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Two Improved Methods Based on Broyden's Newton Methods for the Solution of Nonlinear System of Equations

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Absract: In this study, two improved methods for Quasi Newton's method (QN) called Quasi Modified Newton's (QMN) of type 1 and 2, to obtain an approximate solution for systems of nonlinear equations. The most significant features of these methods are their simplicity and excellent accuracy. Error estimation of the methods was discussed. Some numerical examples are given for comparison reasons and to test the validity of the methods. Superior result shows that the methods are much more efficient and accurate than the other methods.

Key words: Nonlinear systems of equations, Newton method, Broyden's methods, Quasi newton's method, solution, accuracy

INTRODUCTION

In the applied branch of mathematics many physical, engineering and chemical problems lead to a nonlinear system of equations that motivated scientists to solve these problems. Exact solutions for these systems are hardly available; so many researchers (Newton, 1976; Kelley, 1995, 2003; Martinez, 2000; Dennis and Schnabel, 1983) have proposed different strategies of Newton method to find an approximate solution of these problems. Consider the nonlinear system of equation: $f_i(x) = 1, 2, 3, ..., n$, these may be written in matrix form:

$$F(x) = 0 \qquad (1)$$

where, $F(x) = (f_1, f_2,...,f_n)^t$ $F: D\rightarrow R^n$, convex subset of R^n, $x\in D$ and $f_i: D\rightarrow R$ is continuously differentiable in an open neighborhood $D\subseteq R^n$. For any initial vector $x^{(0)}$ close to x^* where x^* is the exact solution of (Eq. 1), Newton-Raphson method generates the sequence of vectors using $\{x^{(k)}\}_{k=0}^{\infty}$ the following iterative scheme:

- Start with an initial guess $x^{(0)}$
- Solve for $J(x^{(k)}+s^{(k)}) = F(x)^k$ or S^k
- Compute $x^{(k+1)} = x^{(k)}+ s^{(k)}$

where, J(k) is the Jacobian matrix of F(x) denoted by J(k) = F'(x). A significant weakness of Newton's method is that, for each iteration a Jacobian matrix must be computed, so this method is very expensive and has the following disadvantages:

- Needs good initial solution $x^{(0)}$ close to the exact solution x^*
- Requires n^2+n function evaluation at each iteration (n^2 for Jacobian matrix and n for F(x))
- $J(x^{(k)})$ must be nonsingular for all k and $J(x^{(*)})$ is invertible
- Need to compute n^2 partial derivative for and $J^{-1}(x^{(k)})$ at each step
- To solve the linear system at each iteration require $O(n^3)$ arithmetic operation

The advantage of this method $\{x^{(k)}\}^{\infty}_{k=0}$ that converge quadratically to x^* and the scheme above is self-corrective when is $J(x^{(k)})$ nonsingular. Many mathematicians (Cordero *et al*., 2010; Homeier, 2005; Weerakoon and Fernando, 2000; Traub, 1976; Sharifi *et al*., 2016) developed the above technique to increase the convergence rate of Newton's method, such as the modified Newton's method of order three (predictor and corrector). We summaries the algorithm as follows:
Start with good initial guess (x^0).

Predictor: Solve $\bar{x}^{(k)} = x^{(k)} - F'(x^{(k)})^{-1}F\, x^{(k)}$ for $x^{(k)}$

Corrector: Solve the following system for $x^{(k+1)}$:

$$x^{(k+1)} = \bar{x}^{(k)} - F'(x^{(k)})^{-1}F(\bar{x}^{(k)}) \qquad (2)$$

The disadvantages of this method are:

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- Needs good initial solution $x^{(0)}$ close to the exact solution $x^{(*)}$
- Needs n^2+2n function evaluation
- The computation of the inverse of Jacobian matrix at each iteration for the predictor is costly
- Requires n^2 partial derivative at each iteration
- To solve the linear systems for the predictor and the corrector at each iteration require $O(n^3)$ arithmetic operation

