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Non-Linear Dynamics of Flexible Curvilinear Bernoulli-Euler Nano-Beams in a Stationary Temperature Field

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Abstract: In this study the mathematical model of non-linear dynamics of flexible curvilinear beams in a stationary temperature field is proposed. On a basis of the variation principles the PDEs governing nonlinear dynamics of curvilinear nano-beams are derived. The proposed mathematical model does not include any requirements for the temperature distribution along the beam thickness and it is defined via solution to the 2D Laplace equation for the corresponding boundary conditions. The governing PDEs are reduced to ODEs employing the finite difference method of a second order and then the counterpart Cauchy problem has been solved using the 4th order Runge-Kutta method. The convergence of reduction from PDEs to ODEs is validated by the Runge principle. In particular, it has been shown that the solutions obtained taking into account the material nano-structural features are more stable in comparison to the case where the micro-effects are neglected.

Key words: Curvilinear Bernoulli-Euler beam, micro and nano-beams, temperature field, non-linear dynamics, PDEs

INTRODUCTION

The micro and nano-sized beams, plates and shells are widely applied in the micro and nano electromechanical systems like sensors measuring the vibration level (Fu and Zhang, 2010) micro-cables (Mojahadi *et al*., 2010) and micro-switchres (Jia *et al*., 2011). It should be mentioned that the dependence of elastic behavior on the body dimensions in the microscales have been observed experimentally in metals (Fleck *et al*., 1994; Nix, 1989) and alloys (Mazza *et al*., 1996) as well as in the polymers (Lam *et al*., 2003) and crystals (Ma and Clark, 1995). In spite of the numerous numerically oriented works devoted to study the mentioned problems being based on the linear modeling, the carried out laboratory experiments imply a need to take into account the occurred non-linearity to fit properly the static and dynamic behavior of the micro and nano-systems (Scheible *et al*., 2002). Unfortunately, a direct application of the classical mechanics of a deformable body cannot allow for a proper understanding, modelling and behaviour prediction of the size dependent features occurred in the scales of micro and submicrons due to lack of a parameter responsible for the exhibited scale effects. In the recent time many novel theories have been proposed to fit properly the scale effects in a continuum like the couple stress theory (Mindlin and Tiersten, 1962; Toupin, 1962), the non-local theory of elasticity (Eringin, 1972), the gradient strain theory (Aifantis, 1999; Fleck *et al*., 1994) as well as the surface elasticity theory (Gurtin *et al*., 1998). Here we are focus only on the works devoted to investigate problems belonging to theory of elasticity, where the so called couple stress theory is employed. The fundamental theoretical background to the couple stress theory has been given in the research by Yang *et al*. (2002) where in spite of two Lame constants there is additional material constant of a higher order. Fleck and Hutchinson (1997) employed the modified couple stress theory to explain a dependence of the elastic behaviour versus the size dependent parameter. In the recent years the latter theory has been successfully used by many researchers for a proper modelling and understanding a size dependent dynamic behaviour of microstructures (Lazopoulos and Lazopoulos, 2010; Asghari *et al*., 2010a, b; Ma *et al*., 2011,

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2010). Study by Kiani (2016) deals with transverse vibration of axially functionally graded tapered nanoscale beams subjected to a longitudinal temperature gradient. Using surface elasticity theory of Gurtin *et al*. (1998), the equations of motion of the nanostructure are derived based on the hypotheses of the Rayleigh, Timoshenko and higher-order beam theory. In study Ghasemi and Mohandes (2016), the effect of finite strain on bending of the geometrically nonlinear of micro laminated compositeEuler-Bernoulli beam based on the Modified Couple Stress Theory (MCST) is investigated in a thermal environment. Study by Li provides a unified and self consistent treatment of a Functionally Graded Material (FGM) micro-beam with varying thermal conductivity subjected to both non-uniform and uniform temperature fields. Our objective is to determine the effect of the microscopic size of the beam, the electrostatic gap, the temperature field and material property on the pull-in voltage of the micro-beam under different boundary conditions. The non-uniform temperature field is obtained by integrating the steady-state heat conduction equation. The governing equations take into account the micro-beam size by introducing an internal material length-scale parameter that is based on the modified couple stress theory. One of the important features of the employed couple stress theory is its ability to explain static and dynamic behaviour of beams. Note that beams belong to widely applied structural elements, where nano-sensors as well as micro and nano-cables and switchers are attached. This implies a need for a deep analysis of non-linear deformations of the size dependent behaviour of beams under both static and dynamic loads.

