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Application of the Method of Least Squares in The Computation of Position at Sea

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Abstract: One of the most commonly used methods for positioning afloat is by measuring ranges from shore stations to the point a float. This technique is widely used on offshore industry (rig placement, pipe lines...etc). The deployment of the shore stations is largely dependent on the configuration of the coastline. The hydrographer finds it is difficult to deploy the desired number of points on the coastline without compromising the factors of strong geometric fix and land path problems. In order to avoid such problems, the acoustic positioning system using satellites and transponders is nowadays used frequently for all offshore engineering projects. In this study, an attempt to determine the optimum range observations to a float point, which could satisfy requirement of coast configuration and meet the accuracy specified.

Key words: Least squares, position fixing, hydrographic surveying, reliability

INTRODUCTION

Hydrographic surveying is one of the modern field sciences nowadays. It became important in the last decades with the increase of the world population. Accordingly, the world started to look at the sea, exploring its resources (minerals, foods… ect) to meet the requirements of the increasing population. Seas are important for recreation, raw materials for industry and for defense purposes (Alan, 1984).

This necessitates a through study of seas and their beds, which led to hydrography (charting of sea beds) and oceanography (study of the nature of the sea).

Position of points at sea can be fixed by means of a number of measured rays to stations with known coordinates (Cross, 1983). It is common practice in offshore surveying to fix the position of a point (such as a moving vessel or oil rig) by measuring the distances to a number of points with known coordinates.

The procedure for computing the required coordinates involves three main stages (Cooper, 1987) viz:

- The projection of the measured distances in to the chosen coordinate system.
- The “adjustment” of these distances to yield the coordinates of the unknown point.
- The assessment of the quality of these coordinates.

We use the observation equations method of least squares because it is the most easiest to apply. The application of the observation equations method of least squares gives estimated parameters and residuals that statistically process certain properties, provided that the observation contains only random errors (Cooper, 1974).

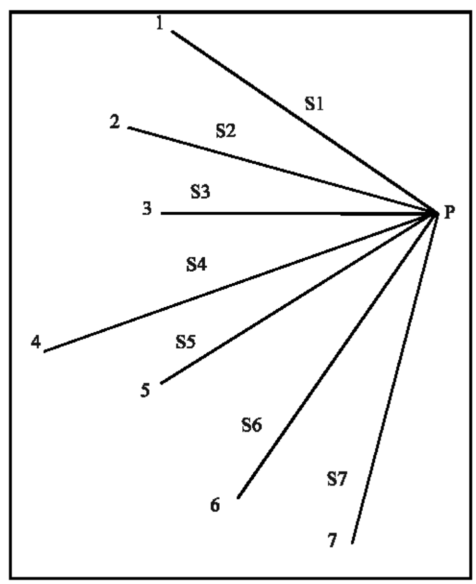

Fig. 1: Seven known off-shore stations and their ranges to the point afloat

This study looks at the complete computational process involved in such a position fix with particular emphasis on the geodetic and statistical aspects of the problem. It includes all the necessary mathematical formulae and a worked example of the computation and analysis (Allan *et al*., 1968) of a fix from seven stations Fig. 1 and Table 1.

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Table 1: Horizontal distances and their standard errors

To station	EDM distances (m)	Standard errors (m)
1	8622.45	0.5
2	5732.17	0.2
3	3069.72	0.2
4	6725.24	0.2
5	9025.57	0.5
6	12138.96	0.5
7	13184.77	0.5

Where:
P is apoint afloat.
1,2,3,……,7 are shore known stations and
s_1, s_2,......, s_7 are the horizontal distances from p to the stations, respectively.

MATERIALS AND METHODS

In this study, we used the method of least squares because it was widely accepted as a method of estimation (Rainsford, 1957), since it possesses the following character:

- Easiness of application to any problem in hand.
- It provides consistency between observed quantities with consideration to all relevant factors.
- The adjustment can be obtained with a little disturbance, as far as possible, to the observations.
- The possibility of checking up the results.
- A assessment of precision and reliability of results.

