Journal of Engineering and Applied Sciences 8 (8): 248-254, 2013
ISSN: 1816-949X

Dynamic Characteristics of Euler-Bernoulli Beams with Multiple Step Changes in Cross Section or Multiple Geometric

Janaina Cunha Vaz Albuquerque and Jose Juliano de Lima Junior
Instituto de Engenharia Mecanica, Universidade Federal de Itajuba, Av. BPS 1303,
C.P. 50, CEP 37500-903, Itajuba/MG, Brazil

Abstract: Stepped beams with elastic end supports have been extensively investigated due to their importance in structural engineering fields, including active structures, structural elements with integrated piezoelectric materials, shaft-disc system components, turbomachinery blades and many other structural configurations. In the present research, a mathematical modeling is proposed to determine the natural frequencies of a stepped beams without the need to research with a large number of discretization elements and different kinds of meshes within the domain. The mathematical modeling is proposed for stepped beams with elastic end supports. The analysis is based on the classical Euler-Bernoulli beam theory. In comparison with the published literature on the transverse vibration of single cross section change beams, there are relatively few works covering beam vibration when there is >1 change in the beam cross section. In the present study, the natural frequencies and the mode shapes of beams with variable geometry or material discontinuities are investigated. The mode shapes of a beam with multiple step changes in cross section are discussed theoretically and experimentally. Numerical results obtained by Euler-Bernoulli beam theory are compared with experimental results.

Key words: Stepped beam, natural frequency, mode shape, Euler-Bernoulli theory, Brazil

INTRODUCTION

A brief review of selected publications on transverse vibration of beams with changes in cross section follows. The frequency equation for a simply supported stepped beam was derived by Taleb and Suppiger (1961) and Levinson (1976). A numerical method to calculate the first natural frequency of simply-supported beams was presented by Heidebrecht (1967). Jang and Bert (1989a, b) were the first to derive the frequency equations for vibrating stepped beams on classical supports. Vibration analysis of stepped beams with one step cross section change constrained by rotational and translational springs at both ends was presented by Maurizi and Belles (1993). De Rosa (1994) studied the vibration of a beam with one step change in cross section with elastic supports at the ends.

Dong *et al*. (2005), presented a scheme to calculate the laminated composite beam flexural rigidity and transverse shearing rigidity based on first order shear deformation theory. A stepped beam model was then developed using Timoshenko's beam theory to analytically predict the natural frequencies and mode shapes of a stepped laminated composite beam. Numerical methods for modal analysis of stepped piezoelectric beams modeled by the Euler-Bernoulli beam theory was studied by Maurini *et al*. (2006).

The present study, presents the transverse vibration of Euler-Bernoulli beams with discontinuous geometry and elastic end supports. The natural frequencies and the mode shapes of stepped beams are investigated. Combinations of the classical clamped, pinned, sliding and free types of elastic end supports are considered. The first 3 frequency parameters of beams with two step changes in cross section are evaluated for selected sets of system parameters and types of end supports. The proposed method can be extended to beams with any number of step changes in cross-section.

MATERIALS AND METHODS

Mathematical formulation: According to Euler-Bernoulli beam theory, the equation of a clamped-free uniform beam in transversal vibration is obtained by applying the static equilibrium equations to sum the forces and moments that act in the beam. The differential equation for the free transverse vibration of a slender beam is as follows (Inman, 2001):

Corresponding Author: Janaina Cunha Vaz Albuquerque, Instituto de Engenharia Mecanica, Universidade Federal de Itajuba, Av. BPS 1303, C.P. 50, CEP 37500-903, Itajuba/MG, Brazil

$$\frac{\partial^2 v(x,t)}{\partial t^2} + c^2 \frac{\partial^4 v(x,t)}{\partial x^4} = 0 \quad (1)$$

$$c = \sqrt{EI / \rho A}$$

Where:

EI = The flexural rigidity (E is Young's modulus of the beam material and I is the cross sectional area moment of inertia)
ρ = The mass density
A = The cross section area
v (x, t) = The deflection of the beam
x = The spatial abscissa
t = The time

