASED ON THE PERTURBATION METHOD

Let us consider now a linear elastic body occupying the region $\Omega\subset\Re^2$ with boundary $\Gamma\equiv\partial\Omega$ and assume that Γ is the Lyapunov surface. Next, let us consider the body forces vector $f=(f_i)$ with surface loadings $p=(p_i)$, which results in a displacement field u(x), strain tensor ε(x) and stresses tensor σ(x). It is well-known that the set of equations posed above can be replaced with the following equivalent integral formulation (Beer, 2001; Brebbia and Dominguez, 1996; Burczyński, 1995)

$$c(x)\, u(x) + \int_{\Gamma} P(x,y)\, u(y)\, d\Gamma(y) = = \int_{\Gamma} U(x,y)\, p(y)\, d\Gamma(y) + \int_{\Omega} U(x,z)\, b(z)\, d\Omega(z) \qquad (52)$$

Where,

$$P(x,y) = \left[P_y\ U(y,x)\right]^T \qquad (53)$$

and P is the stress vector operator. If x ∈ Int(Ω) then c = I and the formulation (52) is known as the Somigliano integral formula. When x∈Γ then we obtain boundary integral equation with c equal to:

$$c(x) = I + \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} P(x,y)\, d\Gamma(y) \qquad (54)$$

where, Γ_ε denotes a sphere with center in point x and radius equal to ε. Generally, coefficient c may take the values

$$c = \alpha\, I\, ;\ \alpha \in (0,1) \qquad (55)$$

α = 0.5 is put for the boundary regular in Lyapunov sense. To obtain a stochastic formulation of the problem we provide a deterministic discretization by the boundary elements. Let us introduce a discretization of the boundary Γ of linear elastic body in the boundary elements Γ^e such that

$$\Gamma = \sum_{e=1}^{K} \Gamma^e \qquad (56)$$

The displacement field u(x) and forces p(x) can be approximated on any element Γ^e in local coordinates $\bar{\xi} = (\xi_i)$ by the use of nodal values $(u)^w_e$ and $(p)^w_e$ as follows:

$$u(x(\xi)) \approx N^w(\xi)\,(u)^w_e\, ;\ p(x(\xi)) \approx \bar{N}^w(\xi)\,(p)^w_e \qquad (57)$$

Where,

$$N^w(\xi) = I\, N_w(\xi),\ \bar{N}^w(\xi) = I\, \bar{N}_w(\xi) \qquad (58)$$

are the shape functions, while $N_w(\xi)$ and $\bar{N}_w(\xi)$ are the interpolation functions. Introducing all these statements into Eq. (52) it is obtained that:

$$c(x)\, u(x) + \sum_{e=1}^{K}\sum_{w=1}^{W_e} (u)^w_e \int_{\Gamma^e} P(x, y(\xi))\, N^w(\xi)\, J(\xi)\, d\Gamma(\xi) = \sum_{e=1}^{K}\sum_{w=1}^{W_e} (p)^w_e \int_{\Gamma^e} U(x, y(\xi))\, N^w(\xi)\, J(\xi)\, d\Gamma(\xi) + B(x) \qquad (59)$$

Where,

$$B(x) = \int_{\Omega} U(x,z)\, b(z)\, d\Omega(z) \qquad (60)$$

and the Jacobian J(ξ) is given as:

$$J(\xi) = \frac{d\Gamma(y)}{d\xi} = \left[\left(\frac{\partial x_k}{\partial \xi}\right)^2\right]^{\frac{1}{2}},\ \text{for } k=1,2,3 \qquad (61)$$

Next, let us introduce the global numbering of boundary nodal points x^β, β = 1, 2,..., W, where, W is the total number of nodes introduced. Thus Eq. (61) can be rewritten as:

$$r(x) = \sum_{\beta=1}^{W} \left[P^\beta(x)\, u_\beta - U^\beta(x)\, p_\beta\right] - B(x) = 0 \qquad (62)$$

with

$$P^\beta(x) = c(x) + \sum_m \int_{\Gamma^m} P(x, y(\xi))\, N^\beta(\xi)\, J(\xi)\, d\Gamma(\xi) \qquad (63)$$

$$U^\beta(x) = \sum_m \int_{\Gamma^m} U(x, y(\xi))\, \bar{N}^\beta(\xi)\, J(\xi)\, d\Gamma(\xi) \qquad (64)$$