Precision: Measures of precision are most conveniently done by the sue of the variance covariance matrix it contains all elements of precision.

The construction of the variance covariance matrix depends on the weight matrix, which is equal to the inverse of the variance covariance matrix of the observations, i.e.

$$W = c^{-1}_{1}$$

Where:
W : Is the weight matrix.
c^{-1}_{1} : Is the inverse of the variance covariance matrix of the observations.

Reliability: The word reliable is defined as "consistently good in quality or performance and so deserving trust". In estimation problems, reliability is meant to be the ability of the system to detect gross error in observation or, as Cross (1983) defined it, a measure of the ease with which gross errors may be detected. The detectable error is generally given the term Marginally Detectable Error (MDE) which, in case of a diagonal weight matrix.

Generally, we can differentiate between two aspects of reliability; internal and external.

Internal reliability: Internal reliability is the one which considers the size of the gross error (assuming normal distribution and the presence of one gross error). If we consider type one error then the simple test can be carried out as follows:

- Specify a level of significance (for type one error).
- Determine w-statistic (w_t) from tables (using a two tailed test).
- Compute w_i using the equation

$$w_i = v_i / \sigma\hat{}_{\hat{vi}}$$

- Compare with w_t

If w_t is larger, then no gross error is present.

The test is applied separately to each observation. This is known as data snooping. If we specify the probability of type two errors

$$\Delta_i^u = \delta_i^u \sigma_i^2 / \sigma_{vi}$$

Where Δ^u_i Is the MDE

δ^u_i Is the value computed from specified probabilities of type one and two errors.

External reliability: It is the effect of an undetected gross errors on the parameters and on the quantities computed from them. In a sense, therefore, external reliability is more important than internal reliability as we don not care too much about the size of an undetected gross error as long as it has no effect on the determined parameters.

After least squares estimation of the parameters, the effect of (MDE) (for each observation) on the parameters is given by:

$$\Delta \hat{x}_i = (A^T wA)^{-1}.A^T w\Delta b_i$$

Where Δb_i is a null vector except for the ith position which is equal to Δ^u_i

The largest elements of $\Delta\hat{x}_i$ will be considered as the a measure of the effect of an undetected gross error of the size of a marginally detectable error on the estimated parameters.

Consider (q) as a quantity estimated from parameters and to be the effect of (MDE) on the derived quantities. It can be shown (Cross, 1983) when W is diagonal would be given by

$$\Delta \hat{q}_i \leq \delta_i^u \gamma_i \hat{\sigma}_{qi}$$

Where:

$$\gamma_i = \hat{\sigma}_i / \sigma_{\hat{v}_i}$$

$\hat{\sigma}_{qi}$	:	Is the standard deviation of q
$\hat{\sigma}$	:	Is standard error of the ith observation
$\hat{\sigma}_{vi}$	:	Is the square root of the ith diagonal element of $C\hat{v}$

For uncorrelated observations, the following could be established:

$$\rho_i = \sigma_i / \sigma \hat{v}_i$$

$$\rho^2_i = \sigma^2_i / \hat{\ }^2_{vi}{}^2$$

$$\gamma_i^2 = \hat{\sigma}_i^2 / \sigma^2_{vi} = \sigma_i^2 / \sigma^2_{vi} - 1$$

$$\therefore \gamma_i^2 = \rho^2_i - 1$$

It follows that if an observation has high internal reliability if must also have high external reliability and conversely low internal reliability reflects low external reliability.

RESULTS

The following tests are carried out as outlined below and the results are obtained. In these tests a land based system was used for positioning. Different ranges were used to determine the optimum number of ranges for a particular position fix. Analysis of precision and reliability is considered.

Three known stations and three ranges: Referring to, Table 1, Fig 1, Three iterations are carried out.