Equation 1 is simplified by assuming a separation of variable solution of the form:

$$v(x,t) = X(x)T(t) \quad (2)$$

By using Eq. 2 in Eq. 1, the equation of motion turns:

$$c^2 \frac{d^4 X(x)}{dx^4(x)} = \frac{d^2 T(t)}{dt^2(t)} = \omega^2 \quad (3)$$

Equation 3 can be rearranged as:

$$\frac{d^4 X(x)}{dx^4(x)} - \left(\frac{\omega}{c}\right)^2 X(x) = 0 \quad (4)$$

By defining:

$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI} \quad (5)$$

Where:

β = The dimensional natural frequency
ω = The natural angular frequency

The general solution of Eq. 4 can be put in the form (Inman, 2001):

$$X(x) = B_1 \sin\beta x + B_2 \cos\beta x + B_3 \sinh\beta x + B_2 \cosh\beta x \quad (6)$$

Where:

X (x) = Represents the mode shapes of beam
B_1-B_4 = The coefficients of general solution
L = The length of clamped-free uniform beam

Based on the classical Euler-Bernoulli beam theory, the general solution of Eq. 6 was imposed for each segment of a stepped beam which is the object of study of this study, as shown in Fig. 1.

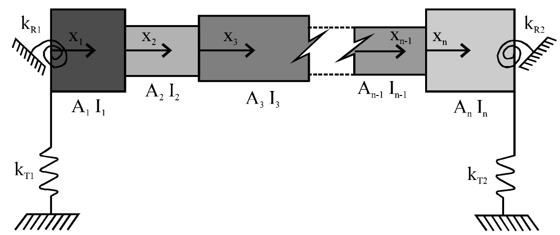

Fig. 1: Stepped beam with multiple step changes in cross section

In Fig. 1, $0 \le x_i \le L_i$, i = 1, 2, ..., n, n is the number of segments of the beam, L_i is the length of the ith segment of the beam, k_{R1} and k_{R2} are rotational spring constants, k_{T1} and k_{T2} are translational spring constants, A is the cross section area of ith segment and I_i is the cross section area moment of inertia. The general solution of the Eq. 4 for each segment of the stepped beam in multiple steps is:

$$X_i(x_i) = B_{indexI} \sin\beta_i x_i + B_{indexII} \cos\beta_i x_i + B_{indexIII} \sinh\beta_i x_i + B_{indexIV} \cosh\beta_i x_i \quad 0 \le x_i \le L_i \quad (7)$$

Where:

I = 1, 2, ..., n
n = The segment number of the beam
k = The number of mode shape
$B_{indexI-IV}$ = The indexes of the coefficients of Eq. 7

The indexes of the coefficients of ith segment of the beam can be expressed as follows:

$$IndexI = 1 + 4(i-1), \; IndexII = 2 + 4(i-1) \quad IndexIII = 3 + 4(i-1), \; IndexIV = 4 + 4(i-1) \quad (8)$$

The equation of mode shape, Eq. 7 contains four unknown coefficients and one natural frequency for each segment of beam. Hence, the solving of solution of Eq. 7 requires four boundary conditions for the ends and also, four boundary conditions for each junction of segments of beam.

The boundary conditions are obtained by examining the deflection, the slope, the bending moment and the shear force at each end of the beam. Besides the boundary conditions, the solution of Eq. 1 requires two initial conditions (in time) to be specified.

The eigenvalue problem must be solved for a particular set of boundary conditions, resulting in expressions for the eigenfunctions $X_{i,k}$ (x_i) and frequencies ω that the structure can accommodate in free vibration. The boundary conditions for the structural system under consideration, Fig. 1 are as follows. At the ends at x = 0. Bending moment:

$$EI_1 \frac{d^2X_1(x_1)}{dx_1^2}\bigg|_{x_1=0} = k_{R1}\frac{dX_1(x_1)}{dx_1}\bigg|_{x_1=0} \qquad (9)$$

Shear force at $x = L_n$:

$$EI_1 \frac{d^3X_1(x_1)}{dx_1^3}\bigg|_{x_1=0} = -k_{T1}X_1(x_1)\big|_{x_1=0} \qquad (10)$$

