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A Method for Designing a Customized Progressive Addition Lens

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Abstract: In this study, we present a method for designing customizable progressive addition lenses (PALs) where, the desired mean curvature map (or power map) can be drawn by the user (optician), interactively. Tensor B-splines are used to approximate the lens surface.

Key words: Design of progressive addition lenses (PALs), tensor B-splines, methods, patient

INTRODUCTION

The research presented in this study was performed as part of a research project with a private Canadian company, which no longer exists, manufacturing a free form surfacing machine used for cutting optical lenses in the ophthalmic industry. The objective was to develop the mathematical method and numerical algorithms which would allow opticians to customize progressive addition lenses (PALs) for their patients. PALs are commonly used in the correction of presbyopia and are characterized by a gradient of increasing lens power, added to a patient's prescription. The gradient starts at a minimum, or no addition power, at the top of the lens and reaches a maximum addition power, magnification, at the bottom of the lens. The disadvantage of PALs is related to the distortion, or aberrations, away from the near region and the far region centers and the optical axis, or progressive corridor, which connects these (Sheedy *et al*., 2005). Because of this, patients require an adaptation period which can limit sales of PALs (Sheedy *et al*., 2006). PAL manufacturers are therefore continually seeking more efficient PALs design methods which reduce distortions along the optical axis and optimize other design features. However, the adaptation to PALs is highly personalized and depends on the user (Sheedy *et al*., 2006). A method which would enable a customizable design in the presence of the patient is therefore, of great interest.

Details of PAL design methods are highly proprietary but insight into recent methods can be obtained from the patent literature. Some of the patents reviewed for this work were Shamir Optical Industries (2001), Sola International Holdings Ltd. (2000; 2001), Carl-Zeiss-Stiftung (1998), Nikon-Essilor Co., Ltd. (2001) and Seiko Epson Corporation (2000). Of interest in these patents were the vastly different design methods and the criteria used for the designs, especially the astigmatism (asphericity) constraints in the progressive corridor. Also, it was interesting to note that the mean curvature is widely used as the means of imposing the optical power of a lens (Shamir, 2001). Also, recent PALs design methods use a lens with 2 progressive surfaces to achieve the desired design criteria (Johnson and Johnson, 2005).

The main difficulty with imposing the mean curvature on a network of grid points is the non linearity of the expression giving the mean curvature which leads to non linear systems of equations which can only be solved using numerical methods. In this research, we will present a method of designing PALs which would allow the opticians to customize the lens in the presence of a patient, interactively. It assumes that opticians will eventually, have access to free form surfacing machines in their offices.

Tensor b-spline representation of a surface: Since, the mean curvature is used in many reviewed patents to impose lens powers, a method of determining the height of a surface over a rectangular grid when its mean curvatures are known at the grid points is required. Following the approach used by Tazeroualti (1994) tensor B-splines were selected as the method of approximating the lens surfaces mainly because of the form of the normalized approximation as shown in Fig. 1.

If the height values of a surface, $f(x_i, y_j) = f_{ij}$, are known on a square grid where, $i = 1, 2, ... n + 6$, $j = 1, 2, ...n+6$, $x_i = x_{min} + (i-3)\Delta x$, $y_j = y_{min} + (j-3)\Delta y$, $\Delta x = (x_{max} + x_{min})/ n$ and $\Delta y = (y_{max} + y_{min})/ n$, where, n is the number of intervals in the x and y directions between grid points 3 and n + 3, as shown in Fig. 1, the following tensor product