Provisional coordinates are estimated graphically (any method would do (Methley, 1986) to be:

$$E' = 33000.00$$
$$N' = 71800.00$$

The least squares estimates of the parameters were obtained as follows:

$$dE = -0.009$$
$$dN = -0.0005$$

And the final coordinates of p are:

$$E = 33028.28-0.009 = 33028.27$$
$$N = 71865.69-0.0005 = 71865.69$$

With least squares estimates of the residuals:
Which was calculated from:

$$\hat{v} = A\hat{x} - b$$

Where:

$\hat{v}$	:	Is the least squares estimates of the residuals and it is (n*1).
A	:	Is the coefficient matrix of dimension and is (n*m) matrix.
$\hat{x}$	:	Is the least squares estimates of the parameters, (m*1).
b	:	Is the observed value of a measured quantity.

With standard error of unit weight

$$\sigma_o = 3.8$$

And variance covariance matrix of residuals:

$$C_{\hat{v}} = \begin{bmatrix} 0.1799 & -0.0406 & 0.0193 \\ '' & 0.009 & -.0043 \\ '' & '' & 0.0021 \end{bmatrix}$$

Which was calculated from:

$$C_{\hat{v}} = w^{-1} - A\left(A^T w A\right)^{-1} A^T$$

Where:

$C_{\hat{v}}$	:	Is asystematic, full, singular idempotent and is an (n*n) matrix.
w	:	Is the weight matrix (n*n) matrix.
A	:	Is the coefficient matrix of dimension and is (n*m) matrix.

After using the above corrections, the reliability is obtained (Table 2) as follows:

Where:

ρ_i	:	Is the external reliability.
σ_i	:	Is the standard error.
$\sigma_{\hat{v}_i}$	:	Is the square root of the ith diagonal element of $C\hat{v}$

Four known stations and four ranges: Referring to, Table 1 and Fig. 1, iterations are carried out as before..
Provisional coordinates are estimated graphically:

$$E' = 33000.0$$
$$N' = 71800.00$$

Table 2: The reliability of three known stations

Observation	σ_i	σ_v	$\rho_i = \sigma_i/\sigma_{vi}$
1-p	0.5	0.42	1.19
2-p	0.2	0.09	2.22
3-p	0.2	0.04	5.0

Table 3: The reliability of four known stations

Observation	σ_i	σ_v	$\rho_i = \sigma_i/\sigma_{vi}$
1-p	0.5	0.43	1.16
2-p	0.2	0.11	1.82
3-p	0.2	0.05	4.0
4-p	0.2	0.48	1.04

The least squares estimates of parameters:

$$dE = -0.01$$
$$dN = 0.0008$$

And the final coordinates of p are:

$$E = 33028.43-0.01 = 33028.4$$
$$N = 71865.68 + 0.0008 = 71865.68$$

$$\hat{v} = \begin{bmatrix} -1.47 \\ 0.48 \\ -0.133 \\ -1.05 \end{bmatrix}$$

$$\sigma_o = 3.1$$

$$C_{\hat{v}} = \begin{bmatrix} 0.1839 & -0.0372 & 0.0204 & -0.0302 \\ '' & 0.012 & -0.0034 & -0.0254 \\ '' & '' & 0.0024 & -0.0083 \\ '' & '' & '' & 0.2263 \end{bmatrix}$$

And the reliability is obtained (Table 3) as follows:

Observed quantities with systematic error: We assume that all measured distances to have a systematic scale error of

SPPm ($1/10^6$), using provisional coordinates:

$$E` = 33028.43$$
$$N` = 71865.68$$

Which are estimated graphically use just one iteration.