The unknown values of displacements u(x) and forces p(x) can be found by r(x) minimization using the following condition:

$$\int_{\Gamma^e} T^\alpha(x)\, r(x)\, d\Gamma(x) = 0 \qquad (65)$$

where, $T^\alpha(x) = IT^\alpha(x)$, x∈$\Gamma^e$, e = 1,2,..., K is the weighted function matrix. In the case of collocation this matrix has the form

$$T^\alpha(x) = I\, \delta(x - x^\alpha) \qquad (66)$$

where, x^α are collocation nodes. If the collocation nodal points are equivalent to the boundary nodes $x = x^\alpha$, α = 1, 2,..., W then the minimization condition (65) is given as:

$$r(x^{\alpha}) = 0 \tag{67}$$

Therefore, it is obtained that

$$c_{\alpha\beta}u_{\beta} + \mathcal{H}_{\alpha\beta}u_{\beta} = G_{\alpha\beta}p_{\beta} + B_{\alpha} \tag{68}$$

where,

$$\mathcal{H}_{\alpha\beta} = \sum_{e} \int_{\Gamma_e} P\left[x^{\alpha}, y(\xi)\right] N^{\beta}(\xi)\, J(\xi)\, d\Gamma(\xi) \tag{69}$$

and

$$G_{\alpha\beta} = \sum_{e} \int_{\Gamma_e} U\left[x^{\alpha}, y(\xi)\right] \bar{N}^{\beta}(\xi)\, J(\xi)\, d\Gamma(\xi) \tag{70}$$

Introducing the following definitions

$$H_{\alpha\beta} = H_{\alpha\beta} \text{ for } \alpha \neq \beta; \alpha, \beta = 1, \ldots, W \tag{71}$$

$$H_{\alpha\beta} = H_{\alpha\beta} + c_{\alpha} \text{ for } \alpha = \beta; \alpha, \beta = 1, \ldots, W \tag{72}$$

we finally derive the fundamental set of the BEM algebraic equations

$$H_{\alpha\beta}u_{\beta} = G_{\alpha\beta}p_{\beta} + B_{\alpha} \;\; \alpha, \;\; \beta = 1, \ldots, W \tag{73}$$

It can be shown that Eq. (73) can be transformed to the classical linear algebra formulation as follows:

$$A_{\alpha\beta}X_{\beta} = F_{\alpha}; \; \alpha, \;\; \beta = 1, \ldots, W \tag{74}$$

which is a starting point for the stochastic formulation provided afterwards. Further, in the discretization of the deterministic problem, the following decomposition is used

$$A \equiv \begin{bmatrix} H^{11} & -G^{11} \\ H^{21} & -G^{21} \end{bmatrix}, \; F \equiv \begin{bmatrix} -H^{11} & G^{12} \\ -H^{21} & G^{22} \end{bmatrix} \begin{Bmatrix} u^1 \\ u^2 \end{Bmatrix} + \begin{Bmatrix} B^1 \\ B^2 \end{Bmatrix} \tag{75}$$

where matrices H^{ij}, G^{ij}, u^{i} and B^{i} for i, j = 1, 2 are submatrices of H, G, u and B matrices equivalent to boundary segments Γ_i, i = 1,2. The column matrix of body forces B is calculated numerically over the region Ω

$$B_a \equiv B(x_a) = \sum_{w=1}^{W} f(y_w) \int_{\Omega^w} U\left(x^a, y\right) d\Omega(y) \tag{76}$$

where, for simplicity function f(y) has the constant value inside the cell Ω^w. Next, the appropriate orders of column matrix F are calculated as:

$$F \equiv \begin{bmatrix} -H^{11} & G^{12} \\ -H^{21} & G^{22} \end{bmatrix} \begin{Bmatrix} u^1 \\ p^2 \end{Bmatrix} + \begin{Bmatrix} B^1 \\ B^2 \end{Bmatrix} \tag{77}$$

To complete this, the subvectors of F are represented by

$$\begin{bmatrix} F^{(1)} \\ F^{(2)} \end{bmatrix} \equiv \begin{bmatrix} -H^{11} & G^{12} \\ -H^{21} & G^{22} \end{bmatrix} \begin{Bmatrix} u^1 \\ p^2 \end{Bmatrix} + \begin{Bmatrix} B^1 \\ B^2 \end{Bmatrix}, \tag{78}$$

where,

$$\begin{aligned} F^{(1)} &= -H^{11}u^1 + G^{12}p^2 + B^1 \\ F^{(2)} &= -H^{21}u^1 + G^{22}p^2 + B^2 \end{aligned} \tag{79}$$