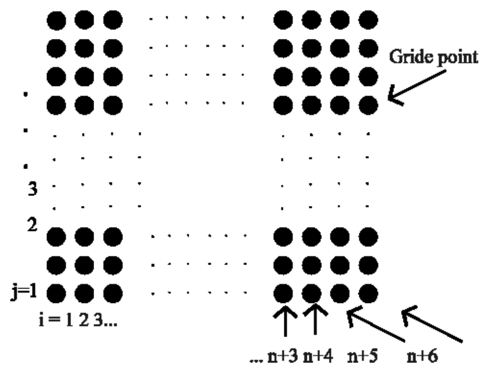

Fig. 1: A square grid of points over which a progressive surface is defined

of 4th degree B-splines centered at (i, j) can be used to approximate F (x, y) as:

$$f(x,y) \approx \sum_{l=i-2}^{i+2} \sum_{k=j-2}^{j+2} A_{l-2}(\varepsilon_x) A_{k-2}(\varepsilon_y) f_{ij}^{I} \qquad (1)$$

Where,

$$\varepsilon_x = \frac{(x - x_i)}{\Delta x}$$

And

$$\varepsilon_y = \frac{(y - y_j)}{\Delta y}$$

are the normalized coordinates with $x_i \le x < x_{ij}$, $y_i \le y < y_{ij}$, $0 < \varepsilon_x < 1$, $0 < \varepsilon_y < 1$ and

$$f_{ij}^{I} = \frac{1}{4}\begin{pmatrix} f(x_i, y_j) + f(x_{i+1}, y_j) + f(x_i, y_{j+1}) \\ + f(x_{i+1}, y_{j+1}) \end{pmatrix} \qquad (2)$$

Where,

$$A_{i-4}(\varepsilon_x) = \frac{1}{24}(1-\varepsilon_x)^4$$

$$A_{i-3}(\varepsilon_x) = \frac{1}{24}(11 - 12\varepsilon_x - 6\varepsilon_x^{\ 2} + 12\varepsilon_x^{\ 3} - 4\varepsilon_x^{\ 4})$$

$$A_{i-2}(\varepsilon_x) = \frac{1}{24}(11 + 12\varepsilon_x - 6\varepsilon_x^{\ 2} - 12\varepsilon_x^{\ 3} + 6\varepsilon_x^{\ 4})$$

$$A_{i-1}(\varepsilon_x) = \frac{1}{24}(1 + 4\varepsilon_x + 6\varepsilon_x^{\ 2} + 4\varepsilon_x^{\ 3} - 4\varepsilon_x^{\ 4})$$

$$A_{i}(\varepsilon_x) = \frac{1}{24}\varepsilon_x^{\ 4}$$

$$A_{j-4}(\varepsilon_y) = \frac{1}{24}(1-\varepsilon_y)^4$$

$$A_{j-3}(\varepsilon_y) = \frac{1}{24}(11 - 12\varepsilon_y - 6\varepsilon_y^{\ 2} + 12\varepsilon_y^{\ 3} - 4\varepsilon_y^{\ 4})$$

$$A_{j-2}(\varepsilon_y) = \frac{1}{24}(11 + 12\varepsilon_y - 6\varepsilon_y^{\ 2} - 12\varepsilon_y^{\ 3} + 6\varepsilon_y^{\ 4}) \qquad (3)$$

$$A_{j-1}(\varepsilon_y) = \frac{1}{24}(1 + 4\varepsilon_y + 6\varepsilon_y^{\ 2} + 4\varepsilon_y^{\ 3} - 4\varepsilon_y^{\ 4})$$

$$A_{j}(\varepsilon_y) = \frac{1}{24}\varepsilon_y^{\ 4}.$$

The advantage of the normalized tensor B-spline above is that it has the same form at every grid point. Derivatives of F (x, y) can also be calculated as:

$$f_x = \frac{\partial f(x,y)}{\partial x} \approx \sum_{l=i-2}^{i+2} \sum_{k=j-2}^{j+2} A'_{l-2}(\varepsilon_x) A_{k-2}(\varepsilon_y)\, f_{ij}^{I}$$

$$f_y = \frac{\partial f(x,y)}{\partial y} \approx \sum_{l=i-2}^{i+2} \sum_{k=j-2}^{j+2} A_{l-2}(\varepsilon_x) A'_{k-2}(\varepsilon_y)\, f_{ij}^{I}$$

