

Pairwise Semi Star Generalized Homeomorphisms

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Abstract: Homeomorphism helps us to determine the properties of complicated spaces easily. In this sequel, the aim of this short communication is introduce and study two new types of homeomorphisms, namely; pairwise s^*g -homeomorphism and pairwise s^*gc -homeomorphism in bitopological spaces. Here researchers proved that the set of all pairwise s^*gc -homeomorphisms forms a group which contains the set of all pairwise homeomorphisms as a subgroup. Furthermore, the relationships between newly introduced and existing homeomorphisms are established.

Key words: Pairwise s^*g -homeomorphism, pairwise s^*gc -homeomorphism, pairwise g -homeomorphism, pairwise sg -homeomorphism, pairwise gs -homeomorphism, sequel, India

INTRODUCTION

Kelly (1963) initiated the study of bitopological space. Thereafter, several researches (Bose, 1995; Rao and Kannan, 2005; Rao *et al.*, 2007; Fukutake, 1986, 1989; Kannan *et al.*, 2010a, b; Ravi and Thivagar, 2006; Maheshwari and Prasad, 1977; John and Sundaram, 2004) has been made for converting topological concepts to the bitopological spaces.

Homeomorphism is one of the important concepts in topological spaces. It helps us to determine the properties of complicated spaces easily. Semi homeomorphisms were introduced by Biswas (1969) and Crossley and Hildebrand (1971).

Generalized homeomorphisms and gc -homeomorphisms in terms of preserving generalized closed sets (Levine, 1970) were first introduced by Maki *et al.* (1991).

Every homeomorphism is a generalized homeomorphism but not vice versa (Maki *et al.*, 1993b). The two concepts coincide when both the domain and the range satisfy the weak separation axiom $T_{1/2}$. The class of gc -homeomorphism is properly placed between the classes of homeomorphism and g -homeomorphism (Maki *et al.*, 1991). Semi continuous mappings (Biswas, 1970; Levine, 1963) and sg -continuous mappings (Caldas, 1995) were studied. Sg -homeomorphisms and gs -homeomorphisms (Maki *et al.*, 1995) and sg -closed maps and gs -closed maps (Maki *et al.*, 1993a) were introduced by Bhattacharya and Lahiri (1987) with the help of sg -closed sets and gs -closed sets. Rao and Joseph (2000) introduced the concepts of semi star

generalized closed sets and semi star generalized homeomorphisms (Rao and Joseph, 2007) in topological spaces. The aim of this short communication is introduce and study two new types of homeomorphisms, namely; pairwise s^*g -homeomorphism and pairwise s^*gc -homeomorphism in bitopological spaces. Here researchers proved that the set of all pairwise s^*gc -homeomorphisms forms a group which contains the set of all pairwise homeomorphisms as a subgroup. Furthermore, the relationships between newly introduced and existing homeomorphisms are established.

PRELIMINARIES

Let (X, τ_1, τ_2) , (Y, σ_1, σ_2) or simply X, Y denote a bitopological space. For any subset $A \subset X$, $\tau_i\text{-int}(A)$ and $\tau_i\text{-cl}(A)$ denote the interior and closure of a set A with respect to the topology τ_i , respectively. A^c denotes the complement of A in X unless explicitly stated.

A set A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$ - g closed {resp. $\tau_1\tau_2$ - sg closed, $\tau_1\tau_2$ - gs closed, $\tau_1\tau_2$ - s^*g closed} if $\tau_2\text{-cl}(A)$ {resp. $\tau_2\text{-scl}(A)$, $\tau_2\text{-scl}(A)$, $\tau_2\text{-cl}(A)$ } $\subset U$ whenever $A \subset U$ and U is τ_1 -open {resp. τ_1 -semi open, τ_1 -open, τ_1 -semi open}. A function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise g -continuous {resp. pairwise sg -continuous, pairwise gs -continuous, pairwise s^*g -continuous} if $f^{-1}(U)$ is $\tau_i\tau_j$ - g closed {resp. $\tau_i\tau_j$ - sg closed, $\tau_i\tau_j$ - gs closed, $\tau_i\tau_j$ - s^*g closed} for each σ_j -closed set U in Y , $i \neq j$ and $i, j = 1, 2$.