MATERIALS AND METHODS

Mathematical modeling of the curvilinear bernoulli-euler beams: In the modified couple stress theory (Yang *et al*., 2002) the accumulated energy of deformation Π in an elastic body occupying the space $\Omega = \{0 \le x \le a; h/2 \le z \le h/2\}$taking into account the infinitely small deformations takes the form:

$$\Pi = \frac{1}{2}\int_{\Omega}(\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij}) \quad (1)$$

Where:

ε = Components of the deformation tensors

χ_{ij} = Components of the symmetric tensor of the curvature gradient

The mentioned components follow:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}) \\ \chi_{ij} = \frac{1}{2}(\theta_{i,j} + \theta_{j,i}), \theta_i = \frac{1}{2}(\mathrm{rot}(u))_i \quad (2)$$

Where:

U_I = Stand for components of the displacement vector

u, θ = The infinitely small rotation vector with components θ_i and δ_{ij}denote the Kronecker symbol

In the case of the linear isotropic elastic material, the stress implied by the kinematic parameters occurring by Fu and Zhang (2010) are defined via the following state Eq. 4:

$$\sigma_{ij} = \lambda_l\varepsilon_{mm}\delta_{ij} + 2\mu\varepsilon_{ij}, m_{ij} = 2\mu l^2\chi_{ij} \quad (3)$$

Where σ_{ij} ε_{ij} m_{ij} and χ_{ij} and denote components of the classical stress tensors deformation tensors σ the deviator part of the symmetric tensor of the higher order moment and the symmetric part of the curvature tensor χ, respectively; λ = Ev/(1+v)(1-2v), μ = E/2(1+v), L ame parameters; E(x, y, z, e_i), v(x, y, z, e_i) Young modulus and Poisson's coefficients, respectively; ρ(x, y, z, e_i) density of beam material; e_i is deformation intensity.

The parameter l occurring in the moment of a higher order m_{ij}, presents and additional independent length parameter associated with the symmetric tensor of the rotation gradient. In this model, in spite of the classical Lame parameters, it is necessary to include one more scale length parameter (Yang *et al*., 2002). This follows a straight forward observation that in the couple stress theory, a density of the energy of deformation is a function only of the deformation tensor and the symmetric curvature tensor.

The latter does not depend explicitly on rotation (non-symmetric part of the deformation gradient) and the non-symmetric part of the curvature tensor (Yang *et al*., 2002). The curvilinear beam has the length L height h the curvature of the beam middle line/and K_x = 1/R_x it possesses the rectangular cross-section (Fig. 1).

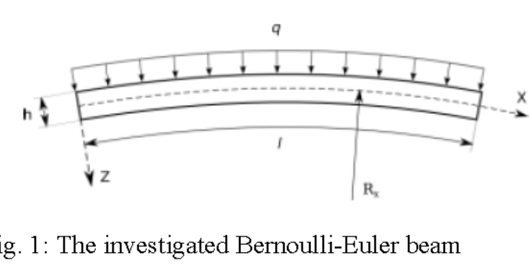

Fig. 1: The investigated Bernoulli-Euler beam

The following hypotheses are taken: Bernoulli-Euler hypothesis; geometric non-linearity in the von Karman form; the beam material is isotropic, elastic and non-homogeneous; the beam curvature is included on a basis of the Vlasov's theory. The equations of motion as well as the boundary and initial conditions are yielded by the energetic Ostrogradskiy-Hamilton principle. Recall that owing to the latter principle two neighbourhood motions from time instant t_0 achieve final positions in time instant t_1:

$$\int_{t_0}^{t_1}(\delta K-\delta\prod+\delta W)dt=0 \tag{4}$$

Where:

