Journal of Engineering and Applied Sciences 8 (6): 215-219, 2013
ISSN: 1816-949X

On the Polar Decomposition and Least Squares Problem

A. Jafarian and A. Mohamadalizadeh
Department of Mathematics, Islamic Azad University, Urmia Branch,
Urmia, Islamic Republic of Iran

Abstract: This research investigate the way of using the polar decomposition of matrix in solving the least squares problem. By using this decomposition, researchers can solve the liner system. Also, this study included the way of obtaining inverse of a non-singular matrix and determinant of a matrix by this decomposition. Finally, the problem of least squares solved by normal equations and Pseudo-inverse matrix by using the polar decomposition. With solving some example, researchers show that the accuracy of this method.

Key words: Linear equations system, least squares problem, polar and SVD decomposition, normal equation, Pseudo-inverse matrix

INTRODUCTION

Linear equations system have many applications in engineering and social sciences and solving them involves bulk of the engineering computation and other disciplines (Zhou and Wei, 2006). They can be solved by various methods such as: Gauss elimination method, inverse of coefficients matrix, Kramer method, Iterative methods of Jacobi and Gauss-Seidel, Analytical method and so on.

In the decomposition method, recommended by Alan Matheson, coefficient matrix is decomposed into two or more simple matrix (Pool, 2006).

The main advantage of this method over other methods is when it is clear that researchers want to solve the system Ax = b for several b. In this case, the number of computations can be dramatically reduced.

Matrices can be decomposed by different methods such as: LU decomposition method, Cholesky, SVD, Polar decomposition and several other methods. The present study aims at finding answer of the least squares problem by using the polar decomposition.

The general form of this decomposition similar to the polar form of a non-zero complex number, $Z = re^{i\theta}$, $r \geq 0$ (Higham, 1986).

If A is a real or complex $m \times n$ matrix (not necessarily m = n) then decomposition A = PU called a polar decomposition of matrix A, where P is positive-semi definite Hermitian matrix and U is unitary matrix ($U^TU = I$). The most important feature of this decomposition is that it can be computed from the singular value decomposition and vice versa (Faβbender and Ikramov, 2008).

If $A = u\sum v^T$ is the singular value decomposition of A, by getting $P = u\sum u^T$ and $U = uv^T$, researchers can calculated the polar decomposition as:

$$A = \left(u \sum u^T\right)\left(uv^T\right)$$

Considering the properties of matrices u, v and Σ, P is a positive-semi definite Hermitian matrix and U is unitary matrix (Golub and van Loan, 1996). The polar decomposition can be used in noise reduction and data compression (Calinger, 1999).

To solve the system Ax = b, researchers must first determine whether it is consistent or inconsistent. The command rref(A) can be used for this purpose. If the above system is consistent, there would be one or infinite answers. Otherwise, it has no real answer and in order to find the approximate answer, the least squares problem must be solved (Calinger, 1999).

The method of least squares is a standard approach to the approximate solution of incompatible system. The means of least squares is that the obtain solution minimizes the sum of the errors made in the results of every single equation. To solve an incompatible system, first the rank of coefficient matrix must be determined so that a suitable method for solving the least squares problem is proposed. If the rank of matrix $A_{m\times n}$ is perfect to find the approximate solution, the normal equations by using the QR and Cholesky decomposition is used (Lay, 1994). Otherwise to solve this system, the Pseudo-inverse matrix by using the singular value decomposition can be used.

Corresponding Author: A. Jafarian, Department of Mathematics, Islamic Azad University, Urmia Branch, Urmia, Islamic Republic of Iran

As mentioned before, polar decomposition can be computed by singular value decomposition. By using this feature, a method is proposed to solve the incompatible systems.

Considering, if the rank of the coefficient matrix in an incompatible system is incomplete to find an approximate answer, the Pseudo-inverse matrix is used.

SOLVING THE LEAST SQUARES PROBLEM BY USING THE PSEUDO-INVERSE MATRIX

One of the methods for solving the least squares problem in incompatible system is using the Pseudo-inverse matrix.

Definition 1: Considering, the real matrix $A_{m\times n}$. $A^{\dagger}_{n\times m}$ matrix which is verified in the following condition (Moore-Penrose condition) is called the Pseudo-inverse of A (Golub and van Loan, 1996).

$$AA^{\dagger}A = A \tag{1}$$

$$A^{\dagger}AA^{\dagger} = A^{\dagger} \tag{2}$$

$$\left(A^{\dagger}A\right)^{T} = A^{\dagger}A \tag{3}$$

$$\left(AA^{\dagger}\right)^{T} = AA^{\dagger} \tag{4}$$

Definition 2: For m×n real matrix A, matrices A^TA and AA^T are Hermitian positive definite. If m<n then second square eigen-values of AA^T and if m>n then the second square of eigenvalues of A^TA called the singular values of A (Golub and van Loan, 1996).