$$f_{xx} = \frac{\partial^2 f(x,y)}{\partial x^2} \approx \sum_{l=i-2}^{i+2} \sum_{k=j-2}^{j+2} A''_{l-2}(\varepsilon_x) A_{k-2}(\varepsilon_y)\, f_{ij}^{I} \qquad (4)$$

$$f_{yy} = \frac{\partial^2 f(x,y)}{\partial y^2} \approx \sum_{l=i-2}^{i+2} \sum_{k=j-2}^{j+2} A_{l-2}(\varepsilon_x) A''_{k-2}(\varepsilon_y)\, f_{ij}^{I}$$

$$f_{xy} = \frac{\partial^2 f(x,y)}{\partial x \partial y} \approx \sum_{l=i-2}^{i+2} \sum_{k=j-2}^{j+2} A'_{l-2}(\varepsilon_x) A'_{k-2}(\varepsilon_y)\, f_{ij}^{I}$$

Where,

$$A'_{i-4}(\varepsilon_x) = -\frac{1}{6\Delta x}(1-\varepsilon_x)^3$$

$$A'_{i-3}(\varepsilon_x) = \frac{1}{24\Delta x}(-12 - 12\varepsilon_x + 36\varepsilon_x^{\ 2} - 16\varepsilon_x^{\ 3})$$

$$A'_{i-2}(\varepsilon_x) = \frac{1}{24\Delta x}(12 - 12\varepsilon_x - 36\varepsilon_x^{\ 2} + 24\varepsilon_x^{\ 3})$$

$$A'_{i-1}(\varepsilon_x) = \frac{1}{24\Delta x}(4 + 12\varepsilon_x + 12\varepsilon_x^{\ 2} - 16\varepsilon_x^{\ 3})$$

$$A'_{i}(\varepsilon_x) = \frac{1}{6\Delta x}\varepsilon_x^{\ 3}$$

$$A''_{i-4}(\varepsilon_x) = -\frac{1}{2\Delta x^2}(1-\varepsilon_x)^2$$

$$A''_{i-3}(\varepsilon_x) = \frac{1}{24\Delta x^2}(-12 + 72\varepsilon_x - 48\varepsilon_x^{\ 2})$$

$$A''_{i-2}(\varepsilon_x) = \frac{1}{24\Delta x^2}(-12\varepsilon_x - 72\varepsilon_x + 72\varepsilon_x^{\ 2})$$

$$
\begin{aligned}
A''_{i-1}(\varepsilon_x) &= \frac{1}{24\Delta x^2}(12+24\varepsilon_x-48\varepsilon_x{}^2)\\
A''_{i}(\varepsilon_x) &= \frac{1}{2\Delta x^2}\varepsilon_x{}^2\\
A'_{j-4}(\varepsilon_y) &= -\frac{1}{6\Delta y}(1-\varepsilon_y)^3\\
A'_{j-3}(\varepsilon_y) &= \frac{1}{24\Delta y}(-12-12\varepsilon_y+36\varepsilon_y{}^2-16\varepsilon_y{}^3)\\
A'_{j-2}(\varepsilon_y) &= \frac{1}{24\Delta y}(12-12\varepsilon_y-36\varepsilon_y{}^2+24\varepsilon_y{}^3)\\
A'_{j-1}(\varepsilon_y) &= \frac{1}{24\Delta y}(4+12\varepsilon_y+12\varepsilon_y{}^2-16\varepsilon_y{}^3)\\
A'_{j}(\varepsilon_y) &= \frac{1}{6\Delta y}\varepsilon_y{}^3\\
A''_{j-4}(\varepsilon_y) &= -\frac{1}{2\Delta y^2}(1-\varepsilon_y)^2\\
A''_{j-3}(\varepsilon_y) &= \frac{1}{24\Delta y^2}(-12+72\varepsilon_y-48\varepsilon_y{}^2)\\
A''_{j-2}(\varepsilon_y) &= \frac{1}{24\Delta y^2}(-12\varepsilon_y-72\varepsilon_y+72\varepsilon_y{}^2)\\
A''_{j-1}(\varepsilon_y) &= \frac{1}{24\Delta y^2}(12+24\varepsilon_y-48\varepsilon_y{}^2)\\
A''_{j}(\varepsilon_y) &= \frac{1}{2\Delta y^2}\varepsilon_y{}^2.
\end{aligned} \tag{5}
$$