K = Beam kinetic energy
Π = Beam potential energy
W = Works of the external forces

Based on the introduced hypotheses and principles, the beam mathematical model consists of the following non-linear PDEs in the non-dimensional form:

$$\frac{\partial^2 u}{\partial x^2}+k_x\frac{\partial w}{\partial x}+L_3(w,w)-\frac{\partial N_x^t}{\partial x}-\frac{\partial^2 u}{\partial t^2}=0 \tag{5}$$

$$\frac{1}{\lambda^2}\left[-\left(\frac{1}{12}+\frac{1}{2}\gamma\right)\frac{\partial^4 w}{\partial x^4}+k_x\left[\frac{\partial u}{\partial x}-k_x w-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2-w\frac{\partial^2 w}{\partial x^2}\right]+ \atop L_1(w,u)+L_2(w,w)+q_0\right]- \\ \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}N_x^T\right)-\frac{\partial^2 M_x^T}{\partial x^2}-\frac{\partial^2 w}{\partial t^2}-\varepsilon\frac{\partial w}{\partial t}=0 \tag{6}$$

In the above:

$$L_1(u,w)=\frac{\partial^2 u}{\partial x^2}\frac{\partial w}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2};\ L_2(w,w)=\frac{3}{2}\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2};$$
$$LL_3(w,w)=\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}$$

are non-linear operators; (x, t) beam deflection in the normal direction; u(x, t) beam element displacement in the longitudinal direction; ε damping coefficient; q_0 amplitude of external load ; λ and γ size dependent coefficients; N_x^T temperature force; M_x^T temperature moments. The PDEs (Eq. 5 and 6) should be supplemented by boundary and initial conditions. Temperature field is defined based on a solution to the 2D Laplace equation of the following form:

$$\nabla^2 T(x,z)=\frac{\partial^2 T(x,z)}{\partial x^2}+\lambda^2\frac{\partial^2 T(x,z)}{\partial z^2}=0 \tag{7}$$

with the corresponding thermal boundary conditions of the 1st-3rd kinds. The temperature moments and forces , appeared in the PDEs (Eq. 5 and 6) are defined by the relations:

$$N_x^T=\int_{-1/2}^{1/2}T(x,z)dz;M_x^T=\int_{-1/2}^{1/2}T(x,z)zdz=0 \tag{8}$$

where, T(x, z) is defined by a solution to PDE (Eq. 7) with corresponding boundary conditions. As itb has been already mentioned Eq. 5-8 are non-dimensuional and the relationals between dimensional and non dimensional parameters follow:

$$\bar{w}=\frac{w}{h},\ \bar{u}=\frac{uL}{h^2},\ \bar{x}=\frac{x}{L},\ \bar{z}=\frac{z}{h}$$
$$\bar{q}=q\frac{L^4}{h^4E},\ c=\sqrt{\frac{Eg}{\rho}},\ \bar{\varepsilon}=\frac{\varepsilon L}{c}$$
$$\bar{t}=\frac{t}{\tau},\tau=\frac{L}{c},\lambda=\frac{L}{h},\lambda=\frac{1}{h}$$
$$\overline{k_x}=\frac{k_xL^2}{h},\ \overline{N_x^T}=\frac{N_x^TL^2}{Eh^3},\ \overline{M_x^T}=\frac{M_x^T}{Eh^2},\ \bar{T}=\alpha T$$

Where:

ρ = Beam denstity
g = Earth acceleration young module
E = Constant
α = Temperature expansion coefficient

Observe that in PDEs (Eq. 5-8) bares overb the non dimensional quantities arev already omitted.

Algorithm of solution: In order to study non-linear dynamics of the flexible curvilinear Bernoulli-Euler nano-beams embedded in the thermal field it is necessary to define thermal forces and moment M_x^t which appear in Eq. 5 and 6. For this purpose we need to solve the 2D Laplace equation for the boundary problems of the 1st-3rd kinds. The problem has been solved using the finite difference method of the second order of accuracy where as the associated set of the algebraic equations has been solved via Gauss method. In addition, the convergence of the employed method has been validated versus a number of the beam partition regarding the co-ordinates x and z (the optimal partition is 11×11). The system of PDEs (Eq. 5 and 6) has been reduced to the system of ODEs of the second order with respect to time using the mentioned already finite difference method of the second order accuracy. The Cauchy problem has been solved via the 6th and 4th order Runge-Kutta method.