A function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise g -irresolute {resp. pairwise sg -irresolute, pairwise gs -irresolute, pairwise s^*g -irresolute} if $f^{-1}(U)$ is $\tau_i\tau_j$ - g closed

{resp. $\tau_i\tau_j$ -sg closed, $\tau_i\tau_j$ -gs closed, $\tau_i\tau_j$ -s*g closed} for each $\sigma_i\sigma_j$ -g closed {resp. $\sigma_i\sigma_j$ -sg closed, $\sigma_i\sigma_j$ -gs closed, $\sigma_i\sigma_j$ -s*g closed} set U in Y , $i \neq j$ and $i, j = 1, 2$. A function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise s*g-closed {resp. pairwise s*g-open} if $f(U)$ is $\sigma_i\sigma_j$ -s*g closed {resp. $\sigma_i\sigma_j$ -s*g open} for each τ_j -closed {resp. τ_j -open} set U in X , $i \neq j$ and $i, j = 1, 2$. A bijection function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise homeomorphism {resp. pairwise g-homeomorphism, pairwise sg-homeomorphism, pairwise gs-homeomorphism} if both f and f^{-1} are pairwise continuous {resp. pairwise g-continuous, pairwise sg-continuous, pairwise gs-continuous}. A bijection function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise gc-homeomorphism {resp. pairwise sgc-homeomorphism, pairwise gsc-homeomorphism} if both f and f^{-1} are pairwise g-irresolute {resp. pairwise sg-irresolute, pairwise gs-irresolute}.

A bitopological space (X, τ_1, τ_2) is called pairwise $T_{1/2}$ -space {resp. pairwise T_b -space} if every $\tau_1\tau_2$ -g closed set {resp. $\tau_1\tau_2$ -gs closed set} is τ_2 -closed and every $\tau_2\tau_1$ -g closed set {resp. $\tau_2\tau_1$ -gs closed set} is τ_1 -closed.

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Definition: A bijection $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise semi star generalized homeomorphism (pairwise s*g-homeomorphism) if f is both pairwise s*g-continuous and pairwise s*g-open.

Example 1: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\phi, X \setminus \{a\}\}$, $\tau_2 = \{\phi, X \setminus \{b, c\}\}$, $\sigma_1 = \{\phi, Y \setminus \{a\}\}$, $\sigma_2 = \{\phi, Y \setminus \{a\}, \{b, c\}\}$. Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a identity mapping. Then f is pairwise s*g-homeomorphism.

Theorem:

- Every pairwise homeomorphism is pairwise s*g-homeomorphism
- Every pairwise s*g-homeomorphism is pairwise g-homeomorphism
- Every pairwise s*g-homeomorphism is pairwise sg-homeomorphism
- Every pairwise s*g-homeomorphism is pairwise gs-homeomorphism

Proof: Obvious from definitions. But the converses of the assertions of the above theorem are not true in general as can be seen from the following examples.

Example 2: In example 1, f is pairwise s*g-homeomorphism. But it is not pairwise continuous and hence, it is not pairwise homeomorphism.

Example 3: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\phi, X \setminus \{a\}\}$, $\tau_2 = \{\phi, X \setminus \{b, c\}\}$, $\sigma_1 = \{\phi, Y \setminus \{a\}\}$, $\sigma_2 = \{\phi, Y \setminus \{a\}, \{b, c\}\}$. Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function defined by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then f is pairwise g-homeomorphism, pairwise gs-homeomorphism but not pairwise s*g-homeomorphism.

Example 4: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\phi, X \setminus \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, $\tau_2 = \{\phi, X \setminus \{a\}\}$, $\sigma_1 = \{\phi, Y \setminus \{b\}\}$, $\sigma_2 = \{\phi, Y \setminus \{a, c\}\}$. Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function defined by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then f is pairwise sg-homeomorphism but not pairwise s*g-homeomorphism.