Having solution of Eq. (74) with definitions (75), the displacements in the interior of the domain Ω are calculated as

$$\begin{aligned} u(x) &= G(x)\,p - H(x)\,u + B(x) \quad ; \; x \in \Omega \\ s(x) &= G_D(x)\,p - H_S(x)\,u + B_D(x)\,; \; x \in \Omega \end{aligned} \tag{80}$$

where, matrices G(x), H(x), $G_S(x)$ and $H_D(x)$ generally depend on boundary integrals with nuclei U(x,y), P(x,y), D(x,y) and S(x,y), respectively, calculated for a fixed point $x \in \Omega$ and for $y \in \Gamma$, while column matrices B(x) and $B_D(x)$ depend on the integrals defined on Ω region with nuclei U(x,y) and D(x,y), respectively. There holds

$$D(x, y) = \sum\nolimits_x [U(x, y)]; \;\; S(x, y) = \sum\nolimits_x [P(x, y)]; \quad x \in \Omega, \; y \in \Gamma \tag{81}$$

where, U(x,y) is the fundamental solution and

$$\sum(f) = \mu\left[\nabla(f) + \nabla^T(f)\right] + \lambda\, I\, \mathrm{div}(f) \tag{82}$$

for any additional function f and λ, μ being Lame constants. It is well known from the SFEM formulations (Kamiñski, 1999, 2001, 2006; Kamiñski and Carey, 2005) that the system of linear equations, which is the basis of the model, can be transformed, due to the second order stochastic perturbation-based technique, to the following systems of equations:

$$\begin{cases} A^0_{\alpha\beta} X^0_\beta = F^0_\alpha \\ A^{,r}_{\alpha\beta} X^0_\beta + A^0_{\alpha\beta} X^{,r}_\beta = F^{,r}_\alpha \\ A^{,rs}_{\alpha\beta} X^0_\beta + 2A^{,r}_{\alpha\beta} X^{,s}_\beta + A^0_{\alpha\beta} X^{,rs}_\beta = F^{,rs}_\alpha \end{cases} \quad (83)$$

where, the matrices with upper index in the form of 0 denote the 0th order matrix functions in the context of the perturbation approach, while $(.)^{,r}$ and $(.)^{,rs}$ denote the first and the second partial derivatives of the respective operators with respect to the components of the input random variables vector. After some algebraic transformations and the extension towards the generalized nth order approach there holds:

$$\begin{cases} A^0_{\alpha\beta} X^0_\beta = F^0_\alpha \\ A^0_{\alpha\beta} X^{,r}_\beta = F^{,r}_\alpha - A^{,r}_{\alpha\beta} X^0_\beta \\ (\ldots) \\ \sum_{k=0}^{n} \binom{n}{k} A^{(k)}_{\alpha\beta} X^{(n-k)}_\beta = F^{(n)}_\alpha \end{cases} \quad (84)$$

where, $(.)^{(n)}$ denotes here *n*th order partial derivative of the operator (.) with respect to the input random variable. Furthermore, it is known from deterministic BEM implementation that the matrix A is represented by the components of the following perturbation orders:

- 0th order components

$$A^0 \equiv \begin{bmatrix} \left(H^{11}\right)^0 & \left(-G^{11}\right)^0 \\ \left(H^{21}\right)^0 & \left(-G^{21}\right)^0 \end{bmatrix}, \quad (85)$$

- nth order derivatives

$$A^{(n)} \equiv \begin{bmatrix} \left(H^{11}\right)^{(n)} & \left(-G^{11}\right)^{(n)} \\ \left(H^{21}\right)^{(n)} & \left(-G^{21}\right)^{(n)} \end{bmatrix} = \frac{\partial^n A}{\partial b^n} \quad (86)$$

In the quite analogous it is possible to obtain the equations describing derivatives of the R.H.S. vector. It yields:

- 0th order equations

$$\begin{cases} \left(F^{(1)}\right)^0 = \left(-H^{11}\right)^0 \left(u^1\right)^0 + \left(G^{12}\right)^0 \left(p^2\right)^0 + \left(B^1\right)^0 \\ \left(F^{(2)}\right)^0 = \left(-H^{21}\right)^0 \left(u^1\right)^0 + \left(G^{22}\right)^0 \left(p^2\right)^0 + \left(B^2\right)^0 \end{cases}, \quad (87)$$