Bending moment:

$$EI_n \frac{d^2X_n(x_n)}{dx_n^2}\bigg|_{x_n=L_n} = k_{R2}\frac{dX_n(x_n)}{dx_n}\bigg|_{x_n=L_n} \qquad (11)$$

Shear force:

$$EI_n \frac{d^3X_n(x_n)}{dx_n^3}\bigg|_{x_n=L_n} = -k_{T2}\frac{dX_n(x_n)}{dx_n}\bigg|_{x_n=L_n} \qquad (12)$$

The continuity conditions at the junctions are; Deflection:

$$X_{p-1}(x_{p-1})\big|_{x_{p-1}=L_{p-1}} = X_p(x_p)\big|_{x_p=0}\ ,\quad p = 2,\ldots,n \qquad (13)$$

Slope:

$$\frac{dX_{p-1}(x_{p-1})}{dx_{p-1}}\bigg|_{x_{p-1}=L_{p-1}} = -\frac{dX_p(x_p)}{dx_p}\bigg|_{x_p=0} \qquad (14)$$

Bending moment:

$$I_{p-1}\frac{d^2X_{p-1}(x_{p-1})}{dx_{p-1}^2}\bigg|_{x_{p-1}=L_{p-1}} = I_p\frac{d^2X_p(x_p)}{dx_p^2}\bigg|_{x_p=0} \qquad (15)$$

Shear force:

$$I_{p-1}\frac{d^3X_{p-1}(x_{p-1})}{dx_{p-1}^3}\bigg|_{x_{p-1}=L_{p-1}} = -I_p\frac{d^3X_p(x_p)}{dx_p^3}\bigg|_{x_p=0} \qquad (16)$$

By applying the boundary conditions, Eq. 9-16 the general solution, Eq. 7 leads to a system of homogeneous equations for the unknown coefficients, $B_{indexI\text{-}indexIV}$. In order to have non-trivial solutions, the determinant of the resulting coefficient matrix must vanish.

RESULTS AND DISCUSSION

The results for two different stepped beams are presented in this study, one of the beams presents a single step change in cross section. The other beam has two step changes. Both beams are supported on elastic ends. Numerical results for the first three natural frequencies for different end support were compared to available literature.

One step change in cross section: Table 1 and 2 show the first dimensionless natural frequencies $\hat{B}_{1,1}$, of stepped beam with one step change in cross section for several supports, as shown in Fig. 2. The calculations were carried out assuming a stepped beam with lengths equal to $L_1 = L_2 = L/2$ and different moments of inertia ratio, starting with $\bar{I}_1 = 0.1$ (where the first segment of stepped beam is ten times thicker than the second segment) and finishing with $\bar{I}_1 = 10$. The moment of inertia ratio is $\bar{I}_1 = I_2/I_1$, where I_1 is the moment of inertia of the first beam cross section and I_2 is the moment of inertia of the second beam cross section. The indexes used in $\hat{B}_{1,k}$ indicate that 1 represents the first segment of the beam for kth natural frequency:

Table 1: First dimensionless natural frequencies of a single stepped beam with elastic support in one end and free in other one

		$\hat{\beta}_{1,1}$		
$R_1 = T_1$	$R_2 = T_2$	$\bar{I}_1 = 0.1$	$\bar{I}_1 = 1$	$\bar{I}_1 = 10$
∞	∞	0	0	0
500	∞	0.34821	0.29263	0.22976
5	∞	1.09088	0.91389	0.71583
0.05	∞	2.17505	1.81072	1.38830
0	∞	2.23550	1.87510	1.43628

Table 2: First dimensionless natural frequencies of a stepped beam clamped ($R_1 = T_1 = 0$) in one of end and with elastic support in other end