Consider incompatible system Ax = b and assume that $A_{m\times n}$ is rank deficient. In this case, normal equations as $x = (A^TA)^{-1} A^Tb$ can not be applied to find the least squares. In this study by using the polar decomposition the pseudo-inverse of the matrix is calculated and by replacing in the equation $\hat{x} = A^{\dagger}b$, the problem will be solved.

First, researchers obtained the polar decomposition of A as:

$$A = \left(u\Sigma u^{T}\right)\left(uv^{T}\right)$$

Where:

u and v = Unitary matrices
Σ = Diagonal matrix

For this purpose, the eigen-values and eigenvectors of the matrices A^TA and AA^T must be calculated. After thinning these vectors, matrices u and v are formed. To obtained the diagonal matrix of Σ, singular values of A^TA or AA^T are calculated and the matrix Σ is formed as $\Sigma = \text{diag}(\sigma_1, ..., \sigma_k)$, then the polar decomposition of A is obtained. Pseudo-inverse matrix from the polar decomposition could be calculated as:

$$A_{m\times n} = \left(u\sum u^{T}\right)\left(uv^{T}\right) \rightarrow A^{\dagger}_{n\times m} = \left[\left(u\sum u^{T}\right)\left(uv^{T}\right)\right]^{\dagger}$$

Where:

$$\sum\nolimits^{\dagger}_{n\times m} = \text{diag}\left(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, ..., \frac{1}{\sigma_k}, 0, ..., 0\right)$$

$$\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_k > 0$$

Now, the solution of the least squares problem can be easily obtain by replaced the Pseudo-inverse matrix on the equation of $\hat{x} = A^{\dagger}b$.

SOLVING THE LEAST SQUARES PROBLEM BY USING THE NORMAL EQUATIONS

One way for solving the least squares problem when the coefficient matrix is full rank is using the normal equations. When A is asymmetric matrix, the system $A^TAx = A^Tb$ called normal equations can be solved instead of the system Ax = b. Therefore, if the vector $\hat{x}$ is the answer for equation Ax = b, then it will be answer for $A^TAx = A^Tb$. For solving the normal equations by using the polar decomposition first, the matrix A decompose in into the multiply of two matrices as A = UP (right polar decomposition) by using the method mentioned by Golub and van Loan (1996) then the normal equations are solved as:

$$Ax = b,\ \ A^TAx = A^Tb$$

$$A = UP \rightarrow (UP)^{T}(UP)x = (UP)^{T}b$$

$$P^TU^TUPx = P^TU^Tb\left(U\,\text{is unitary, i.e.,}\, U^TU = 1\right)$$

Since, P^T is nonsingular, researchers find the least squares solution by solving the triangular system:

$$Px = U^Tb$$

So to getting the vector x in normal equation $A^TAx = A^Tb$, first the polar decomposition of A in the form of A = UP must be obtained, then try to solve the following system of equations in which the equation Px = y can be solved by applying the backward substitution. Researchers have:

$$\begin{Bmatrix} y = U^Tb \\ Px = y \end{Bmatrix}$$

CALCULATING DETERMINANT AND INVERSE OF A MATRIX BY USING THE POLAR DECOMPOSITION

One of the other applications of the polar decomposition is that researchers can use it to calculate the determinant and inverse of the non-singular matrix. As researchers know, the determinant of a square matrix $A_{n\times n}$ is obtained by the following equation:

$$|A| = \pm\Pi_{i=1}^{n}\sigma_i$$

If consider the polar decomposition of matrix A as:

$$A = \left(u\Sigma u^T\right)\left(uv^T\right)$$

Since, v and u are orthogonal matrices, so researchers have:

$$|A| = \left|u\sum u^T\right|\left|uv^T\right| = |u||\Sigma|\left|u^T\right||u|\left|v^T\right| = (\pm1)|\Sigma|(\pm1)(\pm1)(\pm1) = \pm|\Sigma| = \pm\Pi_{i=1}^{n}\sigma_i$$