- Nth order equations

$$\begin{cases} \left(F^{(1)}\right)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \left(-H^{11}\right)^{(k)} \left(u^1\right)^{(n-k)} \\ \qquad + \sum_{k=0}^{n} \binom{n}{k} \left(G^{12}\right)^{(k)} \left(p^2\right)^{(n-k)} + \left(B^1\right)^{(n)} \\ \left(F^{(2)}\right)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \left(-H^{21}\right)^{(k)} \left(u^1\right)^{(n-k)} \\ \qquad + \sum_{k=0}^{n} \binom{n}{k} \left(G^{22}\right)^{(k)} \left(p^2\right)^{(n-k)} + \left(B^2\right)^{(n)} \end{cases} \quad (88)$$

Let us calculate now up to the nth order perturbations of displacement fields

$$u^0(x) = G^0(x)\, p^0 - H^0(x)\, u^0 + B^0(x);\ x \in \Omega \quad (89)$$

as well as:

$$u^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} G^{(k)}(x)\, p^{(n-k)} - \sum_{k=0}^{n} \binom{n}{k} H^{(k)}(x)\, u^{(n-k)} + B^{(n)}(x);\ x \in \Omega \quad (90)$$

Quite analogously the stress tensor components are calculated as:

$$\sigma^0(x) = G^0_D(x)\, p^0 - H^0_S(x)\, u^0 + B^0_D(x); x \in \Omega \quad (91)$$

and

$$\sigma^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} G^{(k)}_D(x)\, p^{(n-k)} - \sum_{k=0}^{n} \binom{n}{k} H^{(k)}_S(x)\, u^{(n-k)} + B^{(n)}_D(x);\ x \in \Omega \quad (92)$$

These equations enable us to calculate any order approximations for any mth probabilistic moments of the structural response. Putting n = 2 we can recover up to the second order approximations for the expectations of displacements and stresses including input cross-correlations as:

$$\begin{aligned} E\left[u(b;x)\right] = {} & G^0(x)\, p^0 - H^0(x)\, u^0 + B^0(x) + \\ & \tfrac{1}{2}\{G^{,rs}(x)\, p^0 + 2G^{,r}(x)\, p^{,s} + \\ & G^0(x)\, p^{,rs} - H^{,rs}(x)\, u^0 - 2H^{,r}(x) u^{,s} \\ & - H^0(x)\, u^{,rs} + B^{,rs}(x)\} \mathrm{Cov}\left(b^r, b^s\right) \end{aligned} \quad (93)$$

as well as:

$$\begin{aligned} E[\boldsymbol{\sigma}(b;x)] &= G_D^0(x)\,p^0 - H_S^0(x)\,u^0 + B_D^0(x) \\ &+\tfrac{1}{2}\{G_D^{,rq}(x)\,p^0 + 2G_D^{,r}(x)\,p^{,q} \\ &+ G_D^{,0}(x)\,p^{,rq} - H_S^{,rq}(x)\,u^0 - 2H_S^{,r}(x)\,u^{,q} \\ &- H_S^0(x)\,u^{,rq} + B_D^{,rq}(x)\}\times \mathrm{Cov}\left(b^r, b^s\right) \end{aligned} \quad (94)$$

It should be underlined that the probabilistic moments obtained refer to discrete values of the displacements and stresses only; following quite analogous considerations we can determine covariance matrices of both functions can be derived.

STOCHASTIC EXTENSION OF THE FINITE DIFFERENCE METHOD

Let us discuss an application of the stochastic perturbation methodology in the Finite Difference Method on the example of the biharmonic partial differential equation:

$$\frac{\partial^2\Phi}{\partial x_1^2}+\frac{\partial^2\Phi}{\partial x_2^2}+\frac{\partial^2\Phi}{\partial x_3^2}=-k, \quad (95)$$

the following probabilistic hierarchical equations are obtained for a second order approach:

- Zeroth order equation, ε^0 terms:

$$\frac{\partial^2\left(\Phi^0\right)}{\partial x_1^2}+\frac{\partial^2\left(\Phi^0\right)}{\partial x_2^2}+\frac{\partial^2\left(\Phi^0\right)}{\partial x_3^2}=-k^0, \quad (96)$$

- First order equation, ε^1 terms:

$$\frac{\partial^2\left(\Phi^{,r}\right)}{\partial x_1^2}+\frac{\partial^2\left(\Phi^{,r}\right)}{\partial x_2^2}+\frac{\partial^2\left(\Phi^{,r}\right)}{\partial x_3^2}=-k^{,r}, \quad (97)$$