		$\hat{\beta}_{1,1}$				
$R_1 = T_1$	$R_2 = T_2$	$\bar{I}_1 = 0.1$	$\bar{I}_1 = 0.5$	$\bar{I}_1 = 1$	$\bar{I}_1 = 5$	$\bar{I}_1 = 10$
0	∞	2.23550	2.00987	1.87510	1.56119	1.43628
0	500	2.23663	2.01208	1.87866	1.57413	1.45941
0	50	2.24656	2.03147	1.90954	1.67694	1.62735
0	5	2.33168	2.18730	2.13952	2.20142	2.29838
0	0.5	2.70056	2.77289	2.87787	3.24695	3.40801
0	0.05	3.42543	3.83194	4.06910	4.56259	4.74954
0	0.005	3.88099	4.42004	4.65386	5.03687	5.20451
0	0	3.94537	4.50112	4.73004	5.09500	5.26124

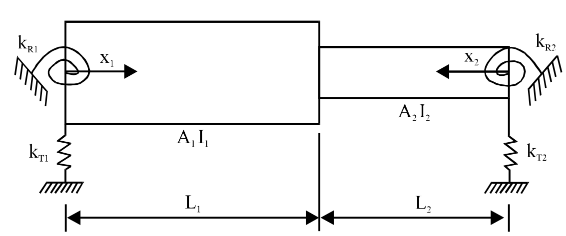

Fig. 2: Beam with one step change in cross section and elastic end supports

$$\hat{\beta}_{1,k} = \beta_{1,k} L \quad (17)$$

$$\omega_k = \left(\frac{\hat{\beta}_{1,k}}{L}\right)^2 \sqrt{\frac{EI_1}{\rho A_1}} \quad (18)$$

Where, ω_k is the angular natural frequency. The classical boundary conditions are shown in Fig. 3, where R and T are the dimensionless flexibility parameters related with the rotational and translational spring constants, k_R and k_T, respectively as defined by Eq. 19:

$$R_1 = \frac{EI_1}{k_{R1}L_1}, \quad T_1 = \frac{EI_1}{k_{T1}L_1^3}, \quad R_2 = \frac{EI_2}{k_{R2}L_2}, \quad T_2 = \frac{EI_2}{k_{T2}L_2^3} \quad (19)$$

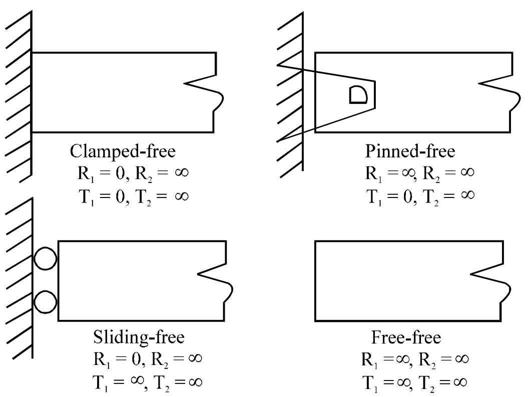

Fig. 3: Types of classical elastic end supports

In order to exemplify the relations present in Eq. 19, Fig. 3 was added with the classical supports. The clamped-free beam has the following dimensionless parameters, $R_1 = 0$ and $T_1 = 0$ in the clamped end. In this case, k_{T1} and k_{R1} are infinite. The same mathematic manipulation is applied to the free end but in this case R_2 and T_2 are infinite.

The first mode shape of uniform beams (Fig. 4a, c, e) and of stepped beams (Fig. 4b, d, f with $\bar{I}_1 = 0.1$) under the same conditions elastic supports are compared in Fig. 4. The changing of boundary conditions is obtained by varying the dimensionless flexibility parameters (R and T). Figure 4a, b have supports ranging from pinned-pinned ($R_1 = \infty$, $T_1 = 0$ to $R_2 = \infty$, $T_2 = 0$) to free-free ($R_1 = \infty$, $T_1 = \infty$ to $R_2 = \infty$, $T_2 = \infty$). Figure 4c, d have supports ranging from clamped-free ($R_1 = 0$, $T_1 = 0$ to $R_2 = \infty$, $T_2 = \infty$) to free-free ($R_1 = \infty$, $T_1 = \infty$ to $R_2 = \infty$, $T_2 = \infty$). Figure 4e, f have supports ranging from free-free ($R_1 = \infty$, $T_1 = \infty$-$R_2 = \infty$, $T_2 = \infty$) to clamped-clamped ($R_1 = 0$, $T_1 = 0$-$R_2 = 0$, $T_2 = 0$. As mentioned previously, the stepped beams in Fig. 4 have $\bar{I}_1 = 0.1$ that is $\bar{I}_1 = 10\,I_2$. In this case, the first segment of stepped beam has flexural rigidity higher than the second segment.