If A = $(u\Sigma u^T)$ (uv^T) is polar decomposition of the square and full rank matrix $A_{n\times n}$ then researchers can say about the inverse of this matrix:

$$A = \left(u\sum u^T\right)\left(uv^T\right) \to A^{-1} = \left[\left(u\sum u^T\right)\left(uv^T\right)\right]^{-1} = \left(uv^T\right)^{-1}\left(u\sum u^T\right)^{-1} = vu^Tu\sum{}^{-1}u^T = v\sum{}^{-1}u^T$$

Where:

$$\sum{}^{-1} = \text{diag}\left(\frac{1}{\sigma_1},\frac{1}{\sigma_2},\ldots,\frac{1}{\sigma_n}\right)$$

NUMERICAL EXAMPLES

In this study, three examples have been solved.

Example 1: Consider the linear system:

$$Ax = b \to A = \begin{bmatrix}1 & 0 & -1\\ 1 & 2 & 1\\ 0 & 1 & 1\end{bmatrix},\ b = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

Since, rank's (A) = 2 so that rank's A is incomplete and system is inconsistent. By solving this system by using the polar decomposition of A. First, calculated the u, v and Σ:

$$u = \begin{bmatrix}-0.0849 & 0.9089 & 0.4082\\ 0.8736 & 0.2650 & -0.4082\\ 0.4792 & 0.3220 & 0.8165\end{bmatrix},$$

$$v = \begin{bmatrix}0.2852 & 0.7651 & 0.5774\\ 0.8052 & 0.1355 & -0.5774\\ 0.5199 & -0.6295 & 0.5774\end{bmatrix}$$

$$\Sigma = \text{diag}\left(\sigma_1,\ldots,\sigma_p\right) = \begin{bmatrix}2.7651 & 0 & 0\\ 0 & 1.5344 & 0\\ 0 & 0 & 0\end{bmatrix} \to \Sigma^{\dagger} = \text{diag}$$

$$\left(\frac{1}{\sigma_1},\ldots,\frac{1}{\sigma_p},0,\ldots,0\right) = \begin{bmatrix}\frac{1}{2.7651} & 0 & 0\\ 0 & \frac{1}{1.5344} & 0\\ 0 & 0 & 0\end{bmatrix}$$

In this step, calculate the Pseudo-inverse matrix, i.e., calculate the $A^{\dagger}$ matrix:

$$A = \left(u\Sigma u^T\right)\left(uv^T\right) \to A^{\dagger} = \left(vu^T\right)\left(u\Sigma^{\dagger}u^T\right) = \begin{bmatrix}0.4444 & 0.2222 & -0.1111\\ 0.0556 & 0.2778 & 0.1111\\ 0.3889 & 0.0556 & 0.2222\end{bmatrix}$$

Now, the answer of the least squares problem with replacing Pseudo-inverse matrix in the equation, $\hat{x} = A^{\dagger}b$ calculated:

$$\hat{x} = A^{\dagger}b \to \hat{x} = \begin{bmatrix}0.4444 & 0.2222 & -0.1111\\ 0.0556 & 0.2778 & 0.1111\\ 0.3889 & 0.0556 & 0.2222\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} \to \begin{bmatrix}0.5556\\ 0.4444\\ -0.1111\end{bmatrix}$$

And error is:

$$\|\varepsilon\| = \|A\hat{x} - b\| \to \|\varepsilon\| = \left\|\begin{bmatrix}-0.3333\\ 0.3333\\ -0.6667\end{bmatrix}\right\| = 0.8165$$

Since, one of the features of the Pseudo-inverse matrix is that $A^{\dagger}A$ and $AA^{\dagger}$ is symmetric, so researchers can proved this feature with calculating the matrix:

$$A^{\dagger}A = \begin{bmatrix}0.6667 & 0.3334 & -0.3333\\ 0.3334 & 0.6667 & 0.3333\\ -0.3333 & 0.3333 & 0.6666\end{bmatrix} \to A^{\dagger}A \text{ is symmertric}$$

Example 2: Consider the linear system:

$$Ax = b \rightarrow A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Since, A is a full rank matrix, so researchers can solve above system by application the normal equations. So that first calculate the right polar decomposition of the matrix A, researchers have:

$$A = PU = \begin{bmatrix} 1.2655 & 0.6186 & 0.1254 \\ 0.6186 & 3.3376 & 1.2156 \\ 0.1254 & 1.2156 & 0.7117 \end{bmatrix} \begin{bmatrix} 0.8836 & 0.3755 & -0.2797 \\ -0.2957 & 0.9107 & 0.2885 \\ 0.3631 & -0.1722 & 0.9157 \end{bmatrix}$$