- Second-order equation, ϵ^2 terms:

$$\frac{\partial^2\left(\Phi^{(2)}\right)}{\partial x_1^2}+\frac{\partial^2\left(\Phi^{(2)}\right)}{\partial x_2^2}+\frac{\partial^2\left(\Phi^{(2)}\right)}{\partial x_3^2}=-k^{,rs}\mathrm{Cov}\left(b^r,b^s\right) \quad (98)$$

Solving these equations for zeroth, first and second orders of the potential function Φ and applying its second order extension:

$$\begin{aligned}\Phi(b) &= \Phi^0\left(b^0\right)+\varepsilon\,\Phi^{,\rho}\left(b^0\right)\Delta b_\rho + \\ &\tfrac{1}{2}\varepsilon^2\,\Phi^{,\rho\sigma}\left(b^0\right)\Delta b^2,\end{aligned} \quad (99)$$

the expected values and cross-covariances for the random potential function Φ are evaluated using the results of Eq. (96-98) as:

$$E[\Phi] = \Phi^0 + \tfrac{1}{2}\Phi^{,\rho\sigma}S_b^{\rho\sigma} \quad (100)$$

Where,

$$S_\Phi^{\alpha\beta} = \Phi^{,\rho}\Phi^{,\sigma}S_b^{\rho\sigma} \quad (101)$$

Let us consider a sufficiently smooth real function y(x) defined discretly by the values y_0, y_1, y_2, …, y_n in uniformly distributed points x=0, δ, 2δ, …, nδ for $\delta\in\Re_+/\{0\}$ and n∈N. The differences of the function values are calculated as:

$$\left(\Delta_1 y\right)_{x=n\delta} = y_{n+1} - y_n \quad (102)$$

The first order derivatives of y(x) can be rewritten in the form of:

$$\left(\frac{dy}{dx}\right)_{x=n\delta} \cong \frac{y_{n+1}-y_n}{\delta}. \quad (103)$$

Analogously, starting from the second differences

$$\begin{aligned}\left(\Delta_2 y\right)_{x=n\delta} &= \left(\Delta_1 y\right)_{x=(n+1)\delta} - \left(\Delta_1 y\right)_{x=n\delta} \\ &= y_{n+1} - 2y_n + y_{n-1}\end{aligned} \quad (104)$$

we obtain approximation for the second order derivatives

$$\left(\frac{d^2y}{dx^2}\right)_{x=n\delta} \cong \frac{\left(\Delta_2 y\right)_{x=n\delta}}{\delta^2} = \frac{y_{n+1}-2y_n+y_{n-1}}{\delta^2};\, n\in N. \quad (105)$$

Further, for the sufficiently smooth real function w = w(x_i) and the rectangular planar network defined by the dimension $\delta\in\Re_+/\{0\}$, the following approximating equations are used:

$$\frac{dw}{dx_1} \cong \frac{w_1 - w_0}{\delta}, \quad (106)$$

$$\frac{dw}{dx_2} \cong \frac{w_2 - w_0}{\delta}, \tag{107}$$

$$\frac{d^2w}{dx_1^2} \cong \frac{w_1 - 2w_0 + w_3}{\delta^2}, \tag{108}$$

$$\frac{d^2w}{dx_2^2} \cong \frac{w_2 - 2w_0 + w_4}{\delta^2}. \tag{109}$$

The point indexed by 0 is located in the center of this network and together with points indexed with 1 (left) and 3 (right) along the axis x_1; the points 2 (bottom), 0 and 4 (top) belong to x_2 axis. Using analogous methodology one can arrive at the higher order derivatives of the function w(x). These equations make it possible to solve numerically all the engineering problems which can be formulated in the form of partial differential equations. As it is known, the method can be used in its relaxation version to assure better efficiency of the computations as well as for curved boundaries of the continua, where some shape correctors for δ are necessary.

Considering the application of the second order perturbation second probabilistic moment approach in the Finite Difference Method, the approximation of zeroth, first and second order derivatives of the potential function Φ being computed, is to be done with respect to random variable vector components $\{b_r(x;\ \theta)\}$. It is obtained for the first partial derivative in x_i direction (assuming the same density of finite difference net in orthogonal directions):

- 0th order terms:

$$\frac{\partial\left(\Phi^0\right)}{\partial x_i} \cong \frac{\Phi_1^0 - \Phi_0^0}{\delta}, \tag{110}$$

