Journal of Engineering and Applied Sciences 4 (1): 77-82, 2009
ISSN: 1816-949X

Applying the WKB Method to the Bifurcation of an Everted Spherical Shell Made of Elastic Varga Material

M. Sanjarani Pour and M. Vakilian
Department of Mathematics, Sistan and Baloochestan University, Zahedan, Iran

Abstract: The WKB method is a powerful tool to obtain solutions for Eigenvalue problems. We apply the WKB method to the bifurcation analysis of everted a spherical shells composed of Varga material. Incompressible cases are considered. The method is degenerate, but we obtain explicit bifurcation criteria and compare with previous numerical approximations.

Key words: Elastic, spherical shells, eversion, bifurcation, asymptotic, WKB method

INTRODUCTION

Eigenvalue problem that results from the linear bifurcation analysis for which, it is possible to write the exact solution are encountered quite seldom. Most often either numerical or asymptotic methods are used for searching the solutions. In many theoretical and applied problems, the possibility of obtaining the asymptotic solution allows to carry out the most complete analysis of a problem. Therefore, hardly there is a necessity to explain in detail importance of creating and investigating asymptotic methods for the solving the Eigenvalue problem by Bush (1992). The Wentzel-Kramers-Brillouin quasi classical approximation (or WKB method) is one of basic and most universal asymptotic methods of solving problems of theoretical and mathematical physics In a series of studies, Haughton and Orr (1995), Chen and Haughton (1997) and Haughton and Chen (1999) were investigated various aspects of the problem of everting isotropic hyperelastic shells. In particular for the bifurcation problem for these shells it turns out that there is a critical mode number, which gives the thickest shell for which, we can expect the shape of the undeformed shell to be maintained upon eversion. For different materials these mode numbers maybe finite or infinite. The numerical analysis of the bifurcation problem seems to cope very well. Finite mode numbers are found as accurately as we want while, the numerical methods appear to approach an asymptote for the infinite mode number. However, it would be interesting and useful to know what the asymptote actually is, if only to confirm that the numerical solution is approaching the correct value. Fu and Sanjarani (2002) and Sanjarani (2001) and have shown that with the aid of a symbolic manipulation package, it is possible to apply the WKB method to the incompressible cylinder problem discussed by Haughton and Orr (1995). They found that the WKB method was degenerate for the Varga material.

In the study, we apply the WKB method to the bifurcation analysis of everted spherical shells of Varga materials and compare with numerical approximations (Haughton and Chen, 1999). We show efficiency of WKB in such case.

Spherical shell: The undeformed spherical shell occupies the region:

$$0<A\le R\le B,\quad 0\le\Theta\le\pi,\quad 0\le\Phi\le 2\pi \tag{1}$$

in spherical polar coordinates (R, Θ, Φ) where, A and B are the inner and outer radii of the undeformed cylinder. The spherical shell undergoes the deformation:

$$r=r(R),\quad \theta=\pi-\Theta,\quad \phi=\Phi \tag{2}$$

where,

r, θ, ϕ = Also spherical polar coordinates and
r(R) = A smooth, strictly decreasing function

The everted shell occupies the region

$$0<a\le r\le b,\quad 0\le\theta\le\pi,\quad 0\le\phi\le 2\pi \tag{3}$$

and the principal stretches are given by Eq. 4:

$$\lambda_1=-r'(R),\ \lambda_2=\lambda_3=\frac{r}{R}=\lambda \tag{4}$$

where, a dash indicates differentiation with respect to R. However, we shall employ an Eulerian formulation of the problem so we make appropriate changes of variables when needed.