Two step changes in cross section: Table 3-5 show the first 3 dimensionless natural frequencies of a two cross section step change beam, i.e., with three different segments. The beam lengths are $L_1 = 0.200$, $L_2 = 0.300$ and $L_3 = 0.500$ m. The main dimensions related to cross section depend on beam type, i.e., type 1 and 2 are rectangular cross section beam and type 3 is circular cross section beam displayed in Table 6.

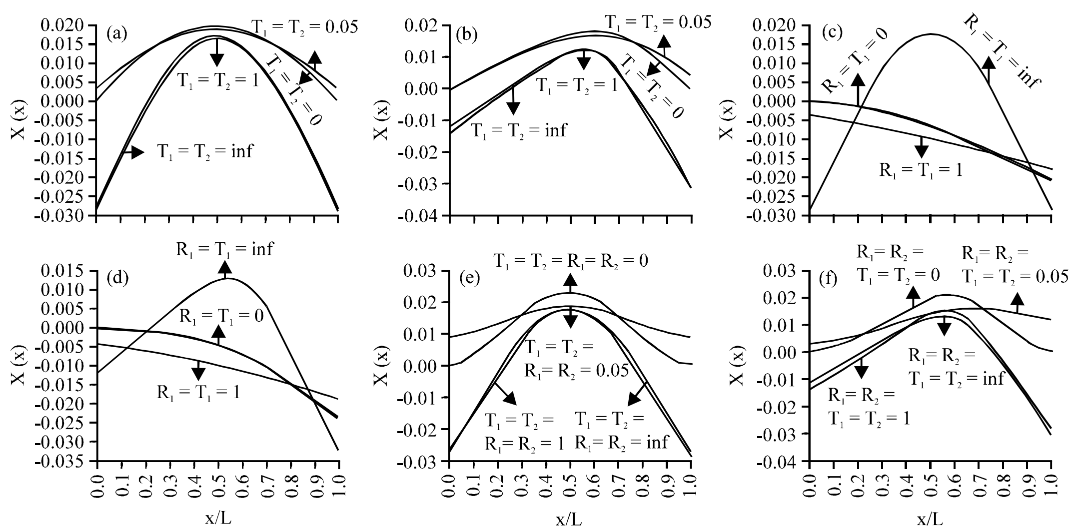

Fig. 4: Types of classical elastic end supports: a, c, e) 1st mode shape: Uniform beam; b, d, f) 1st mode shape: Stepted beam

Figure 5 shows the dynamic behavior of first three mode shapes of a stepped beam in three different segments on classical supports. Figure 5a displays of clear way the mode shapes of a clamped-free beam, i.e., with R = T = 0 for the clamped end and R = T = ∞ for the free end. In this case, the clamped end presents high rotational and translational spring constants and the free end presents non-null values.

Table 3: First 3 dimensionless frequencies of a stepped beam with two step changes in cross sections type 1

Classical supports	R_1	T_1	R_2	T_2	Type 1 $\beta_{1,1}$	$\beta_{1,2}$	$\beta_{1,3}$
Clamped-ree	0	0	∞	∞	1.66100	4.56222	7.84841
Free-free	∞	∞	∞	∞	0	4.77621	7.91134
Clamped-sliding	0	0	0	∞	2.20800	5.42355	8.62546
Clamped-inned	0	0	∞	0	3.80416	7.04505	10.17390

Table 4: First 3 dimensionless frequencies of a stepped beam with two step changes in cross sections type 2

Classical supports	R_1	T_1	R_2	T_2	Type 2 $\beta_{1,1}$	$\beta_{1,2}$	$\beta_{1,3}$
Clamped-free	0	0	∞	∞	1.71452	5.16922	9.41405
Free-free	∞	∞	∞	∞	0	5.57601	9.51969
Clamped-sliding	0	0	0	∞	2.57846	6.32019	10.26300
Clamped-pinned	0	0	∞	0	4.39742	8.43703	11.88250

