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Solutions for the Perturbed Sine-Gordon Equation

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Abstract: This document is based on the search of solutions for the perturbed Sine-Gordon equation using Hamiltonian systems and Fourier transform. Also, with inverse integrant factor we guarantee the non existence of periodic orbits in some regions of the plane for the dynamical system and we generalize such system.

Key words: Dynamical systems, Dulac functions, periodic orbits, perturbed Sine-Gordon equation

INTRODUCTION

Almanza-Vasquez *et al*. (2015) was showed a solution of the cubic non linear dissipative Klein-Gordon equation, the perturbed Sine-Gordon equation is a generalization of the before equation for small solutions. Kolpak and Ivanov (2016) was treated the Sine-Gordon equation but we work with the perturbed Sin-Gordon equation. In (Marin *et al*., 2013a, b) was encountered a generalization of a gradient system. Cao was made a generalization of a Birkhoffian system. Chen and Mei was made gradient representations for generalized Birkhoff systems. Chen was studied a combined gradient system. Chen generalized gradient systems were proposed. Li generalized gradient representations were studied. Adeniyi, Aliyu and Kayode was showed methods for solving second order differential equations. Marin-Ramirez *et al*. (2014) was constructed an asymptotic for resonance of a wave function associated with the Klein-Gordon equation in presence of a potential barrier, this equation is a linear equation our equation is a nonlinear equation. Ortiz *et al*. (2012) was proposed a simple method for constructing asymptotics of eigenvalues for the Klein-Gordon equation in the presence of a shallow potential well. Marin *et al*. (2013a, b) was studied the Klein-Gordon equation, reducing the initial problem to an integral equation and then by applying the method of Neumann series to solve it. Marin *et al*. (2013a, b) was encountered an asymptotics of eigenfunctions for discrete Klein-Gordon equation. Salas and Castillo (2012) gave exact solutions to perturbed Sine-Gordon equation. We show there exists a solution and construct a system without periodic orbits.

MATERIALS AND METHODS

The perturbed Sine-Gordon equation: There are three mathematical not linear models to shape the dynamics of the DNA: that of Sine-Gordon, that of Yakushevich and that of Peyrand and Bishop. The model of Sine-Gordon, proposed in 1980 by Englander and collaborators: it appears in such a way that it is obtained as description of the dynamics of the DNA but experimental later results put it in question. Like that, this one is not too useful to understand the dynamics of the DNA though it had merit for being the first one and motivating the introduction of other two. From the equation disturbed Sine-Gordon we want to seek for solutions and to look if these can contribute a better model of the non linear dynamics of the DNA. Using different methods one tries to show solutions of the differential perturbed Sine-Gordon equation:

$$\alpha u_{tt} + \beta u_{xx} + \gamma u_t + \delta \sin(u) = 0 \qquad (1)$$

where, $\alpha \neq 0$, $\beta \neq 0$, $\delta \neq 0$ and γ is a real constant. The first one consists of realizing a change of variable to transform (Eq. 1) into a differential ordinary equation that can be solved by classic methods. In the second one, it is applied transformed of Fourier and a series of steps follows then to apply transformed inverse of Fourier, obtaining a solution. Then, there is constructed a dynamic system associated with (Eq. 1) which does not possess periodic orbits and to check this fact simply connected regions of the plane are constructed bearing in mind the Bendixson-Dulac criterion. Also, a solution is going to calculate route Hamiltonian systems.

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Preliminary notes

Theorem 3.1: Bendixson-Dulac Criterion (Osvaldo *et al*., 2015). Let f_1 (x_1, x_2), f_2 (x_1, x_2) and h_1 (x_1, x_2) be functions C^1 in a simply connected domain $D \subset R^2$ such that $\partial(f_1 h)/\partial x_1 + \partial(f_2 h)/\partial x_2$ does not change sign in D and vanishes at most on a set of measure zero. Then the system:

$$\begin{cases} x_1' = f_1(x_1, x_2) \\ x_2' = f_2(x_1, x_2), (x_1, x_2) \in D \end{cases} \quad (2)$$

does not have periodic orbits in D. We consider the following equation to obtain Dulac functions:

$$f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} = h\left[C(x_1, x_2) - \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\right] \quad (3)$$

For a dynamic system as (Eq. 2) (Osvaldo *et al*., 2015).