Corresponding Author: M. Sanjarani Pour, Department of Mathematics, Sistan and Baloochestan University, Zahedan, Iran

MATERIALS AND METHODS

Incompressible material: When, the material is incompressible we require Eq. 5:

$$J = \lambda_1 \lambda_2 \lambda_3 = 1 \tag{5}$$

and then Eq. (4) gives Eq. (6):

$$r^3 - a^3 = B^3 - R^3 \tag{6}$$

We also note that:

$$B^3 - a^3 = B^3 - A^3 \tag{7}$$

The sphere is composed of a homogeneous, isotropic and hyper elastic material, which is associated with a strain-energy function that depends on the deformation gradient through the principal stretches:

$$W = W(\lambda_1, \lambda_2, \lambda_3)$$

For incompressible materials the principal components of the Cauchy stress tensor σ are given by Eq. (8):

$$\sigma_{ii} = \sigma_i - p = \lambda_i W_i - p, \; i = 1,2,3, \text{ no sum} \tag{8}$$

where, p is the hydrostatic pressure and $W_i = \partial W/\partial \lambda_i$.

In the absence of body forces, the equilibrium of the deformed spherical shell requires the Cauchy stress tensor to be divergence free. By Eq. (2) with Eq. (4 and 5), the only non-trivial equilibrium Eq. (9) is:

$$\frac{d}{dr}\sigma_{11} + \frac{2}{r}(\sigma_{11} - \sigma_{22}) = 0 \tag{9}$$

In general, we assume, that the cavity does not close so that we have a>0 and the appropriate boundary conditions are then zero traction on the inner and outer surfaces:

$$\sigma_{11} = 0, \text{ for } R = A, R = B \tag{10}$$

Chen and Haughton (1997) and Haughton and Chen (1999) for further discussion of boundary conditions. For this problem, the equilibrium Eq. (9) can be integrated to give the hydrostatic pressure to within an arbitrary constant. The two boundary conditions Eq. (10) determine this arbitrary constant and the deformed inner radius a. In this study, we are concerned with the Varga material where in the incompressible case, we have:

$$W(\lambda_1, \lambda_2, \lambda_3) = 2\mu(\lambda_1 + \lambda_2 + \lambda_3 - 3) \tag{11}$$

where, μ is the positive ground state shear modulus. In this case, the equilibrium Eq. (9) reduces to:

$$\frac{dp}{dr} = -\frac{8\mu}{R(r)} \tag{12}$$

Using Eq. (6), this can then be integrated. The constant of integration is found by applying Eq. (10). We can then write down Eq. (10), which gives an equation for a in terms of A and B.

Bifurcation: The equations describing incremental deformations are well known, Ogden (1997). For completeness, we give a brief description. Full details for the eversion of incompressible spherical shells can be found by Haughton and Chen (1999). In the absence of body forces, the incremental equilibrium equations can be written as:

$$\operatorname{div}\chi = 0 \tag{13}$$

where:

div = The divergence operator in the current configuration

χ = The increment in the nominal stress in the current configuration

Since, no loading is imposed on the surface of the body the incremental boundary conditions are given by Eq. (14):

$$\chi^T n = 0 \tag{14}$$

where, n is the unit outward normal to the surface of the everted shell. The incremental constitutive law is Eq. (15):

$$\chi = B V^T + p\, V - \dot{p} I \tag{15}$$

where:

B = The fourth order tensor of instantaneous moduli in the current configuration

I = The identity and we have written V for $\dot{F}_0$

The non-zero components of B for a general isotropic material can be written as Eq. (16):

$$\begin{gathered} B_{ijij} = \lambda_i^2 \frac{\sigma_i - \sigma_j}{\lambda_i^2 - \lambda_j^2}, \; \lambda_i \neq \lambda_j \\ B_{iijj} = B_{jjii} = \frac{\lambda_i \lambda_j}{J} \frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j}, \\ B_{ijij} - B_{ijji} = B_{ijij} - B_{jiij} = \sigma_i, \; i \neq j \end{gathered} \tag{16}$$

where, for incompressible materials, σ_i are defined in Eq. (8) and we set J = I. For this problem, we have $\lambda_1 = \lambda_2$ and we obtain Eq. (17):