Table 5: First 3 dimensionless frequencies of a stepped beam with two step changes in cross sections type 3

Classical supports	R_1	T_1	R_2	T_2	Type 3 $\beta_{1,1}$	$\beta_{1,2}$	$\beta_{1,3}$
Clamped-free	0	0	∞	∞	1.50383	4.93207	9.42708
Free-free	∞	∞	∞	∞	0	5.54988	9.59866
Clamped-sliding	0	0	0	∞	2.42957	6.28097	10.18050
Clamped-pinned	0	0	∞	0	4.18529	8.50978	11.75500

Table 6: Different types of cross section

Types	Cross section	Dimensions
1	Rectangular with constant height	$b_1 = 0.005$ m, $b_2 = 0.006$ m and $b_3 = 0.009$ m
2	Rectangular width constant width	$h_1 = 0.005$ m, $h_2 = 0.006$ m and $h_3 = 0.009$ m
3	Circular	$d_1 = 0.005$ m, $d_2 = 0.006$ m and $d_3 = 0.009$ m

In order to verify the accuracy of the adopted model, two cylindrical beams were manufactured to be tested in a test rig. The beams were designed with only one step change in cross section and were machined from aluminum blankets as shown in Fig. 6. The aluminum density and modulus of elasticity are approximately $\rho = 2683$ kg m^{-3} and E = 68.11 GPa, respectively. The assumptions of Euler-Bernoulli beam model were maintained because the ratio between the length of each section and its diameter was set equal or less than ten. The assumptions used in this model consider the beam: Slender of linear, homogeneous and isotropic material, such that the plane of symmetry of the beam is also the plane of vibration and the rotary inertia and shear deformation were neglected. In this case, it is recommended to work with thin or slender beams and low frequency.

To simulate free-free boundary conditions, a wire of naylon was fixed to the ends of the beam and attached to the ceiling. In order to measure, the natural frequencies of the beams under study the beams were put in free oscillation by using an instrumented hammer. The vibration signal of the beam was captured by the laser vibrometer (Ometron, model VQ-500-D) and transmitted to the signal analyzer as illustrated in Fig. 7.

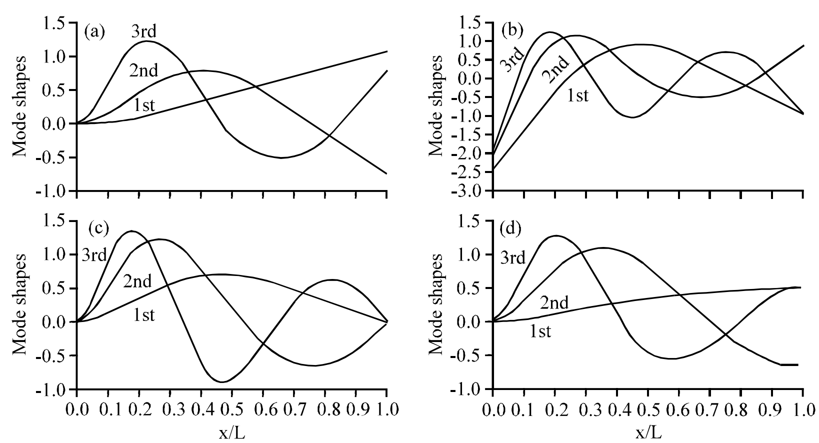

Fig. 5: Comparison of the first 3 mode shapes of beams with three different segments on classical supports. The classical supports correspond to: a) Clamped-free; b) Free-free; c) Clamped-pinned; d) Clamped-sliding. The different segments are represented by solid line (1st segment), dashed line (2nd segment) and dash-dot (3rd segment)

Table 7: Comparison between analytical and experimental results

Mode shape*	Experimental*	Numerical*	Difference* (%)	Experimental**	Numerrical**	Difference** (%)
1st	319	324	1.54	176	177	0.56
2nd	989	985	-0.41	544	545	0.18
3nd	1825	1835	0.54	992	1011	1.88