Since, the results yielded by two methods coincide we have finally chosen the 4th order Runge-Kutta Method (Awrejcewicz *et al*., 2013) (its computational time is twice less). Researchesr have also investigated a convergence of the solution of the obtained ODEs versus a number of partition with respect to the co-ordinate x (the optimal number of partition is 80). Therefore, we have employed mesh 80×11 while solving the 2D Laplace equations. Practically, it means that we investigated nonlinear dynamics of the curvilinear Berenoulli-Euler nano-beams embedded into temperature field as a system with infinite number of degrees-of-freedom. The problem of solution stability has been solved via the Runge principle.

RESULTS AND DISCUSSION

Numerical experiment: Researchers consider stability of flexible curvilinear Bernoulli-Euler nano-beams in the stationary temperature field, i.e., researchers assume that the Young modulus and the Poisson coefficient do not depend on temperature and they are constant. The following parameters are fixed while carrying out the numerical experiment $E = 2.06 \times 10^5$ MPa, $\alpha = 12.5 \times 10^{-6}$ l/grad. As the boundary conditions we take fixed support on the beam ends:

$$W(0, t) = u(0, t) = M_x(0, t) = w(1, t) = u(1, t) = M_x(1, t) = 0 \quad (10)$$

whereas the initial conditions for the beam follow:

$$W(x, 0) = \dot{w}(x., 0) = u(x, 0) = \dot{u}(x, 0) = 0 \quad (11)$$

The boundary conditions for the heat transfer are reported in Table 1.

The upper beam part is subjected to action of the constant load uniquely distributed. We assume the medium damping factor . In order to find a stationary solution the relaxation method proposed by Tikhonov and Arsenin (1977) and then employed by Feodosev (1963) for the problem of theory of shells has been employed. This method belongs to very effective and suitable to solve the stated problem (Krysko *et al*., 2005). Among the approximate methods devoted to solution of the problem the key role play iteration methods yielding a solution with the regarding accuracy. If an iteration process is considered as a result of the limiting solution to a certain time dependent process, the iteration methods can be considered as those allowing a continuation regarding a given parameter.

Researchers are aimed on a study of stability of the curvilinear Bernoulli-Euler beams ($\gamma = 0$) and the Bernoulli-Euler curvature nano-beams ($\gamma = 0.3$; 0.5) with/without action of the temperature field. The dependencies $q^{+}_{crit}(K_x)$ are given in Fig. 2.

For the fixed values of the parameter, $\gamma$is the curvilinear nano-beam and Bernoulli-Euler beam have been initially heated up to the temperature 50°C. The dependencies q_0 vs. W(0.5)for the parameters $k_x = 12$, 24, 36, 48 are reported in Fig. 3. As the reference stability criterions those proposed earlier by Volmir and Kantor are taken.

Table 1: Boundary condition for the heat transfer

Parameters	Young module	Poisson coefficeient
T (x, z) = T	z = -1/2	0<x<1
T (x, z) = 0	z = 1/2	0<x<1
T (x, z) = 0	z = 1	-1/2<z<1/2
T (x, z) = 0	z = 0	-1/2<z<1/2