The least squares estimates of the parameters were obtained as follows:

$$dE = -0.92$$
$$dN = -1.10$$
$$S = 210$$

Table 4: The reliability of four known stations with scale error

Observation	σ_i	σ_v	$\rho_i = \sigma_i/\sigma_{vi}$
1-p	0.5	0.371	1.35
2-p	0.2	0.197	1.02
3-p	0.2	0.195	1.03
4-p	0.5	0.376	1.33

And the final coordinates of p are:

$$E = 33028.43 - 0.92 = 33027.51$$
$$N = 71865.68 - 1.10 = 71864.58$$
$$\sigma_o = 0.67$$

$$\hat{v} = \begin{bmatrix} -0.32 \\ 0.02 \\ -0.04 \\ 0.33 \end{bmatrix}$$

And the reliability is obtained (Table 4) as follows:

Five known stations and five ranges: Referring Table 1, Fig1, iteration methods are obtained just one time. Provisional coordinates are estimated graphically to be:

$$E` = 33028.42$$
$$N` = 71865.68$$

The least squares estimates of the parameters were obtained as follows:

$$dE = 0.002$$
$$dN = -0.001$$

And the final coordinates of p are:

$$E = 33028.42 + 0.002 = 33028.42$$
$$N = 71865.68 - 0.001 = 71865.68$$

$$\hat{v} = \begin{bmatrix} -1.46 \\ -0.002 \\ 0.48 \\ -0.13 \\ -1.05 \end{bmatrix}$$

$$\sigma_o = 2.50$$

$$C_{\hat{v}} = \begin{bmatrix} 0.2216 & 0.0253 & -0.0148 & 0.0113 & -0.0115 \\ '' & 0.0171 & -0.0150 & 0.0061 & -0.0125 \\ '' & '' & 0.0253 & -0.0088 & -0.0143 \\ '' & '' & '' & 0.0046 & -0.0128 \\ '' & '' & '' & '' & 0.02355 \end{bmatrix}$$

Table 5: The reliability of five known stations

Observation	σ_i	σ_v	$\rho_i = \sigma_i/\sigma_{vi}$
1-p	0.5	0.47	1.06
2-p	0.2	0.13	1.54
3-p	0.2	0.16	1.25
4-p	0.2	0.07	2.86
5-p	0.5	0.49	1.02

And the reliability is obtained (Table 5) as follows:

Six known stations and six ranges: Referring to Table 1, Fig 1, iteration methods are obtained just one time.
Using

$$E` = 33028.42$$
$$N` = 71865.68$$

Which are estimated graphically.

The least squares estimates of the parameters were obtained as follows:

$$dE = 0.002$$
$$dN = -0.00$$

And the final coordinates of p are:

$$E = 33028.42 + 0.002 = 33028.42$$
$$N = 71865.68 - 0.00 = 71865.68$$

$$\hat{v} = \begin{bmatrix} -1.460 \\ -0.002 \\ 0.48 \\ -0.130 \\ -0.004 \\ -1.050 \end{bmatrix}$$

$$\sigma_o = 2.20$$

$$C_{\hat{v}} = \begin{bmatrix} 0.2217 & -0.0252 & -0.0146 & 0.0116 & -0.0033 & -0.0113 \\ '' & 0.0171 & -0.0147 & 0.006 & -0.0057 & -0.0122 \\ '' & '' & 0.0259 & -0.0078 & -0.0120 & -0.0136 \\ '' & '' & '' & 0.0064 & -0.0204 & -0.0117 \\ '' & '' & '' & '' & 0.2333 & -0.0136 \\ '' & '' & '' & '' & '' & 0.2363 \end{bmatrix}$$

And the reliability is obtained (Table 6) as follows:

Table 6: The reliability of six known stations

Observation	σ_i	σ_v	$\rho_i = \sigma_i/\sigma_{vi}$
1-p	0.5	0.47	1.06
2-p	0.2	0.13	1.54
3-p	0.2	0.16	1.25
4-p	0.2	0.08	2.50
5-p	0.5	0.48	1.04
6-p	0.5	0.49	1.02

Seven known stations and seven ranges: Referring to Table 1 and Fig. 1, iteration methods are obtained just one time.
Using

$$E` = 33028.42$$
$$N` = 71865.68$$

Which are estimated graphically.