To overcome these disadvantages and to increase the convergence rate many scientists extend Newton's method to solve a nonlinear systems of equations (Chun and Lee, 2013; Bi *et al*., 2009; Soleymani *et al*., 2014; Papakonstantinou, 2009; Eyert, 1996; Fang and Saad, 2009) such as the Quasi Newton methods. These methods approximate the Jacobian matrix or it's inverse with another matrix, (i.e., $F'_k = B_k$ or $F^{-1}{}_k = H_k$ where F_k is the Jacobian matrix evaluated at the kth iteration, B_k and H_k are easily computed. A well-known Quasi Newton method proposed by Broyden (1965) called Broyden's method or secant method; we can use it when Jacobian matrix is unknown or it is difficult to compute. Now, we will describe the two methods of Broyden's. The first method is approximating the Jacobian matrix as follows. Start with the initial solution $x^{(0)}$. Set, $B_0 = F'(x^{(0)}$ in some cases they set, where $B_0 = I$ is the identity matrix. For k = 1, 2, ..., m do the following Set:

$$x^{(k)} = x^{(k-1)} - B_{k-1}{}^{-1}F(x^{(k-1)}) \tag{3}$$

Compute:

$$s^{(k)} = x^{(k)}-y^{(k-1)},\ \ y^{(k)} = F(x^{(k)})-F(x^{(k-1)})$$

Compute:

$$B_k^{-1} = B_{k-1}^{-1} + \frac{(s^{(k)} - B_{k-1}^{-1}y^{(k)})(s^{(k)})^t B_{k-1}^{-1}}{(s^{(k)})^t B_{k-1}^{-1}y^{(k)}}$$

Where m is the maximum number of iterations allowed. The second method is approximating the Jacobian inverse as follows. Start with the initial solution $x^{(0)}$. Set, $H_0 = (F(x^{(0)}))^{-1}$ in some cases they set $H_0 = I$, where I is the identity matrix. For k = 1, 2, ..., m do the following set:

$$x^{(k)} = x^{(k-1)} - H_{k-1}F(x^{(k-1)}) \tag{4}$$

Compute:

$$s^{(k)} = x^{(k)}-y^{(k-1)},\ y^{(k)} = F(x^{(k)})-F(x^{(k-1)})$$

Compute:

$$H_k = H_{k-1} + \frac{(s^{(k)} - H_{k-1}s^{(k)})}{\parallel y^{(k)} \parallel^2}(y^{(k)})^t$$

Where m is the maximum number of iterations allowed. The main benefits of updating formula is reducing the number of function evaluation at each step from n^2+n to just n and require $O(n^2)$ arithmetic operation per iteration. Dennis and Schnabel (1983) proved the super-linearity of convergence of the sequence $\{x^{(k)}\}^{\infty}_{k=0}$ and updating the inverse of B_k from B_{k-1} using Sherman-Morrison formula (Deng, 2011).

MATERIALS AND METHODS

We will propose two improved methods based on Broyden's methods B_k and H_k modified Newton's method and discuss how we approximate and for the Jacobian matrix and it's inverse respectively. Our purpose is to accelerate the rate of convergence of Quasi Newton's method.

Quasi Modified Newton's method of type one (QMN1): We approximate Jacobian matrix using the first method of Broyden's on the modified Newton's method Eq.2 as follows predictor. Set:

$$\tilde{x}^{(k)} = x^{(k)} - B_k{}^{-1}F(x^{(k)})$$

Corrector compute:

$$x^{(k+1)} = \tilde{x}^{(k)} - B_k{}^{-1}F(\tilde{x}^{(k)}) \tag{5}$$

where B_k as in section one, to describe how Broyden's determined B_k first we use the secant formula to obtain:

$$B_{k+1}(x^{(k+1)} - x^{(k)}) = F(x^{(k+1)}) - F(x^{(k)}) \tag{6}$$

Now, any nonzero vector in R^n can be expressed as a combination of $s^{(k)}$ and the orthogonal complement of $s^{(k)}$ say q, to uniquely defined the matrix B_k, we also need to specify how it acts on q.No information is available about the change in F in a direction of q, so we specify that no change be made in this direction (B_k-B_{k-1}) q = 0 this assumption (no change condition) (Magrenana and Argyros, 2015), implies:

$$B_k\ q = B_{k-1}\ q, \forall q \in R^n \text{ s. t. } (x^{(k)} - x^{(k-1)})q = 0 \tag{7}$$