$$B_{2323} = (B_{2222} - B_{2233} + \sigma_2)/2 \qquad (17)$$

It was shown by Haughton and Chen (1999) that without loss of generality, we may consider an incremental displacement Eq. (18):

$$X_0 = (u(r, \theta), v(r, \theta), 0) \qquad (18)$$

with respect to spherical coordinates. Hence, V has components Eq. (19):

$$V = \begin{bmatrix} u_r & \frac{u_\theta - v}{r} & 0 \\ v_r & \frac{u + v_\theta}{r} & o \\ 0 & 0 & \frac{u + v\cot\theta}{r} \end{bmatrix} \qquad (19)$$

where, subscripts denote partial derivatives. Since, the material is incompressible, we have Eq. (20):

$$\text{tr } V \equiv u_r + \frac{\mu}{r} + \frac{v_\theta + v\cot\theta}{r} = 0 \qquad (20)$$

Substituting Eq. (18) with Eq. (19) into (13), we obtain Eq. (21):

$$\begin{aligned} p_r &= B_{1111} u_{rr} + (rB'_{1111} + rp' + 2B_{1111})\frac{u_\theta}{r} \\ &+ B_{2121}\frac{u_{\theta\theta} + u_\theta\cot\theta}{r^2} + 2(rB'_{1122} + B_{1122} - B_{2222} - B_{2233}) \\ &\frac{v_\theta + v\cot\theta}{r^2} + (B_{1122} + B_{1221})\frac{v_{r\theta} + v_r\cot\theta}{r} \end{aligned} \qquad (21)$$

$$\begin{aligned} \dot{p}_\theta &= rB'_{1221} + rp' + B_{2222} + B_{2233} + B_{1221} + B_{2121})\frac{u_\theta}{r} \\ &+ (B_{1122} + B_{1221})u_{r\theta} + rB_{1212}v_{rr} + (rB'_{1212} + 2B_{1212})v_r \\ &-(rB'_{1221} + rp' + B_{1221} + B_{2121} + B_{2233} + \cot^2\theta B_{2222})\frac{v}{r} \end{aligned} \qquad (22)$$

$$\dot{p}_\phi = 0 \qquad (23)$$

If, we look for separable solutions and write as Eq. (24):

$$\begin{aligned} u &= f_n(r)p_n[\cos n\theta] \\ v &= g_n(r)\frac{\partial p_n}{\partial\theta}[\cos n\theta] \\ \dot{p} &= k_n(r)p_n[\cos n\theta] \end{aligned} \qquad (24)$$

where, there is implied summation from n = 0...∞ and p_n ($\cos\theta$) are legendre polynomials.

Substituting Eq. (24) with Eq. (21) and also m = n (n + 1):

$$\begin{aligned} &k' = B_{1111}f'' + (rB'_{1111} + rp' + 2B_{1111}) \\ &\frac{f}{r} + 2(rB'_{1122} + B_{1122} - B_{2121} - B_{2222} - B_{2233} - \\ &m\frac{B_{2121}}{r})\frac{f}{r^2} + m(rB'_{1122} + B_{1122} - B_{2121} - B_{2222} - B_{2233}) \\ &\frac{g}{r^2} - m(B_{1122} + B_{1221})\frac{g'}{r} \end{aligned} \qquad (25)$$

and as past case by substituting Eq. (24) with Eq. (22) and also m = n (n + 1):

$$\begin{aligned} k &= (rB'_{1221} + rp' + B_{2222} + B_{2233} + B_{1221} + B_{2121})\frac{f}{r} \\ &+ (B_{1122} + B_{1221})f' + rB_{1212}g'' - (rB_{1221} + rp' \\ &+ B_{1221} + B_{2121} + B_{2233}) + (m-1)B_{2222}\frac{g}{r} \end{aligned} \qquad (26)$$