Natural frequencies (Hz): *L_1 = L/2; ϕ_1 = 22.10 mm, L_2 = L/2; ϕ_2 = 16.32 mm; **L_1 = L/2; ϕ_1 = 12.64 mm, L_2 = L/2; ϕ_2 = 9.10 mm

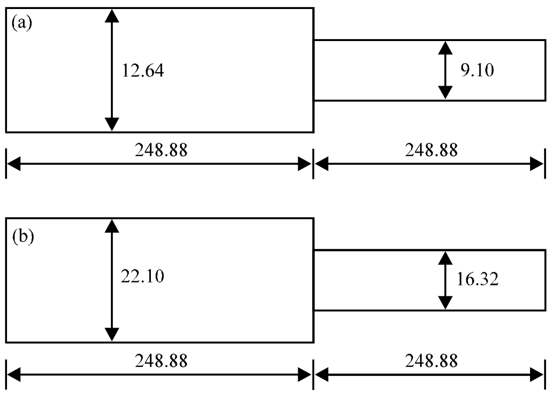

Fig. 6: Representation of the dimensions (mm) of the experimental beams

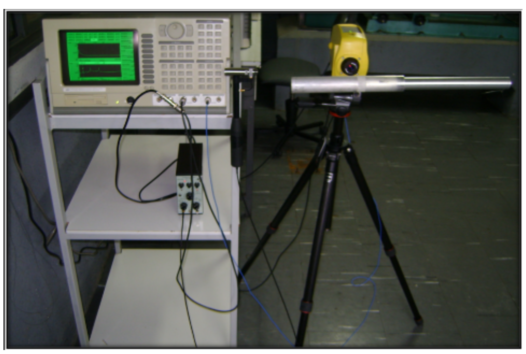

Fig. 7: Experimental beam, laser vibrometer and signal analyzer

The signal analyzer (SRS-Stanford Research Systems, model SR 780) is able to receive and store a large number of vibrometer signals in short time intervals, allowing to obtain the frequency spectrum. The resolution used to obtain the frequency spectrum was 15.96 Hz and the sampling frequency was 12,800 kHz. The impact tests were performed with a modal hammer (Bruel and Kjaer, model 8202). Three series of impact tests were performed along the length of the entire structure and final results were obtained from an average of the trials: Resonant frequencies were obtained by average of 10 recorded impacts applied in three different locations of beam. Figure 8 and 9 show the frequency spectrum of the studied beams. Figure 8 refers to stepped beam sketched in Fig. 6a. Figure 9 refers to stepped beam sketched in Fig. 6b.

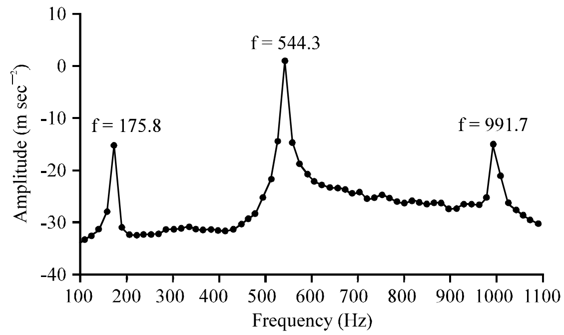

Fig. 8: Frequency spectrum of the stepped beam of Fig. 6a

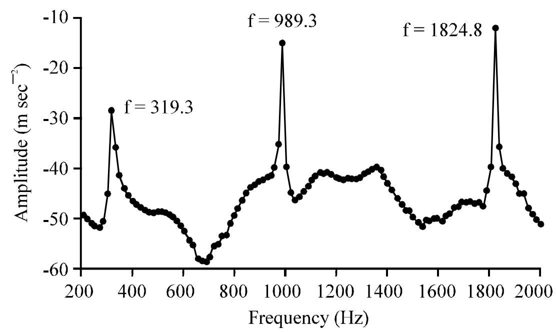

Fig. 9: Frequency spectrum of the stepped beam of Fig. 6b

Table 7 compares the three first natural frequencies obtained experimentally and numerically. The results show good agreement, since the relative differences are small.