However, every pairwise s*g-homeomorphism from a pairwise $T_{1/2}$ -space onto itself is a pairwise homeomorphism and every pairwise g-homeomorphism from a pairwise $T_{1/2}$ -space onto itself is a pairwise s*g-homeomorphism. Also, every pairwise sg-homeomorphism {resp. pairwise gs-homeomorphism} from a pairwise T_b -space onto itself is a pairwise s*g-homeomorphism. Concerning the compositions of the maps, the composition of two pairwise s*g-homeomorphisms is not a pairwise s*g-homeomorphism. Also, one can easily verify that pairwise s*g-homeomorphisms and pairwise gc-homeomorphisms {resp. pairwise sgc-homeomorphisms and pairwise gsc-homeomorphisms} are independent each other.

But if X and Y are pairwise $T_{1/2}$ -spaces then the function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise gc-homeomorphism if and only if f is pairwise s*g-homeomorphism. Also if X and Y are pairwise T_b -spaces then the map $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise s*g-homeomorphism if and only if f is pairwise sgc-homeomorphism {resp. pairwise gsc-homeomorphism}. Moreover for any bijection $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following are equivalent:

- $f^{-1}: (Y, \sigma_1, \sigma_2) \rightarrow (X, \tau_1, \tau_2)$ is pairwise s*g-continuous
- f is pairwise s*g-open
- f is pairwise s*g-closed

Also for any bijection and pairwise s*g-continuous function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following are equivalent:

- f is pairwise s*g-open
- f is pairwise s*g-homeomorphism
- f is pairwise s*g-closed

Definition: A bijection $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise s*gc-homeomorphism if f and f^{-1} are pairwise s*g-irresolute.

Example 5: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\phi, X \{a\}\}$, $\tau_2 = \{\phi, X \{b, c\}\}$, $\sigma_1 = \{\phi, Y \{b\}\}$, $\sigma_2 = \{\phi, Y \{a, c\}\}$. Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function defined by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then f is pairwise s^*gc -homeomorphism.

Theorem: Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a map. Then the following are true:

- Every pairwise homeomorphism is pairwise s^*gc -homeomorphism
- Every pairwise s^*gc -homeomorphism is pairwise g -homeomorphism
- Every pairwise s^*gc -homeomorphism is pairwise sg -homeomorphism
- Every pairwise s^*gc -homeomorphism is pairwise gs -homeomorphism
- Every pairwise s^*gc -homeomorphism is pairwise s^*g -homeomorphism

Proof: Obvious from definitions. But the converses of the assertions of the above theorem are not true in general as can be seen from the following examples.

Example 6: In this example, f is pairwise g -homeomorphism, pairwise gs -homeomorphism but not pairwise s^*gc -homeomorphism. In example 1, f is pairwise s^*g -homeomorphism. But it is not pairwise s^*gc -homeomorphism. In example 4, f is pairwise sg -homeomorphism. But it is not pairwise s^*gc -homeomorphism.

However, every pairwise s^*gc -homeomorphism from a pairwise $T_{1/2}$ -space onto itself is a pairwise homeomorphism every pairwise g -homeomorphism from a pairwise $T_{1/2}$ -space onto itself is a pairwise s^*gc -homeomorphism and every pairwise s^*g -homeomorphism from a pairwise $T_{1/2}$ -space onto itself is a s^*gc -pairwise homeomorphism. Also, every pairwise sg -homeomorphism {resp. pairwise gs -homeomorphism} from a pairwise T_b -space onto itself is a pairwise s^*gc -homeomorphism. Concerning the compositions of the maps, the composition of two pairwise s^*gc -homeomorphisms is always a pairwise s^*gc -homeomorphism. Also, one can easily verify that pairwise s^*gc -homeomorphisms and pairwise gc -homeomorphisms {resp. pairwise sgc -homeomorphisms and pairwise gsc -homeomorphisms} are independent each other.