It is now possible to eliminate g_n and k_n to obtain a single equation for f_n (r). In general, there is no particular advantage in doing this but for the asymptotic analysis, it greatly simplifies the calculations. In this case, the incremental equilibrium equation and boundary conditions for an arbitrary incompressible material can be written as:

$$\begin{aligned} &B_{1212}r^4f^{(IV)} + 2r^3[4B_{1212} + rB'_{1212}]f''' \\ &+[\{2B_{1122} - B_{2222} - B_{1111} + 2B_{1221}\}m + r^2B''_{1212} \\ &+10rB'_{1212} + 10B_{1212} - B_{2233} - B_{1221} + B_{2121} + B_{2222}]r^2f'' \\ &+[\{2(rB'_{1122} + rB'_{1221} + 2B_{1122} + 2B_{1221} - B_{2222} - B_{1111}) \\ &-rB'_{1111} - rB'_{2222}\}m + 2(r^2B''_{1212} + 2rB'_{1212}) + rB'_{2222} \\ &-rB'_{2233} + rB'_{2121} - rB'_{1221}\,2(B_{2121} - B_{2233} + B_{2222} \\ &-2B_{1212} - B_{1221})]rf' + [(m+1)B_{2121} + rB'_{2233} \\ &-rB'_{2222} - B_{1221} - B_{2233} + B_{2222} \\ &\;r^2B''_{1212} - rB'_{2121} + rB'_{1221} + 2rB'_{1212} - 2B_{1212}] \\ &\;(m-2)f = 0 \end{aligned} \qquad (27)$$

where, m = n (n + 1) and we have omitted the subscript n from f_n (r). The boundary conditions Eq. (14) can be written as:

$$\begin{aligned} &B_{1212}r^3f''' + 4r^2B_{1212}f'' + [B_{2222} + B_{2121} - B_{1221} - B_{2233} \\ &+m(2B_{1221} + 2B_{1122} - B_{1212} - B_{2222} - B_{1111})]rf' \\ &+[2B_{1221} + 2B_{1122} - B_{1212} - B_{2222} - B_{1111}](m-2)f = 0, \end{aligned} \qquad (28)$$

on the boundary r = a, r = b and

$$r^2 f'' + 2rf' + (m-2)f = 0 \quad (29)$$

We now have a homogeneous system for f (r). The bifurcation criterion is that there should be non-trivial solutions to this system.

RESULTS AND DISCUSSION

Asymptotic results for n>>1: Following Fu and Sanjarani (2002, 2001), apply the WKB method and look for solutions of the form:

$$f(r) = T(r)e^{\int_a^r nS(x)dx} \quad (30)$$

Where:

$$T = T_0 + \frac{T_1}{n} + \frac{T_2}{n^2} + \frac{T_3}{n^3} + \cdots, \quad (31)$$

and S(r) is to be determined. For the aim, it is sufficient to look at the leading two terms only. Substituting Eq. (30) and (31) into the incremental equilibrium Eq. (27) gives to leading order in n:

$$\begin{aligned} &B_{1212}\, r^4S^4 - (B_{2222} - 2(B_{1221} + B_{1122}) \\ &+ B_{1111})r^2S^2 + B_{2121}]T_0 = 0 \end{aligned} \quad (32)$$

independent solutions s^i = 1, 2, 3, 4. When, we go to the next highest order in the equilibrium equation, we then obtain a simple algebraic equation for T_0, which depends on S (r) . Substituting the four solutions for S(r) gives four solutions for T_0 (r). This might be thought of as the standard WKB method. Combining T_0 and S through Eq. (30), then provides the four independent solutions that we require for f (r).

Looking at subsequent orders of n gives similar algebraic equations for T_1, T_2, etc., where, the equations involve derivatives of the functions already found.