MATERIALS AND METHODS

With the derivatives calculated as above, the mean curvature, h (x_i, y_j), at the grid points (x_i, y_j) can be calculated using:

$$\hat{h}(x,y)=\frac{1}{2}\frac{\left((1+f_x^2)f_{yy}-2f_xf_yf_{xy}+(1+f_y^2)f_{xx}\right)}{\left(1+f_x^2+f_y^2\right)^{\frac{3}{2}}} \tag{6}$$

The mean curvature, h (x_i, y_j), = h_{ij} to be imposed at every grid point can be specified by drawing the desired mean curvature level curves using any software with drawing and interpolation capabilities. Once the level curves are drawn by the user and interpolated by the software, the mean curvature map at the grid points is read by the PALs design program. This approach worked very well with MATLAB which has all the necessary drawing and interpolation capabilities.

In order to mathematically impose the mean curvatures and other design criteria at the grid points, the following error function can be minimized:

$$
\begin{aligned}
&E(f^I_{11},f^I_{12},\ldots,f^I_{n+5n+4},f^I_{n+5n+5})=\\
&\sum_{i=1}^{n+5}\sum_{j=1}^{n+5}\omega_h(\hat{h}_{ij}-h_{ij})^2+\omega_a[(M_{ij}E_{ij}-L_{ij}F_{ij})^2+\\
&(N_{ij}E_{ij}-L_{ij}G_{ij})^2+(M_{ij}G_{ij}-N_{ij}F_{ij})^2]+\omega_f(\hat{f}_{\alpha\beta}-f_{\alpha\beta})^2+\\
&\omega_{f_x}(\hat{f}_{x\alpha\beta}-f_{x\alpha\beta})^2+\omega_{f_y}(\hat{f}_{y\alpha\beta}-f_{y\alpha\beta})^2+\omega_{f_{xy}}(\hat{f}_{xy\alpha\beta}-f_{xy\alpha\beta})^2,
\end{aligned} \tag{7}
$$

where, $\hat{h}_{ij}$ is the calculated mean curvature at grid point ij, h_{ij} is the imposed mean curvature at grid point ij, $f_{\alpha\beta}$, $f_{x\alpha\beta}$, $f_{y\alpha\beta}$, $f_{xy\alpha\beta}$ are the calculated values of f, its first derivative with respect to x, its first derivative with respect to y and its cross derivative, respectively, at grid point αß, $f_{\alpha\beta}$, $f_{x\alpha\beta}$, $f_{y\alpha\beta}$, $f_{xy\alpha\beta}$ are the imposed values of f and its derivatives at grid point αß, ω_h, ω_a, ω_f, ω_{fx}, ω_{fy}, ω_{fxy} are the mean curvature, astigmatism, function value, derivative with respect to x, derivative with respect to y and cross derivative weights, respectively.

E_{ij}, F_{ij}, G_{ij} are the first fundamental coefficients of the surface z = f (x, y) at grid point ij given by

$$
\begin{aligned}
E_{ij} &= 1+f_{xij}^2\\
F_{ij} &= f_{xij}f_{yij}\\
G_{ij} &= 1+f_{yij}^2,
\end{aligned} \tag{8}
$$

and L_{ij}, M_{ij}, N_{ij} are the second fundamental coefficients at point ij given by

$$
\begin{aligned}
L_{ij} &= \frac{f_{xxij}}{(1+f_{xij}^2+f_{yij}^2)^{\frac{1}{2}}}\\
M_{ij} &= \frac{f_{xyij}}{(1+f_{xij}^2+f_{yij}^2)^{\frac{1}{2}}}\\
N_{ij} &= \frac{f_{yyij}}{(1+f_{xij}^2+f_{yij}^2)^{\frac{1}{2}}}.
\end{aligned} \tag{9}
$$