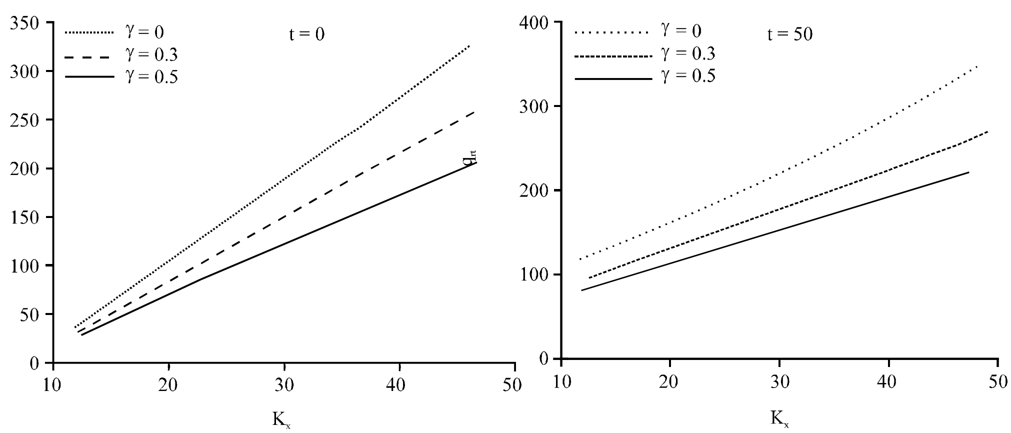

Fig. 2: Stability of curvilinera Bernouli Euler beams

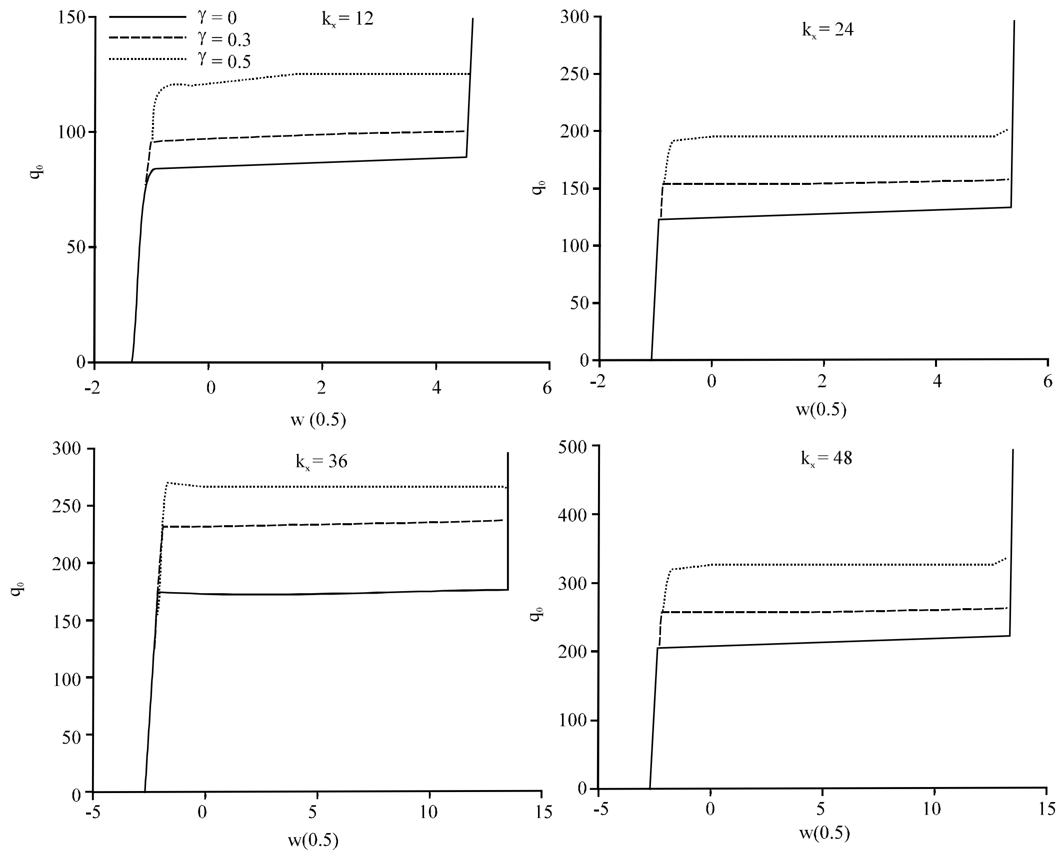

Fig. 3: Dependencies for the parameters

CONCLUSION

The fundamental conclusion states that the value of the parameter and the temperature field essentially influence the critical load values of the curvilinear Bernoulli-Euler nano-beams.

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