Definition 3.2: We can take the following quasi-differential equation:

$$f_1 \frac{\partial V}{\partial x} + f_2 \frac{\partial V}{\partial y} = V\left(\frac{\partial f}{\partial x} + \frac{\partial f_2}{\partial y}\right) \quad (4)$$

Where V is an inverse integrating factor of the system 9 (Laura *et al*., 2011).

Definition 3.3: Let p.q.k, $l \in Z^+$. A real function f: $R^2 \to R$ is called a p-q-quasi-homogeneous function of weighted degree k if f ($\alpha^p x_1$, $\alpha^q x_2$) = α^kf (x_1, x_2),..., all $\alpha \in R/\{0\}$. A vector field F = $f_1\partial/\partial x_1 + f_2\partial/\partial x_2$ is called a p-q-quasi homogeneous vector field of weighted degree l, if f_1 and f_2 are p-q-quasi-homogeneous functions of weight degree p+l-1 and q+l-1, respectively. A p-q-quasi-homogeneous differential system of weighted degree l is determined by a p-q-quasi-homogeneous vector field (Laura *et al*., 2011).

Theorem 3.4: Given a p-q-quasi-homogeneous vector field F = $f_1\partial/\partial x_1 + f_2\partial/\partial x_2$ then V = $qx_2f_1 - px_1f_2$ is an inverse integrating factor of the system (Laura *et al*., 2011).

Theorem 3.5: If a non-zero p-q-quasi-homogeneous polynomial of weighted k is an inverse integrating factor of the system (Eq. 9), then it has no limit cycles (Laura *et al*., 2011).

Traveling wave equation: We make the change of variable:

$$u = u(x,t) = 2\tan^{-1}\upsilon(\mu(x + \lambda t + \xi_o))\lambda$$
$$\xi_o \in \text{RoC and let } \xi = \mu(x + \lambda t + \xi_o)$$

Then:

$$u_t = \frac{2\mu\lambda\upsilon'(\xi)}{1+\upsilon^2(\xi)};\ \mu_{tt} = \frac{2(\mu\lambda)^2\upsilon''(\xi)(1+\upsilon^2(\xi)) - (2\mu\lambda\upsilon'(\xi))^2\upsilon(\xi)}{\left[1+\upsilon^2(\xi)\right]^2}$$

$$u_x = \frac{2\mu\upsilon'(\xi)}{1+\upsilon^2(\xi)};\ u_{xx} = \frac{2\mu^2\upsilon''(\xi)(1+\upsilon^2(\xi)) - (2\mu\upsilon'(\xi))^2\upsilon(\xi)}{\left[1+\upsilon^2(\xi)\right]^2}$$

$$\sin(u) = \sin(2\tan^{-1}\upsilon(\xi)) = \frac{2\upsilon(\xi)}{1+\upsilon^2(\xi)}$$

Hence:

$$\begin{aligned} &\mu^2(\alpha\lambda^2+\beta)\left(\upsilon(\xi)\right)^2+1)(\upsilon')+ \\ &\left[\left(-2\mu^2\right)(\alpha\lambda^2+\beta)[\upsilon'(\xi)]^2 + \gamma\lambda\mu\upsilon(\xi)\upsilon'(\xi) + \delta\upsilon^2(\xi) + \delta\right] \\ &\upsilon(\xi) + \gamma\lambda\mu\upsilon'(\xi) = 0 \end{aligned}$$

Dynamical system: Now, we let $\upsilon'(\xi)$ then $\upsilon''(\xi)$:

$$\begin{aligned} &\mu^2(\alpha\lambda^2+\beta)\left(\upsilon(\xi)\right)^2+1)y''+ \\ &\left[\left(-2\mu^2\right)(\alpha\lambda^2+\beta)y^2 + \gamma\lambda\mu\upsilon(\xi)y + \delta\upsilon^2(\xi) + \delta\right] \\ &(\upsilon(\xi)) + \gamma\lambda\mu y = 0 \end{aligned}$$

If x = υ (ξ) we obtain:

$$\begin{aligned} &\mu^2(\alpha\lambda^2+\beta)\left(x^2+1\right)y' = \\ &-\left[\left(-2\mu^2\right)(\alpha\lambda^2+\beta)y^2 + \gamma\lambda\mu xy + \delta x^2 + \delta\right] \\ &x - \gamma\lambda\mu y \end{aligned} \quad (5)$$