It then follow that the non-zero elastic moduli are given by Eq. (33):

$$B_{ijij} = \frac{\lambda_i^2}{\lambda_i + \lambda_j}\ (i \neq j), \qquad B_{ijji} = -\frac{\lambda_i\lambda_j}{\lambda_i + \lambda_j}\ (i \neq j) \quad (33)$$

However, the problem is rather different for the Varga material Eq. (11). In this case, the equilibrium equation Eq. (31) reduces to Eq. (34):

$$T_0(r^4\lambda^4S^4 - 2r^2\lambda^7S^2 + \lambda^{10}) = 0 \quad (34)$$

and so we have only two independent solutions for S(r). Using the notation introduced above we write as:

$$\begin{aligned} S_1 = S_3 &= \sqrt{\lambda^3/r^2} = \sqrt{r/R(r)^2}, \\ S_2 = S_4 &= -\sqrt{\lambda^3/r^2} = -\sqrt{r/R(r)^2} \end{aligned} \quad (35)$$

Looking at the next order in Eq. (27), for the Varga material, we find that the equation is identically zero. This Effectively removes the possibility of an algebraic expression for T_0, which is to be expected since, S is deficient. The third order Eq. (36) is:

$$\begin{aligned} &T_0'' - (rS - 3 + 2\lambda^3)T_0' - (4rS(7-\lambda^3) \\ &- 7 + 5\lambda^6 + 22\lambda^3)T_0/16 = 0 \end{aligned} \quad (36)$$

having used Eq. (32). Substituting in the two independent solutions for S(r) gives the required four solutions for T_0. We denote these four independent solutions by T^k_0 (r), k = 1, 2, 3, 4. Similarly in principal, we will have a hierarchy of equations successively giving T_1, T_2 etc. We can now replace Eq. (30 and 31) by Eq. (37):

$$f(r) = \sum_{i=1}^{4} C^k T^k(r)E^{(k)}(r) \quad (37)$$

Where:

$$T^k = T_0^k(r) + \frac{T_1^k(r)}{n} + \cdots \text{ and } E^k(r) = e^{\int_a^r nS^k(x)dx} \quad (38)$$

for some constants C^k, k = 1, 2, 3, 4. As we shall shown below a leading order analysis of the bifurcation criterion does not require an explicate valuation of T_0.

On substituting Eq. 37 into the boundary conditions Eq. (28 and 29), we obtain Eq. (39):

$$\sum_{i=1}^{4} C^k\alpha^{(i)}(r)E^{(i)}(r) = 0, \sum_{i=1}^{4} C^k\gamma^{(i)}(r)E^{(i)}(r) = 0 \quad (39)$$

Where to O (l/n), we have:

$$\begin{aligned} \alpha^{(i)} &= \alpha_0^{(i)} + \frac{1}{n}\alpha_1^{(i)} + O\left(\frac{1}{n^2}\right), \\ \alpha^{(i)} &= \gamma_0^{(i)} + \frac{1}{n}\gamma_1^{(i)} + O\left(\frac{1}{n^2}\right) \end{aligned} \quad (40)$$

and

$$\begin{aligned} \alpha_0^{(i)} &= rS^{(i)}T_0^{(i)}[r^2(S^{(i)})^2 - 2\lambda^3 - 1] \\ \alpha_1^{(i)} &= rS^{(i)}T_1^{(i)}(r^2(S^{(i)})^2 - 2\lambda^3 - 1) + 3r^3(S^{(i)})^2(T_0^{(i)})' \\ &+3r^3S^{(i)}(S^{(i)})'T_0^{(i)} + 4r^2(S^{(i)})^2\, T_0^{(i)} - r(2\lambda^3+1)(T_0^{(i)})' \\ &+ rS^{(i)}(2\lambda^3+1)T_0^{(i)} - (\lambda^6 + \lambda^3)T_0^{(i)} \end{aligned} \quad (41)$$

and

$$\gamma_0^{(i)} = (r^2(S^{(i)})^2 + 1)T_0^{(i)}$$
$$\gamma_1^{(i)} = (r^2(S^{(i)})^2 + 1)T_1^{(i)} + 2r^2S^{(i)}(T_0^{(i)})' + r^2(S^{(i)})'T_0^{(i)} + 2rS^{(i)}T_0^{(i)} + T_0^{(i)} \quad (42)$$