Now, researchers solve the equations:

$$\begin{Bmatrix} y = u^T b \\ px = y \end{Bmatrix}$$

$$y = \begin{bmatrix} 0.8836 & 0.3755 & -0.1797 \\ -0.2957 & 0.9107 & 0.2885 \\ 0.3631 & -0.1722 & 0.9157 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow y = \begin{bmatrix} 1.4841 \\ 0.6102 \\ 0.6520 \end{bmatrix}$$

In this stage, researchers solve the equation px = y:

$$px = y \rightarrow \begin{bmatrix} 1.2655 & 0.6186 & 0.1254 \\ 0.6186 & 3.3374 & 1.2156 \\ 0.1254 & 1.2156 & 0.7117 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1.4841 \\ 0.6102 \\ 0.6520 \end{bmatrix} \rightarrow x = \begin{bmatrix} 1.3742 \\ -0.8396 \\ 2.1079 \end{bmatrix}$$

Example 3: In this example, researchers calculate the inverse of matrix A by using the polar decomposition:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

If A = $(u\Sigma u^T)$ (uv^T) is a polar decomposition of A then according to the relation $A^{-1} = u\Sigma^{-1}u^{T}$, researchers have:

$$A = \left(u\Sigma v^T\right)\left(uv^T\right)$$

$$A = \left(\begin{bmatrix} 0.2253 & -0.9674 & 0.1152 \\ 0.9095 & 0.1662 & -0.3810 \\ 0.3493 & 0.1910 & 0.9173 \end{bmatrix} \begin{bmatrix} 3.9577 & 0 & 0 \\ 0 & 1.1345 & 0 \\ 0 & 0 & 0.2227 \end{bmatrix} \begin{bmatrix} 0.2253 & -0.9674 & 0.1152 \\ 0.9095 & 0.1662 & -0.3810 \\ 0.3493 & 0.1910 & 0.9173 \end{bmatrix}^T \right)$$

$$A = \left(\begin{bmatrix} 0.2253 & -0.9674 & 0.1152 \\ 0.9095 & 0.1662 & -0.3810 \\ 0.3493 & 0.1910 & 0.9173 \end{bmatrix} \begin{bmatrix} 0.0569 & -0.8527 & 0.5192 \\ 0.8346 & -0.2448 & -0.4935 \\ 0.5479 & 0.4614 & 0.6978 \end{bmatrix}^T \right)$$

$$A^{-1} = v\Sigma^{-1}u^T$$

$$A^{-1} = \begin{bmatrix} 0.0569 & -0.8527 & 0.5192 \\ 0.8346 & -0.2448 & -0.4935 \\ 0.5479 & 0.4614 & 0.6978 \end{bmatrix} \begin{bmatrix} 3.9577 & 0 & 0 \\ 0 & 1.1345 & 0 \\ 0 & 0 & 0.2227 \end{bmatrix}^{-1} \begin{bmatrix} 0.2253 & -0.9674 & 0.1152 \\ 0.9095 & 0.1662 & -0.3810 \\ 0.3493 & 0.1910 & 0.9173 \end{bmatrix}^T$$

$$\rightarrow A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & -1 & 3 \end{bmatrix}$$

CONCLUSION

In this study, a method to obtain the least squares solution by using the polar decomposition is presented. With using this method, one can solve the least squares problem by using the normal equations and Pseudo-inverse matrix.

REFERENCES

Calinger, R., 1999. A Contextual History of Mathematics. Upper Saddle River, Prentice Hall, New Jersey.

Faβbender, H. and K.D. Ikramov, 2008. A note on an unusual type of polar decomposition. Linear Algebra Appl., 429: 42-49.

Golub, G.H. and C.F. van Loan, 1996. Matrix Computations. 3rd Edn., Johns Hopkins University Press, Baltimore, USA., Pages: 694.

Higham, N.J., 1986. Computing the polar decomposition-with applications. SIAM J. Sci. Stat. Comput., 7: 1160-1174.

Lay, D.C., 1994. Linear Algebra and its Applications. Addison-Wesley, Reading, MA., USA., ISBN-13: 9780201520330, Pages: 256.

Pool, D., 2006. Linear Algebra: A Modern Introduction. 2nd Edn., Thomson Brooks/Cole, Belmont, CL., USA.

Zhou, J. and Y. Wei, 2006. A two-step algorithm for solving singular linear systems with index one. Applied Math. Comput., 175: 472-485.