- 1st order terms (r = 1,…,R):

$$\frac{\partial\left(\Phi^{,r}\right)}{\partial x_i} = \frac{\partial^2\Phi}{\partial x_i \partial b_r} \cong \frac{\Phi_1^{,r} - \Phi_0^{,r}}{\delta} = \frac{1}{\delta}\frac{\partial}{\partial b_r}\left(\Phi_1 - \Phi_0\right), \tag{111}$$

- nth order terms (for single input random variable):

$$\frac{\partial}{\partial x_i}\left(\frac{\partial^n\Phi}{\partial b^n}\right) = \frac{\partial^{n+1}\Phi}{\partial x_i \partial b^n} \cong \frac{1}{\delta}\left(\frac{\partial^n\Phi_1}{\partial b^n} - \frac{\partial^n\Phi_0}{\partial b^n}\right) = \frac{1}{\delta}\frac{\partial^n}{\partial b^n}\left(\Phi_1 - \Phi_0\right) \tag{112}$$

Quite analogously, the second order derivatives with respect to variable x are calculated as:

- 0th order terms:

$$\frac{d^2\left(\Phi^0\right)}{dx_i^2} \cong \frac{\Phi_1^0 - 2\Phi_0^0 + \Phi_3^0}{\delta^2}, \tag{113}$$

- 1st order terms (r = 1,…, R):

$$\frac{\partial^2\left(\Phi^{,r}\right)}{\partial x_i^2} = \frac{\partial^3\Phi}{\partial x_i^2 \partial b_r} \cong \frac{\Phi_1^{,r} - 2\Phi_0^{,r} + \Phi_3^{,r}}{\delta^2} = \frac{1}{\delta^2}\frac{\partial}{\partial b_r}\left(\Phi_1 - 2\Phi_0 + \Phi_3\right) \tag{114}$$

- nth order terms (for single random variable):

$$\frac{\partial^2}{\partial x_i^2}\left(\frac{\partial^n\Phi}{\partial b^n}\right) = \frac{\partial^{n+2}\Phi}{\partial x_i^2 \partial b^n} \cong \frac{1}{\delta^2}\left(\frac{\partial^n\Phi_1}{\partial b^n} - 2\frac{\partial^n\Phi_0}{\partial b^n} + \frac{\partial^n\Phi_3}{\partial b^n}\right) = \frac{1}{\delta^2}\frac{\partial^n}{\partial b^n}\left(\Phi_1 - 2\Phi_0 + \Phi_3\right) \tag{115}$$

Using the above approximation, the second order perturbation second central probabilistic moments equations for the Stochastic Finite Difference Method (SFDM) is obtained as follows:

- 0th order one FDM equation:

$$\frac{1}{\delta^2}\left(\Phi_1^0 + \Phi_2^0 + \Phi_3^0 + \Phi_4^0 - 4\Phi_0^0\right) = -k^0, \tag{116}$$

- 1st order terms (r = 1,…, R):

$$\frac{1}{\delta^2}\left(\Phi_1^{,r} + \Phi_2^{,r} + \Phi_3^{,r} + \Phi_4^{,r} - 4\Phi_0^{,r}\right) = -k^{,r}, \tag{117}$$

- nth order terms (for a single random variable):

$$\frac{1}{\delta^2}\frac{\partial^n}{\partial b^n}\left(\Phi_1^{,rs} + \Phi_2^{,rs} + \Phi_3^{,rs} + \Phi_4^{,rs} - 4\Phi_0^{,rs}\right) = -\frac{\partial^n k}{\partial b^n} \tag{118}$$

To reduce the total number of second order equations for instance, the additional equation is multiplied by the cross-covariances of the input random variables to get:

$$\frac{1}{\delta^2}\left(\Phi_1^{(2)}+\Phi_2^{(2)}+\Phi_3^{(2)}+\Phi_4^{(2)}-4\Phi_0^{(2)}\right)=-k^{,rs}S_b^{rs} \quad (119)$$

Where,

$$\Phi_i^{(2)}=\Phi_i^{,rs}S_b^{rs};\ i=0,\dots,4 \quad (120)$$

Solving Eq. (117-119) for the 0th, first and second order potential function random fields, the expected values and covariances of this function are derived next. There holds:

$$E\left[\Phi(x_i;\omega)\right]=\Phi^0(x_i;\omega)+\tfrac{1}{2}\Phi^{,rs}(x_i;\omega)S_b^{rs} \quad (121)$$

and for the second order moments

$$\begin{aligned}&\mathrm{Cov}\left(\Phi\left(x_i^{(1)};\omega\right);\Phi\left(x_i^{(2)};\omega\right)\right)\\&=\Phi^{,r}\left(x_i^{(1)};\omega\right)\Phi^{,s}\left(x_i^{(1)};\omega\right)S_b^{rs},\end{aligned} \quad (122)$$