We consider the following system to (Eq. 5):

$$\begin{cases} x' - y \\ y' = \dfrac{-\begin{bmatrix}\left(-2\mu^2\right)(\alpha\lambda^2+\beta)y^2 + \\ \gamma\lambda\mu xy + \delta x^2 + \delta\end{bmatrix}x - \gamma\lambda\mu y}{\mu^2(\alpha\lambda^2+\beta)\left(x^2+1\right)} \end{cases} \quad (6)$$

We take θ = μ^2 ($\alpha\lambda^2+\beta$), then:

$$\begin{cases} x' = y \\ y' = \dfrac{-\begin{bmatrix}\left(-2\theta y^2\right) + \gamma\lambda\mu xy \\ +\delta x^2 + \delta\end{bmatrix}x - \gamma\lambda\mu y}{\theta(x^2+1)} \end{cases} \quad (7)$$

Solutions through Hamiltonian system: We consider (Eq. 6) and do:

$$\begin{Bmatrix} x' = H_y \\ y' = -H_x \end{Bmatrix} \quad (8)$$

Integrating with respect to y the first equation of the system and with respect to x the second equation, we have that:

$$\begin{Bmatrix} H = \frac{y^2}{2} + C(x) \\ H = -y^2 \ln(x^2+1) + \frac{\gamma\lambda\mu}{\theta}xy + \frac{\delta}{2\theta}x^2 + P(y) \end{Bmatrix} \quad (9)$$

where, C(x) and P(y) are constant that depend of x and y, respectively, then:

$$H = \left[\frac{1}{2} - \ln(x^2+1)\right]y^2 + \frac{\gamma\lambda\mu}{\theta}yx + \frac{\delta}{2\theta}x^2$$

The solution curves are given by H (x, y) = K where K is a constant.

Non existence of periodic orbits

Theorem 7.1: The system (Eq. 6) has no periodic orbits in a region of the plane.

Proof: To ensure the non existence of periodic orbits we will make use of the Poincare-Bendixson theorem. Taking $x_1 = x$, $x_2 = y$, $f_1(x_1, x_2)$ and:

$$f_2(x_1,x_2) = f(x,y) = \frac{-\left[\left(-2\theta y^2\right) + \gamma\lambda\mu xy + \delta x^2 + \delta\right]x - \gamma\lambda\mu y}{\theta(x^2+1)}$$

We suppose $\partial h/\partial x_1 = 0$ and $\partial h/\partial x_2 = h$, substituting in Eq. 3 we obtain; $h(y) = e^y$ and:

$$\frac{-[-2\theta y^2 + \gamma\mu\lambda xy + \delta x^2 + \delta]x - \gamma\lambda\mu y}{\theta(x^2+1)} = C(x,y) + \frac{[-4\theta y + \gamma\mu\lambda x]x + \gamma\mu\lambda}{\theta(x^2+1)}$$

In consequence:

$$C(x,y) = \frac{(2\theta x)y^2 + \left[4\theta x - (x^2+1)\gamma\lambda\mu\right] y - \left[\delta x^3 + \gamma\lambda\mu x^2 + \delta x + \gamma\lambda\mu\right]}{\theta(x^2+1)} \quad (10)$$

Considering the numerator as a function of y in Eq. 9 then we have a quadratic function with discriminant given by:

$$D = \left[4\theta x - (x^2+1)\gamma\lambda\mu\right]^2 + 4(2\theta x)\left[\delta x^3 + \gamma\lambda\mu x^2 + \delta x + \gamma\lambda\mu\right]$$

That is to say:

$$D = (8\theta\delta - \gamma^2\lambda^2\mu^2)x^4 + (16\theta^2 + 8\theta\delta^2 + 8\theta\delta + 2\gamma^2\lambda^2\mu^2)x^2 + \gamma^2\lambda^2\mu^2$$

Hence, C (x, y)>0 in the regions:

$$R_1 = \left\{ (x,y) \in R^2 : x>0, y < H \frac{-4\theta x + (x^2+1)\gamma\lambda\mu - \sqrt{D}}{4\theta x} \right\}$$

$$R_2 = \left\{ (x,y) \in R^2 : x > 0, y<H \frac{-4\theta x + (x^2+1)\gamma\lambda\mu + \sqrt{D}}{4\theta x} \right\}$$

Always that; $\theta>0$, $\gamma>0$, $\lambda>0$, $\mu>0$ and $\delta>0$.