Let $y^{(k)} = F(x^{(k)})-F(x^{(k-1)})$ and $s^{(k)} = x^{(k)}-y^{(k-1)}$. From Eq. 6 and 7, we obtain the updating Equation:

$$B_k = B_{k-1} + \frac{(y^{(k)} - B_{k-1}s^{(k)})}{\parallel s^{(k)} \parallel_2^2}(s^{(k)})^t$$

Using Sherman-Morrison formula, we obtain:

$$B_k^{-1} = B_{k-1}^{-1} + \frac{(s^{(k)} - B_{k-1}^{-1} y^{(k)})(s^{(k)})^t B_{k-1}^{-1}}{(s^{(k)})^t B_{k-1}^{-1} y^{(k)}}$$

The main benefits of this; reducing number of function evaluation at each step from n^2+n to just n and require $O(n^2)$ arithmetic operation per iteration, and reduce the number of iterations required. The algorithm of (QMN1).

Alogrithm:
Input, $x^{(0)}$ Tol
Let $B_0 = J\, x^{(0)}$
Compute $\bar{x}^{(1)} = x^{(0)} - B_0^{-1} F(x^{(0)})$
Compute $x^{(1)} = \bar{x}^{(1)} - B_0^{-1} F(\bar{x}^{(1)})$

For k=1

Compute $s^{(k)} = x^{(k)} - x^{(k-1)}$
Compute $y^{(k)} = F(x^{(k)}) - F(x^{(k-1)})$

Compute
$$B_k^{-1} = B_{k-1}^{-1} + \frac{(s^{(k)} - B_{k-1}^{-1} y^{(k)})(s^{(k)})^t B_{k-1}^{-1}}{(s^{(k)})^t B_{k-1}^{-1} y^{(k)}}$$
Compute $\bar{x}^{(k+1)} = x^{(k)} - B_k^{-1} F(x^{(k)})$

Compute $x^{(k+1)} = \bar{x}^{(k+1)} - B_k^{-1} F(\bar{x}^{(k+1)})$

If $\left\| x^{(k+1)} - x^{(k)} \right\|_2 \le Tol$, stop.
otherwise set k=k+1, continue looping

Quasi Modified Newton's method type two (QMN2): We follow the same steps in the previous subsection. We apply the second method of Broyden's to approximate F^{t-1} by H^k so, the iteration Eq. 2 becomes:

$$\hat{x}^{(k)} = x^{(k)} - H_k F(x^{(k)}) x^{(k+1)} = \hat{x}^{(k)} - H_k F(\hat{x}^{(k)}) \quad (8)$$

From Eq. 6 we have:

$$(x^{(k+1)} - x^{(k)}) = H_{k+1} F(x^{(k+1)}) - F(x^{(k)}) \quad (9)$$

So, the no change condition Eq. 7 becomes:

$$H_k q = H_{k-1}\, q, \forall q \in R^n \text{ s. t. } (F(x^{(k)}) - F(x^{(k-1)}))\, q = 0 \quad (10)$$

From Eq. 9 and Eq. 10, the updating formula is uniquely given by:

$$H_k = H_{k-1} + \frac{(s^{(k)} - H_{k-1} s^{(k)})}{\| y^{(k)} \|_2^2} (y^{(k)})^t \quad (11)$$

The algorithm of QMN2.

Alogrithm:
Input, $x^{(0)}$ Tol
Let $B_0 = J\, x^{(0)}$
Compute $\bar{x}^{(1)} = x^{(0)} - H_0 F(x^{(0)})$
Compute $x^{(1)} = \bar{x}^{(1)} - H_0 F(\bar{x}^{(1)})$

For k=1

Compute $s^{(k)} = x^{(k)} - x^{(k-1)}$

Compute $y^{(k)} = F(x^{(k)}) - F(x^{(k-1)})$

Compute
$$H_k = H_{k-1} + \frac{(s^{(k)} - H_{k-1} s^{(k)})}{\left\| y^{(k)} \right\|_2^2} (y^{(k)})^t$$
Compute $\bar{x}^{(k+1)} = x^{(k)} - H_k F(x^{(k)})$

Compute $x^{(k+1)} = \bar{x}^{(k+1)} - H_k F(\bar{x}^{(k+1)})$

If $\left\| x^{(k+1)} - x^{(k)} \right\|_2 \le Tol$, stop.
otherwise set k=k+1, continue looping

RESULTS AND DISCUSSION

Performance evaluation and comparisons: We give some numerical examples to test the validity of the proposed methods and for comparison reasons. For all the examples, we implement all the methods described above.