The boundary conditions Eq. (39) yield a matrix equation of the from:

$$\sum_{i=1}^{4} M_{ij}\, C^j = 0, \ (i = 1, 2, 3, 4) \quad (43)$$

Where:

$$(M_{ij}) = \begin{bmatrix} \alpha^{(1)}(a) & \alpha^{(2)}(a) & \alpha^{(3)}(a) & \alpha^{(4)}(a) \\ \gamma^{(1)}(a) & \gamma^{(2)}(a) & \gamma^{(3)}(a) & \gamma^{(4)}(a) \\ E_1\alpha^{(1)}(b) & E_2\alpha^{(2)}(b) & E_3\alpha^{(3)}(b) & E_4\alpha^{(4)}(b) \\ E_1\gamma^{(1)}(b) & E_2\gamma^{(2)}(b) & E_3\gamma^{(3)}(b) & E_4\gamma^{(4)}(b) \end{bmatrix} \quad (44)$$

and $E_i = E^{(i)}$ (b) (i = 1, 2, 3, 4). A non-trival solution for C requires:

$$\det\,(M_{ij}) = 0 \quad (45)$$

In the case, E_1, E_3 are exponentially large whereas, E_2, E_4 are exponentially small. Thus, we have

$$\frac{\det\,(M_{ij})}{E_1\ E_3} = \begin{bmatrix} \alpha^{(1)}(b) & \alpha^{(3)}(b) \\ \gamma^{(1)}(b) & \gamma^{(3)}(b) \end{bmatrix}\begin{bmatrix} \alpha^{(2)}(b) & \alpha^{(4)}(a) \\ \gamma^{(2)}(a) & \gamma^{(4)}(a) \end{bmatrix} + \text{EST} \quad (46)$$

Where, EST stands for exponentially small terms. Thus, the Eq. (43) can be replaced with an error, which is exponentially small by two matrix Eq. (47):

$$\begin{bmatrix} \alpha^{(1)}(b) & \alpha^{(3)}(b) \\ \gamma^{(1)}(b) & \gamma^{(3)}(b) \end{bmatrix}\begin{bmatrix} C^1 \\ C^3 \end{bmatrix} = 0 \quad (47)$$

$$\begin{bmatrix} \alpha^{(2)}(b) & \alpha^{(4)}(a) \\ \gamma^{(2)}(a) & \gamma^{(4)}(a) \end{bmatrix}\begin{bmatrix} C^2 \\ C^4 \end{bmatrix} = 0 \quad (48)$$

By using Eq. (41 and 42), we note that Eq. (47 and 48) can be replaced, respectively by Eq. (49):

$$\begin{bmatrix} \hat{\alpha}^{(1)}(b) & \hat{\alpha}^{(3)}(b) \\ \hat{\gamma}^{(1)}(b) & \hat{\gamma}^{(3)}(b) \end{bmatrix}\begin{bmatrix} C^1\,T^{(1)}(b) \\ C^3\,T^{(3)}(b) \end{bmatrix} = 0 \quad (49)$$

$$\begin{bmatrix} \hat{\alpha}^{(2)}(b) & \hat{\alpha}^{(4)}(a) \\ \hat{\gamma}^{(2)}(a) & \hat{\gamma}^{(4)}(a) \end{bmatrix}\begin{bmatrix} C^2T^{(2)}(a) \\ C^4T^{(2)}(a) \end{bmatrix} = 0 \quad (50)$$

Where:

$$[\hat{\alpha}^{(i)}(a), \hat{\gamma}^{(i)}(a)] = [\alpha^{(i)}(a), \gamma^{(i)}(a)] / T^{(i)}(a) \quad (51)$$

And

$$[\hat{\alpha}^{(i)}(b), \hat{\gamma}^{(i)}(b)] = [\alpha^{(i)}(b), \gamma^{(i)}(b)] / T^{(i)}(b) \quad (52)$$

To highest order in n both 2×2 determinants are identically zero. To the next order, we get tow symmetrical conditions:

$$(3a^3 - B^3)\left(T_0^2\,(T_0^4)' - T_0^4\,(T_0^2)'\right)|_{r=a} = 0 \quad (53)$$

$$(3b^3 - A^3)\left(T_0^1\,(T_0^3)' - T_0^3\,(T_0^1)'\right)|_{r=b} = 0 \quad (54)$$

The asymptotic bifurcation criterion is then either Eq. (52) and (53).