All directions at a point ij are principal directions if and only if

$$\frac{L_{ij}}{E_{ij}}=\frac{M_{ij}}{F_{ij}}=\frac{N_{ij}}{G_{ij}}=\text{const}$$

which means the second term in the error function above controls the sphericity (or astigmatism) of the surface. In order for the problem to be well-posed, the height the surface has to be specified at a certain point αß. The derivatives of f are also specified at this point also to obtain a well-posed problem but these derivatives are related to the prism (angle) of the lens which is part of a lens prescription. The function values

$$(f^I_{11}, f^I_{12}, \ldots, f^I_{n+5n+4}, f^I_{n+5n+5})$$

will minimize the error function if

$$\frac{\partial E}{\partial f^I_{sp}} = \sum_{i=1}^{n+5}\sum_{j=1}^{n+5}\omega_h(\hat{h}_{ij}-h_{ij})\frac{A''_s(\varepsilon_{xij})A_p(\varepsilon_{yij})}{\left(1+f^2_{xij}\right)\left(1+f^2_{xij}+f^2_{yij}\right)^{\frac{1}{2}}}+$$

$$\omega_a\left\{\begin{array}{l}(M_{ij}E_{ij}-L_{ij}F_{ij})\left(\frac{\left(1+f^2_{xij}\right)}{\left(1+f^2_{xij}+f^2_{yij}\right)^{\frac{1}{2}}}A'_s(\varepsilon_{xij})A'_p(\varepsilon_{yij})-\frac{f_{xij}f_{yij}}{\left(1+f^2_{xij}+f^2_{yij}\right)^{\frac{1}{2}}}A''_s(\varepsilon_{xij})A_p(\varepsilon_{yij})\right)+\\(N_{ij}E_{ij}-L_{ij}G_{ij})\left(\frac{\left(1+f^2_{xij}\right)}{\left(1+f^2_{xij}+f^2_{yij}\right)^{\frac{1}{2}}}A_s(\varepsilon_{xij})A''_p(\varepsilon_{yij})-\frac{\left(1+f^2_{yij}\right)}{\left(1+f^2_{xij}+f^2_{yij}\right)^{\frac{1}{2}}}A''_s(\varepsilon_{xij})A_p(\varepsilon_{yij})\right)+\\(M_{ij}G_{ij}-N_{ij}F_{ij})\left(\frac{\left(1+f^2_{yij}\right)}{\left(1+f^2_{xij}+f^2_{yij}\right)^{\frac{1}{2}}}A'_s(\varepsilon_{xij})A'_p(\varepsilon_{yij})-\frac{\left(1+f^2_{yij}\right)}{\left(1+f^2_{xij}+f^2_{yij}\right)^{\frac{1}{2}}}A_s(\varepsilon_{xij})A''_p(\varepsilon_{yij})\right)\end{array}\right\}+ \quad (10)$$

$$\omega_f(\hat{f}_{\alpha\beta}-f_{\alpha\beta})A_s(\varepsilon_{xij})A_p(\varepsilon_{yij})+\omega_{f_x}(\hat{f}_{x\alpha\beta}-f_{x\alpha\beta})A'_s(\varepsilon_{xij})A_p(\varepsilon_{yij})+$$
$$\omega_{f_y}(\hat{f}_{y\alpha\beta}-f_{y\alpha\beta})A_s(\varepsilon_{xij})A'_p(\varepsilon_{yij})+\omega_{f_{xy}}(\hat{f}_{xy\alpha\beta}-f_{xy\alpha\beta})A'_s(\varepsilon_{xij})A'_p(\varepsilon_{yij})=0.$$

Expressing condition (10) at $1< s < n+5, 1 < p < n + 5$ leads to an $(n + 5) \times (n + 5)$ system of nonlinear equations which is difficult to solve, even numerically. Therefore, an iterative method based on the approach used by Tazeroualti is used where, by the values of the derivatives f_x and f_y are assumed known and treated as constants at the first iteration and are updated iteratively until they no longer change significantly. The corresponding system of linear equations can be solved using a standard linear system solver. The following algorithm can be used to calculate the

$$(f^I_{11}, f^I_{12}, \ldots, f^I_{n+5n+4}, f^I_{n+5n+5})$$

values that minimize the above error function.