which completes general description of the probabilistic extension of the FDM in case of the biharmonic equation solution. As we know, the nature of the perturbation method can result in secular terms in the final solution of Eq. (116-120). This phenomenon is independent from interrelations between various probabilistic moments and characteristics of the input random field. Some theoretical methods to eliminate secularity are displayed in the literature (Kleiber and Hien, 1992; Schenk *et al*., 2005); in the case of stochastic dynamics for instance, the secular terms may be eliminated by the use of discrete or the Fast Fourier Transform (FFT).

CONCLUSION

- As it was demonstrated above, an application of the stochastic perturbation technique to the fundamental discrete numerical methods, i.e. Finite and Boundary Element Methods as well as to the Finite Difference Method is not a very complicated process. The classical single Taylor series expansion of all state functions and parameters made possible to find a solution to partial differential equations system with a single input random variable or with a random vector with uncorrelated or even correlated random components. It reflects basic needs of computational modelling, where some system parameters comes from experimentally driven statistical estimation. In all these cases up to the nth order equations of static or dynamic equilibrium must be formed on the basis of the fundamental equation solved for the deterministic problem, where mean values of random input are frequently put instead of deterministic values. These up to *n*th order equations have also deterministic nature, they include up to *n*th order partial derivatives of system parameters and state functions with respect to a given random input parameter(s). As it was derived before, thanks to the generalized perturbation approach, it is possible to compute output probabilistic moments of any given order with a priori defined accuracy by increasing expansion order or, alternatively, by an additional modification of the perturbation parameter ε (Kamiñski, 2006; Kamiñski and Carey, 2005).
- The essential advantage of the generalized perturbation method presented here is a lack of limitation of the input random dispersion level on a final precision for the computed state functions. This preference is of a special importance considering well known discrepancies of the previously used second order technique and, on the other hand, the real engineering and scientific stochastic problems. The second point here is that contrary to the other stochastic methods, the deterministic equations of an increasing order (and especially their solutions) enable final computations of the desired probabilistic moments.
- Although, differential equations or variational principles worked out above deal with the boundary value problems, an extension towards transient formulations and their solutions should be relatively easy (Kamiñski, 2001; Kleiber and Hien, 1992). As one can expect, this perturbation methodology can find its application in the other computer methods like the coupled Boundary Finite Element Method (Wolf, 2003), the Complex Variable Boundary Element Method (Hromadka and Lai, 1986), some meshless approaches or even the analytical solutions to specific engineering or scientific problems with random parameters. The second issue, not discussed in details here, is a discretization of stochastic processes (Feller, 1967) by the perturbation technique and its further application to the solution of some stochastic differential equations (Sobczyk, 1993). Partial answer to this temporarily unsolved problem may be given by a replacement of a single Taylor expansion used up to now by a double Taylor expansion. The basic power of this idea would be to approximate all the functions appearing in the partial differential equations around their expectations with respect to 2 independent parameters like spatial coordinate (or temperature) and time, respectively. Further mathematical and numerical investigations will help to effectively verify this opportunity.

REFERENCES

Babuška, I., R. Tempone and G.E. Zouranis, 2005. Solving elliptic boundary value problems with uncertain coefficients by the finite element method: The stochastic formulation. Comput. Meth. Applied Mech. Eng., 194: 1251-1294.

Beer, G., 2001. Programming the Boundary Element Method. An Introduction for Engineers. Wiley.

Bendat, J.S. and A.G. Piersol, 1971. Random Data: Analysis and Measurement Procedures. Wiley.

Berger, J.R. and V.K. Tewary, 1997. Boundary integral equation formulation for interface cracks in anisotropic materials. Comput. Mech., 20: 261-266.

Brebbia, C.A. and J. Dominguez, 1996. Boundary Elements. An Introductory Course. Comput. Mech. Publ.

Burczyński, T., 1995. The Boundary Element Method in Structural Mechanics (in Polish). WNT, Warsaw.

Chatterjee, K. and J. Poggie, 2007. A two-dimensional stochastic algorithm for the solution of the non-linear Poisson-Boltzmann equation: Validation with finite-difference benchmarks. Int. J. Num. Meth. Engrg. (in Press).

Cheng, H.D. and O.E. Lafe, 1991. Boundary element solution for stochastic groundwater flow: Random boundary conditions and recharge. Water Research, 27 (2): 231-242.