The least squares estimates of the parameters were obtained as follows:

$$dE = -0.004$$
$$dN = -0.004$$

And the final coordinates of p are:

$$E = 33028.42 - 0.004 = 33028.42$$
$$N = 71865.68 - 0.004 = 71865.68$$

$$\hat{v} = \begin{bmatrix} -1.530 \\ -0.056 \\ 0.42 \\ -0.0186 \\ -0.065 \\ -1.1100 \\ 1.12 \end{bmatrix}$$

$$\sigma_o = 2.20$$

$$C_{\hat{v}} = \begin{bmatrix} 0.2222 & -0.0247 & -0.0140 & 0.0121 & 0.0027 & -0.0107 & -0.0108 \\ '' & 0.0177 & -0.0141 & 0.0071 & -0.0051 & -0.0115 & -0.0116 \\ '' & '' & 0.0266 & 0.0072 & -0.0113 & -0.0129 & -0.0129 \\ '' & '' & '' & 0.0069 & -0.0198 & -0.0111 & -0.0110 \\ '' & '' & '' & '' & 0.2340 & -0.0129 & -0.0128 \\ '' & '' & '' & '' & '' & 0.2370 & -0.0130 \\ '' & '' & '' & '' & '' & '' & 0.2370 \end{bmatrix}$$

And the reliability is obtained (Table 7) as follows:

Table 7: The reliability of seven known stations

Observation	σ_i	σ_v	$\rho_i = \sigma_i/\sigma_{vi}$
1-p	0.5	0.47	1.06
2-p	0.2	0.13	1.54
3-p	0.2	0.16	1.25
4-p	0.2	0.08	2.50
5-p	0.5	0.48	1.04
6-p	0.5	0.49	1.02
7 -p	0.5	0.49	1.02

DISCUSSION

It should be emphasized that the number of iterations is not related to the accuracy of the fix. A large

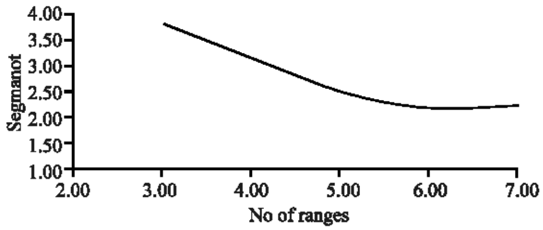

Fig. 2: Graph represents the standard errors of all seven ranges

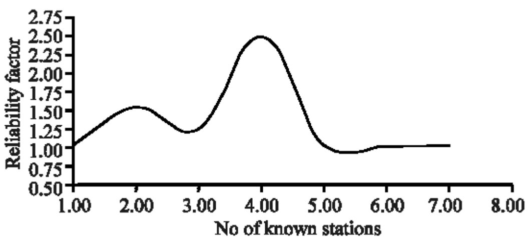

Fig. 3: Graph represents the internal reliability of seven known off-shore stations

number of iterations simply indicate poorly chosen provisional coordinates.

- It is very difficult to the hydrographer to deploy the desired number of points on the coast line because of the configuration of the shore line , so we must find number of points on the coast line putting in mind two problems:

- Land path.
- Strong of the geometric fix.

After this scarce and limited calculation of least squares estimates, we find that after six ranges σ_o remain steady, so any additional ranges are nonsense and the graph below show this fact, Fig. 2.

- A position fix is said to be reliable if we are sure that it does not contain gross error.

After computing the internal reliability for seven known off-shore stations, Fig. 3 drawn to show that a station having a low reliability is rejected.

So, according to the above graph the known station number four must be rejected.

CONCLUSION

The main conclusions drawn from the study are as follows:

- The optimum number of ranges that gives an accurate and reliable fix is six ranges.
- The model errors are eliminated when using the method of least squares with additional parameters. This is clear from the change in the standard error of unit weight and the sizes of residuals.

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