Since, the function $T^1{}_0(r)$ and $T^3{}_0(r)$ are linearly independent solution of the same second order different Eq. (31), the second factor in Eq. (52) (the Wronskian of $T^1{}_0(r)$ and $T^3{}_0(r)$) is not zero. Similarly, for the second factor in Eq. (54). There then seem to be two possible bifurcation criteria. If we consider:

$$b^3 = A^3/3 \quad (55)$$

The incompressibility result Eq. (7) coupled with a>0 leads to the requirement that

$$A^3/B^3 > \frac{3}{4} \quad (56)$$

Hence, only thin shells are allowable. Further in the interval $3/4<A^3/B^3<1$, we find from Eq. (55) that the deformed inner radius a lies in the interval that the defamed inner radius a lies in the interval:

$$0 < a \leq (1/3)^{1/3} \approx 0.693$$

However, from the analysis outlined in above Eq. (11) for the Varga material Eq. (11), it turns out that everted shells with $A^3/B^3>3/4$ have a deformed inner radius significantly greater than that given by Eq. (43). Hence, the only possible asymptotic bifurcation criterion is:

$$a^3 = B^3/3 \quad (57)$$

In Haughton and Chen (1999), there is a Table 1 of bifurcation results for the Varga material giving critical values of A/B . Here, we repeat the table but include values for a/B. Clearly, we can shown how

$$a/B \to 3^{-1/3} \approx 0.69336127$$

Table 1: Critical values of A/B and corresponding values of a/B for various mode numbers in respect of the incompressible Varga material

Mode n	A/B	a/B
5	0.419483	0.682664
10	0.427109	0.684236
15	0.440293	0.687050
20	0.447465	0.688633
25	0.451753	0.689598
50	0.460106	0.691517
100	0.464099	0.692453
150	0.465393	0.692760
200	0.466033	0.692911
250	0.466414	0.693002
300	0.466668	0.693062
350	0.466848	0.693105
400	0.466983	0.693137
450	0.467087	0.693162
500	0.467171	0.693182
550	0.467240	0.693198
600	0.467297	0.693212
650	0.467345	0.693223

We shows that the numerical results gave the correct asymptotic result to three decimal places despite being very slowly convergent.

CONCLUSION

We have shown how it is generally possible to apply the WKB method to obtain a first order approximation to the bifurcation criterion for a variety of problems. It seems that those materials where, the asymptotic mode number is the critical one will also have a degenerate WKB analysis.

REFERENCES

Bush, A.W., 1992. Perturbation methods for engineers and scientist. School of Coputing and Mathematics Teeside polytechnic Middlesborough, UK.

Chen, Y.C. and D.M. Haughton, 1997. On the existence of exact solutions for the eversion of elastic cylinders J. Elasticity, 49: 79-88.

Fu, Y. and M.S. Pour, 2002. WKB method with repeated roots and its application to the buckling analysis of an everted cylindrical tube. SIAM J. Applied Mathe., 62: 1856-1876.

Haughton, D.M. and A. Orr, 1995. On the eversion of incompressible elastic cylinders. Int. J. Non-Linear Mech., 30: 81-95.

Haughton, D.M. and Y.C. Chen, 1999. On the eversion of incompressible elastic spherical shells. ZAMP, 50: 312-326.

Ogden, R.W., 1997. Non-linear elastic deformations. Dover Publications, New York.

Sanjarani, P.M.S., 2001. Application of the WKB method to some of the buckling problems in finite elasticity Ph.D Thesis Keele.