RESULTS AND DISCUSSION

Method of the fourier transform: In the low-amplitude case $\mu \to 0$ then $\sin(u) \approx u$. Now Eq. 1 takes the form:

$$\alpha u_{tt} + \beta u_{xx} + \gamma u_t + \delta u = 0 \quad (11)$$

Theorem 8.1: The solution for (Eq. 11) is (Eq. 14):

Proof: Applying the Fourier transform from x to p we have:

$$\alpha\tilde{u}_{tt} + \gamma\tilde{u}_t + (\delta - p^2\beta)\tilde{u} = 0 \quad (12)$$

which can be seen as an ordinary differential equation in $\tilde{u}(p,t)$ whose solution is given by:

$$u(p,t) = k_1 e^{-t\left(\frac{\gamma + \sqrt{r^2 - 4\alpha(\delta - p^2\beta)}}{2\alpha}\right)} + k2 e^{t\left(\frac{-\gamma + \sqrt{r^2 - 4\alpha(\delta - p^2\beta)}}{2\alpha}\right)} \quad (13)$$

Where k_1 and k_2 are constants. Using the theorem of inverse fourier transform in Eq. 12, we obtain:

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\begin{matrix} k_1 e^{-t\left(\frac{\gamma+\sqrt{r^2-4\alpha(\delta-p^2\beta)}}{2\alpha}\right)} \\ +k_2 e_t^{-t\left(\frac{-\gamma+\sqrt{r^2-4\alpha(\delta-p^2\beta)}}{2\alpha}\right)} \end{matrix}\right] e^{ipx}dp \quad (14)$$

Method of inverse integrating factor: Taking $x=\tilde{u}$ and $x=\tilde{u}_t$ we have:

$$\begin{Bmatrix} \dot{x}=y \\ \dot{y}=ay+bx \end{Bmatrix} \quad (15)$$

where, $\alpha = \gamma/a$ and $b = \delta\text{-}p^2\beta/\alpha$. Taking p = q, we have f_1 $(\alpha^p x, \alpha^p y) = \alpha^p y = \alpha^p f_1$ (x, y) then f_1 is p-q-quasi-homogeneous of weighted degree p and f_2 $(\alpha^p x, \alpha^p y) = \alpha^p$ (ay+bx) = $a^p f_2$ (x, y) then f_2 is p-q-quasi-homogeneous of weighted degree p.

Theorem 9.1: V = p $(-bx^2+y^2-axy)$ is a non-zero p-p-quasi-homogeneous polynomial of weighted 2p and is an inverse integrating factor of the system (Eq. 15), then it has no limit cycles (Laura *et al*., 2011).

Proof: Now using the inverse integrating factor V = p $(-bx^2+y^2-axy)$ a p-q-quasi-homogeneous function of weight degree 2p and satisfies the quasi-differential Eq. 4. Then this system does not have periodic orbits in R^2.

Generalization of the system: The Hamiltonnian system:

$$\begin{Bmatrix} \dot{x}=-H_y \\ \dot{y}=H_x \end{Bmatrix} \quad (16)$$

has this solution H = $-y^2/2+ayx+bx^2/2$ = K for some constant K. For the next result we are going to use the Poincare-Bendixson theorem and the following quasi-differential (Eq. 3):

Theorem 10.1: The dynamical system (Eq. 15) can be generalized to (Eq. 17) and both do not have periodic orbits for; y+a<0 and x∈R.

Proof: Taking $\dot{x}=y$ and supposing that ∂h/∂y = 0, C (x, y) = a+y<0. Using Eq. 3, we have y∂h/∂y = h [C (x, y-a)] and ∂h/∂y = h. So, h = e^x. If $\partial f_2/\partial y$ = then f_2 = ay C_2 (x). From Eq. 3, we get the ordinary differential equation $f_1 = a+y-(\partial f_1/\partial x+\alpha)$. Then, its solution is $f_1 = C_1$ (y). We obtain the generalized dynamical system:

$$\begin{Bmatrix} \dot{x}=C_1(y) \\ \dot{y}=ay+C_2(x) \end{Bmatrix} \quad (17)$$

CONCLUSION

We found solutions for the perturbed Sine-Gordon equation using Hamiltonian systems and Fourier transform. Also, with inverse integrant factor we guarantee the non existence of periodic orbits in some regions of the plane for the dynamical system and we generalize such system.

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