Example 1: Alipanah and Dehghan (2007), Mirzaee and Bimest (2015) and Borzabadi and Fard (2009). Consider the nonlinear Fredholm integral equation:

$$g(s) = (s - \frac{\pi}{8}) + \frac{1}{2} \int_0^1 \frac{1}{1 + g^2(t)} dt$$

Solution: Using Simpson's rule with 11 equally spaced nodes (n = 10) to approximate the integral part, so, h = 0.1, then we have:

$$\frac{1}{2} \int_0^1 \frac{1}{1 + g^2(t)} dt \cong \frac{0.1}{2(3)} [\frac{1}{1 + g^2(0)} + \frac{1}{1 + g^2(1)} + 4 \sum_{i=1}^{5} \frac{1}{1 + g^2(t_{2i-1})} + 2 \sum_{i=1}^{4} \frac{1}{1 + g^2(t_{2i})}] \quad (12)$$

Substitute in Eq.12 to obtain the corresponding nonlinear system:

$$F(\xi) = (f_0(\xi), f_1(\xi), \ldots, f_{10}(\xi))^t = 0$$

where:

$$f_m(\xi) = (s_m - \frac{\pi}{8}) + \frac{0.1}{2(3)} [\frac{1}{1 + g^2(0)} + \frac{1}{1 + g^2(1)} + 4 \sum_{i=1}^{5} \frac{1}{1 + g^2(t_{2i-1})} + 2 \sum_{i=1}^{4} \frac{1}{1 + g^2(t_{2i})}] \quad (13)$$

Table 1: The error analysis for example 1

Method	E_3	E_4	E_5
Newton's method	8.7844×10^{-4}	2.3245×10^{-8}	2.4768×10^{-16}
Broyden's method 1	8.1263×10^{-3}	4.2469×10^{-5}	1.0503×10^{-8}
Broyden's method 2	6.4953×10^{-3}	3.3565×10^{-5}	6.5622×10^{-9}
Proposed QMN1	6.0196×10^{-5}	4.2781×10^{-11}	8.9425×10^{-17}
Proposed QMN2	3.6977×10^{-5}	2.6257×10^{-11}	7.9462×10^{-17}

Table 2: The error analysis for example 2

Method	E_6	E_7	E_8
Broyden's method 1	5.2766×10^{-5}	4.9476×10^{-6}	8.6654×10^{-7}
Broyden's method 2	4.6086×10^{-5}	5.4138×10^{-6}	1.1486×10^{-6}
Proposed QMN1	4.4228×10^{-8}	1.6971×10^{-9}	1.0000×10^{-15}
Proposed QMN2	7.6575×10^{-8}	2.2398×10^{-9}	2.8000×10^{-14}

$\xi = (g_0(0), ..., g(n))^t$, $t_m = s_m = mh\ 0.1\ m$, $m = 0, 1, 2, ..., 10$, Start with. $\xi(0) = (0.1, 0.1, ..., 0.1)^t$. Table 1 shows the absolute error.

Example 2: Broyden Tridiagonal Function (More *et al*., 1981). Consider the nonlinear system H(x) = 0 where:

$$h_1(x) = (3-2x_1)x_1 - 2x_2 + 1$$
$$h_2(x) = (3-x_2)x_2 - x_1 - 2x_3 + 1$$
$$h_3(x) = (3-x_3)x_3 - x_2 + 1$$

set $x^{(0)} = (1, 1, 1)^t$. Table 2 shows the absolute error $E_k = \|x^{(k)} - x^{(k-1)}\|$ for some iterations of our proposed methods and some other methods. The numerical results in Table 1 and 2 of the above two examples show that the proposed methods (QMN1 and 2) is very comparable and competitive to Newton's and Broyden's methods. Also, we observe that the errors of our proposed methods decreases rapidly as number of iterations increases.

CONCLUSION

Researchers proposed two improved methods based on Quasi Newton methods called Quasi Modified Newton's methods type one and two. The proposed methods need not to compute the inverse or partial derivative. The given numerical examples have illustrated the efficiency and accuracy of the proposed methods. The proposed methods converge faster than the Quasi Newton method and more attractive than Newton's method.

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