Chou, Y.F., 1991. Applications of Discrete Functional Analysis to the Finite Difference Method. Elsevier.

Falsone, G., 2005. An extension of the Kazakov relationship for non-Gaussian random variables and its use in the non-linear stochastic dynamics. Prob. Engrg. Mech., 20: 45-56.

Feller, W., 1967. An Introduction to Probability Theory and Applications. Wiley.

Ghanem, R.G. and P.D. Spanos, 1991. Stochastic Finite Elements: A Spectral Approach. Springer-Verlag, New York.

Grzywiński, M. and A. Sluzalec, 2000. Stochastic convective heat transfer equations in finite difference method. Int. J. Heat Mass Transfer, 43: 4003-4008.

Honda, R., 2005. Stochastic BEM with spectral approach in elastostatic and elastodynamic problems with geometrical uncertainty. Eng. Anal. Bound. Elem., 29: 415-427.

Hromadka, Th. and Ch. Lai, 1986. The Complex Variable Boundary Element Method in Engineering Analysis. Springer-Verlag, New York.

Hurtado, J.E. and A.H. Barbat, 1998. Monte-Carlo techniques in computational stochastic mechanics. Arch. Comput. Meth. Eng., 5 (1): 3-30.

Kamiński, M. and G.F. Carey, 2005. Stochastic perturbation-based finite element approach to fluid flow problems. Int. J. Num. Meth. Heat Fluid Flow 15 (7): 671-697.

Kamiński, M., 2006. On generalized stochastic perturbation-based finite elements. Comm. Num. Meth. Eng., 22 (1): 23-31.

Kamiński, M., 2001. Stochastic perturbation approach in vibration analysis using Finite Difference Method. J. Sound Vibr., 251 (4): 651-670.

Kamiński, M., 1999. Stochastic second order perturbation BEM formulation. Engrg. Anal. Bound. Elem., 2: 123-130.

Keese, A. and H.G. Matthies, 2005. Hierarchical parallelisation for the solution of stochastic finite element equations. Comput. Struct., 83: 1033-1047.

Kleiber, M. and T.D. Hien, 1992. The Stochastic Finite Element Method. Wiley.

Kunz, K.S. and R.J. Luebbers, 1993. The Finite Difference Time Domain Method for Electromagnetics. CRC Press.

Liu, W.K., T. Belytschko and A. Mani, 1986. Random field finite elements. Int. J. Num. Meth. Eng., 23: 1831-1845.

Manolis, G.D. and R.P. Shaw, 1996. Boundary integral formulation for 2D and 3D thermal problems exhibiting a linearly varying stochastic conductivity. Comput. Mech., 17: 406-417.

Minkowycz, M.J. *et al.*, 1998. Handbook of Numerical Heat Transfer. Wiley-Interscience.

Mitchell, A.R. and D.F. Griffiths, 1980. The Finite Difference Method in Partial Differential Equations. Wiley.

Nayfeh, A., 1973. Perturbation Methods. Wiley.

Oden, J.T. *et al.*, 2005. Theory and methodology for estimation and control of errors due to modeling, approximation and uncertainty. Comput. Applied Mech. Eng., 194: 195-204.

Papadopoulos, V. and M. Papadrakakis, 2005. The effect of material and thickness variability on the buckling load of shells with random initial imperfections. Comput. Applied Mech. Eng., 194: 1405-1426.

Papadrakakis, M., 1993. Solving Large-Scale Problems in Mechanics: The Development and Application of Computational Solution Methods. Wiley.

Schenk, C.A., H.J. Pradlwarter and G.I. Schueller, 2005. Non-stationary response of large, non-linear finite element systems under stochastic loading. Comput. Struct., 83: 1086-1102.

Sobczyk, K., 1993. Stochastic Differential Equations: Applications in Physics, Engineering and Mechanics. Kluwer Acad. Publ.

Taflove, A., 1998. Advances in Computational Electrodynamics: The Finite-Difference Time-Domain Method. Artech House.

Vanmarcke, E., 1983. Random Fields. Analysis and Synthesis. MIT Press.

Wang, H.F. and M.P. Anderson, 1995. Introduction to Groundwater Modeling: Finite Difference and Finite Element Methods. Academic Press.

Wolf, J.P., 2003. The Scaled Boundary Finite Element Method. John Wiley and Sons.

Zienkiewicz, O.C., 1971. The Finite Element Method in Engineering Science. McGraw-Hill, London.