CONCLUSION

This study presents, Euler-Bernoulli beam theory in order to evaluate the natural frequencies and the mode shapes of stepped beams in multiple parts. The proposed method is applicable to beams with any number of changes in cross section, different geometries and on classical and/or elastic supports. Numerical simulations were carried out to illustrate beams with one and two step changes in cross section. The first three natural frequencies of 3 types of beams (of rectangular and circular cross section) on classical combinations of supports are tabulated in Table 4-6. In order to verify the accuracy of the adopted model, experiment verifications were made to validate the numerical results. The comparison between calculated and measured frequencies is shown in Table 7 and it shows good agreement, since the relative differences are small.

ACKNOWLEDGEMENTS

The researchers are grateful to the financial support from FAPEMIG (Research Foundation of the State of Minas Gerais) by Proc. TEC-1670/05 and from CAPES (Coordination of Improvement of Higher Education Personnel) by scholarship support.

NOMENCLATURE

A	=	Area cross section (m^2)
B	=	Coefficient of the general solution of Eq. 4
b	=	Height of the rectangular cross section (m)
cos	=	Cosine
cosh	=	Hyperbolic cosine
d	=	Diameter of the circular cross section (m)
d/dx	=	Spatial derivative
d/dt	=	Time derivative
E	=	Young's modulus (GPa)
EI	=	Flexural rigidity (GPa.m^4)
f	=	Measured natural frequency (Hz)
h	=	Width of rectangular cross section (m)
i	=	Locates the segment of the stepped beam
index I-Iv	=	Indexes of the coefficients of the general solution of Eq. 8
I	=	Area moment of inertia (m^4)
$\bar{I}$	=	Relation between adjacent moments of inertia
L	=	Length of the beam (m)
n	=	Number of segments of the beam
k_R	=	Rotational spring constant (N/m)
k_T	=	Translational spring constant (N/m)
k	=	Number of mode shape
R	=	Dimensionless flexibility parameter inversely proportional to k_R
sin	=	Sine
sin h	=	Hyperbolic sine
T	=	Dimensionless flexibility parameter inversely proportional to k_T
T	=	Function of time
t	=	Time (sec)
v (x, t)	=	Deflection of the beam (m)
X	=	Mode shape
x	=	Spatial abscissa
β	=	Calculated dimensional natural frequency (1/m)
$\hat{\beta}$	=	Calculated dimensionless natural frequency
ρ	=	Mass density (kg m^{-3})
ω	=	Natural angular frequency (rad sec^{-1})
ϕ	=	Diameter of the experimental beam (mm)

REFERENCES

De Rosa, M.A., 1994. Free vibration of stepped beams with elastic ends. J. Sound Vibration, 173: 557-565.

Dong, X.J., G. Meng, H.G. Li and L. Ye, 2005. Vibration analysis of a stepped laminated composite Timoshenko beam. Mech. Res. Commun., 32: 572-581.

Heidebrecht, A.C., 1967. Vibration of non-uniform simply-supported beams. J. Eng. Mech. Division, 93: 1-15.

Inman, D.J., 2001. Engineering Vibration. 2nd Edn., Prentice Hall, New Jersey, ISBN: 9780137261420, Pages: 621.

Jang, S.K. and C.W. Bert, 1989a. Free vibration of stepped beams: Higher mode frequencies and effects of steps on frequency. J. Sound Vibration, 132: 164-168.

Jang, S.K. and C.W. Bert, 1989b. Free vibration of stepped beams: Exact and numerical solutions. J. Sound Vibration, 130: 342-346.

Levinson, M., 1976. Vibrations of stepped strings and beams. J. Sound Vibration, 49: 287-291.

Maurini, C., M. Porfiri and J. Pouget, 2006. Numerical methods for modal analysis of stepped piezoelectric beams. J. Sound Vibration, 298: 918-933.

Maurizi, M.J. and P.M. Belles, 1993. Natural frequencies of one-span beams with stepwise variable cross-section. J. Sound Vibration, 168: 184-188.

Taleb, N.J. and E.W. Suppiger, 1961. Vibration of stepped beam. J. Aerospace Eng., 28: 295-298.