Algorithm:

- Define grid as in Fig. 1.
- Read in mean curvature map (and weights maps).
- Start with initial guesses for f_x and f_y at every grid point.
- Express $(n + 5) \times (n + 5)$ system of linear equations by evaluating the condition

$$\frac{\partial E}{\partial f^I_{sp}}$$

  at every grid point.
- Solve linear system for

$$(f^I_{11}, f^I_{12}, \ldots, f^I_{n+5n+4}, f^I_{n+5n+5})$$

- Calculate f_x and f_y at every grid point using the tensor B-splines and the values calculated at step 4.
- Repeat steps 3-5 until the values f_x and f_y no longer change significantly.

RESULTS AND DISCUSSION

The method presented in this paper has been tested with MATLAB and has been proven very successful for the design of a progressive lens based on a mean curvature map specified by a user (optician). We have

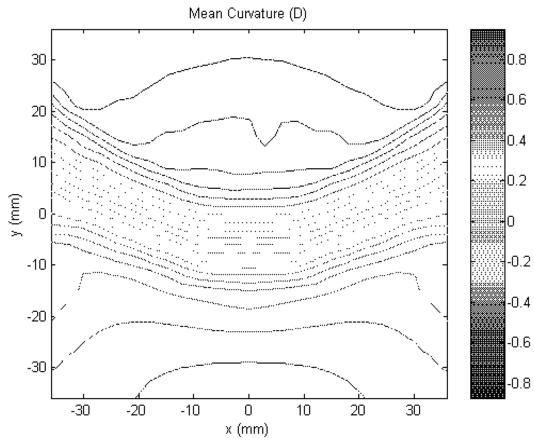

Fig. 2: Mean curvature level curves for-1 D base curve with a +2 D add

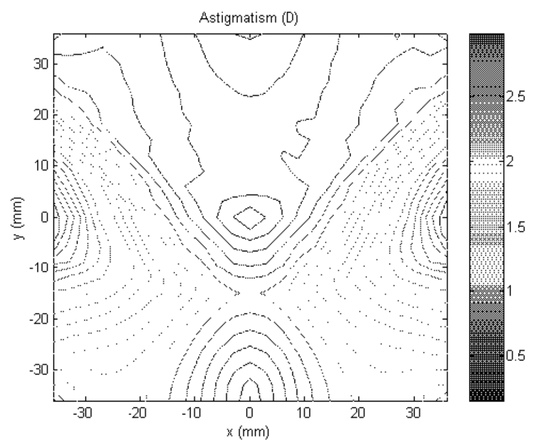

Fig. 3: Astigmatism (asphericity) level curves for-1 D base curve with a +2 D add

also adapted the method to accept a weight map for ω_h, ω_a, ω_f, ω_{fx}, ω_{fy}, ω_{fxy} which enables the designer to control the grid points at which a certain design parameter is more important in relation to the other parameters. These weight maps can also be drawn as level curves and read into the program. This approach also worked very well. A sample design is shown below in Fig. 2 and 3. The results are shown for a design where, a mean curvature map was imposed for a-1 D base curve with a +2 D add, minimizing astigmatism everywhere and imposing the height of a-1 D sphere at the centre of the far region. The values of the weights for this design were: $\omega_h = 1$, $\omega_a = 1$, $\omega_f = 1$, $\omega_{fx} = 1$, $\omega_{fy} = 1$, $\omega_{fxy} = 1$. The results in Fig. 2 and 3 are for a very coarse grid of only 25 by 25 points. A diopter (D) is a unit of measurement of the optical power of a lens which is equal to the reciprocal of the focal length measured in metres.

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