But if X and Y are pairwise $T_{1/2}$ -spaces then the function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise gc -homeomorphism if and only if f is pairwise s^*gc -homeomorphism. Also if X and Y are pairwise T_b -spaces then the map $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise s^*gc -homeomorphism if and only if f is pairwise sgc -

homeomorphism {resp. pairwise gsc -homeomorphism}. In order to state the algebraic structure of the set of all pairwise s^*g -homeomorphisms and pairwise s^*gc -homeomorphisms, we introduce the following notations:

- a) $ph(X, \tau_1, \tau_2) = \{f: f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \text{ is pairwise homeomorphism}\}$
- b) $ps^*gch(X, \tau_1, \tau_2) = \{f: f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \text{ is pairwise } s^*gc\text{-homeomorphism}\}$
- c) $ps^*gh(X, \tau_1, \tau_2) = \{f: f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \text{ is pairwise } s^*g\text{-homeomorphism}\}$
- d) $pgh(X, \tau_1, \tau_2) = \{f: f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \text{ is pairwise } g\text{-homeomorphism}\}$
- e) $psgh(X, \tau_1, \tau_2) = \{f: f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \text{ is pairwise } sg\text{-homeomorphism}\}$
- f) $pgsh(X, \tau_1, \tau_2) = \{f: f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \text{ is pairwise } gs\text{-homeomorphism}\}$

Theorem:

- a) $ph(X, \tau_1, \tau_2) \subseteq ps^*gch(X, \tau_1, \tau_2) \subseteq ps^*gh(X, \tau_1, \tau_2) \subseteq pgsh(X, \tau_1, \tau_2)$
- b) $ps^*gh(X, \tau_1, \tau_2) \subseteq psgh(X, \tau_1, \tau_2)$
- c) $ps^*gh(X, \tau_1, \tau_2) \subseteq pgh(X, \tau_1, \tau_2)$
- d) the set $ps^*gch(X, \tau_1, \tau_2)$ is a group which contains $ph(X, \tau_1, \tau_2)$ as a subgroup

Proof: a-c are obvious and d for $f, g \in ps^*gch(X, \tau_1, \tau_2)$, we define a binary operation $\mu: ps^*gch(X, \tau_1, \tau_2) \times ps^*gch(X, \tau_1, \tau_2) \rightarrow ps^*gch(X, \tau_1, \tau_2)$ such that $\mu(f, g) =$ the composition of f and g , namely; $g \circ f$. Since, the composition of two pairwise s^*gc -homeomorphisms is always a pairwise s^*gc -homeomorphism, it is obvious that the closure and associative properties are true under μ . The proof of the existence of identity and inverse property is left to the reader. Since, $ph(X, \tau_1, \tau_2) \subseteq ps^*gch(X, \tau_1, \tau_2)$ and $ph(X, \tau_1, \tau_2)$ is also a group with μ , $ph(X, \tau_1, \tau_2)$ is a sub group of $ps^*gch(X, \tau_1, \tau_2)$.

Theorem: Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise s^*gc -homeomorphism. Then, it induces an isomorphism from the group $ps^*gch(X, \tau_1, \tau_2)$ onto $ps^*gch(Y, \sigma_1, \sigma_2)$.

Proof: The pairwise homomorphism $f: ps^*gch(X, \tau_1, \tau_2) \rightarrow ps^*gch(Y, \sigma_1, \sigma_2)$ is induced from f by $f^*(h) = f \circ h \circ f^{-1}$ for every $h \in ps^*gch(X, \tau_1, \tau_2)$. The f^* is a bijection and also is an pairwise isomorphism by usual argument.

CONCLUSION

Thus, the study of two new types of homeomorphisms, namely; pairwise s^*g -homeomorphism and pairwise s^*gc -homeomorphism in bitopological

spaces are done in this study. Bitopological spaces have some applications in the study of digraphs and hence, this study of these two homeomorphisms may yield some improvements and preserving theorems in this area. The further research in this direction is undergoing.

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