Journal of Engineering and Applied Sciences 11 (12): 2750-2754, 2016
ISSN: 1816-949X


# Time Fractional Wave Equation in the Caputo Sense

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**Abstract:** In this study, we consider the wave equation of time fractional order in the sense of Caputo with initial conditions, Neumann boundary conditions and force term.

**Key words:** Modeling, fractional partial differential equation, Neumann boundary conditions, force, Colombia

## INTRODUCTION

In this study, a nonhomogeneous initial boundary value problem for the time fractional wave equation was treated. This problem was obtained from the nonhomogeneous wave equation by replacing the second order time derivative by a fractional derivative of order $1<\alpha<2$ in Caputo's sense. In this research, we solve the nonhomogeneous wave equation with fractional time, initial conditions and Neumann boundary conditions. The classical model of the wave equation is based on the Newton's second law and sometimes we cannot apply this law and we have to use the nonhomegeneous wave equation with time fractional. The objective is to make use of the concept of fractional derivative of Caputo in the solution of the wave equation, then we construct the formal solution of the problem of initial conditions and boundary conditions by the method of separation of variables and Fourier series for characterizing a complete orthonormal system. Finally, we will apply a theorem for finding a unique solution to the problem. The initial-boundary value problem for partial differential equations of higher-order involving the Caputo fractional derivative was studied by Amanov and Ashyralyev. Kahlout *et al.* (2008), a time fractional partial differential equation was considered where the fractional derivative is defined in the Caputo sense. Hemeda (2012), they presented an efficient treatment of the homotopy perturbation method for linear and nonlinear partial differential equations with fractional order. Marin *et al.* (2014a-c), they treated a nonhomogeneous subdiffusion heat equation of fractional order with different initial-boundary conditions. Parsian (2012) they proposed a new approach in time fractional wave equation, the equation is more general than their equation because we work in two dimensions and with force term. Salman was treated fractional order differential equations with an Adomian decomposition method. Olayiwola was proved the generalized Taylor series in terms of fractional order derivatives.

## MATERIALS AND METHODS

**Preliminary notions:** In this study, we present some basic definitions and preliminary data that are used throughout the document.

**Definition 2.1:** Here we define the following functions for complex argument $z\epsilon C$ . The Mittag-Leffler type functions are defined by:

$$E_{\alpha}(z):=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+\beta)}$$

$$E_{\alpha,\beta}(z):=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+\beta)}$$

$$e_{\alpha}^{\varsigma z}:=z^{\alpha-1}E_{\alpha,\alpha}(\varsigma z^{\alpha})$$

where, $\zeta\epsilon C$ and $\alpha$, $\beta>0$ is Euler's Gamma function defined for any complex number $\Gamma(.)$ with a positive real part as:

$$\Gamma(z):=\int_0^{\infty}t^{z-1}dt$$

Note that these functions are generalizations of the exponential function base e as $e^z=\Sigma_{j=0}^{\infty}z^j/j!$ and $j!=\Gamma(j+1)$.

**Definition 2.2:** If $g(t)\epsilon C[a, b]$ and $\alpha>0$ then its Riemann-Liouville fractional integral is defined by:

$$I_{a+}^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_{\alpha}^{t}\frac{g(s)}{(t-s)^{1-\alpha}}ds$$

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**Definition 2.3:** The Caputo-Djrbashyan fractional derivative of order $\alpha>0$ of a continuous function g: (a, b)→R is defined by:

$$\left(\frac{d}{dt}\right)^{\alpha} g(t) = I_{0+}^{n-\alpha} g^{(n)}(t)$$

where n = [α]+1 (the notation [α] denotes the largest integer not >α).

**Lemma 2.4:** Srivastava and Trujillo (2006), Let $q\geq 0$ and $\phi(t)$ a function of absolute value integrable on an interval [0, T] (namely, $|\phi(t)|$ is integrable on [0, T] or $\phi(t)\in L_1[0, T]$). Then:

$$I_{0+}^{p} I_{0+}^{q} \phi(t) = I_{0+}^{p+q}\phi(t) = I_{0+}^{q} I_{0+}^{p}\phi(t) \quad (1)$$

is satisfied almost everywhere (i.e., except in a set of measure 0) on [0, T]. Further, if $\phi(t)$ is continuous in the interval [0, T] ($\phi(t)\in C$ [0, T] ), then (Eq. 1) is true and:

$$\left(\frac{d}{dt}\right)^{\alpha} I_{0+}^{\alpha}\phi(t) = \phi(t)$$

for all $t\in[0, T]$ and $\alpha>0$.

**Theorem 2.5:** Goren *et al.* (2014), Let $\phi(t)\in L_1(0, T)$. Then, the integral equation:

$$\phi(t) = \phi(t) + \frac{\gamma}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\varphi(\tau)d\tau$$

has a unique solution $\varphi(t)$ defined by the following formula:

$$\varphi(t) = \phi(t) + \gamma\int_0^t e_{\alpha}^{\gamma(t-\tau)}\phi(\tau)d\tau$$

where $e_{\alpha}^{\gamma z}$ is a Mittag-Leffler type function given in Definition 2.1.

**Wave equation model:** We consider the equation:

$$\left(\frac{d}{dt}\right)^{\alpha} W = \eta^2\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right) + F(x,y,t) \quad (2)$$

where 0<x<a, 0<y<b, 0<t<Tα<2 is the differential operator in the sense of Caputo, $(d/dt)^{\alpha}$ is the propagation speed of the wave, η is the force term. The problem is to find the solution F (x, y, t) of Eq. 2 which satisfies the conditions:

$$\begin{aligned}\frac{\partial W(0,y,t)}{\partial x} &= \frac{\partial W(a,y,t)}{\partial x} = 0, 0\leq y\leq b, 0\leq t\leq T\\ \frac{\partial W(x,0,t)}{\partial y} &= \frac{\partial W(x,b,t)}{\partial y} = 0, 0\leq x\leq a, 0\leq t\leq T\end{aligned} \quad (3)$$

$$\begin{aligned}&W(x,y,0) = \phi(x,y),\\ &W(x,y,0) = \phi(x,y), 0\leq x\leq a\leq y\leq b\end{aligned} \quad (4)$$

## RESULTS AND DISCUSSION

**Theorem 4.1:** If the differential Eq. 2 satisfies the boundary and initial conditions (Eq. 3 and 4). Then the solution of the problem is unique and has the form:

$$\begin{aligned}W(x,y,t) = \sum_{k=1}^{\infty}\Big(&E_{\alpha,1}(-\Lambda_k t^{\alpha})\phi_k + tE_{\alpha,2}\big((-\Lambda_k t^{\alpha}\big)\varphi_k +\\ &\int_0^t E_{\alpha,\alpha}\big(-\Lambda_k(t-\tau)^{\alpha-1}\big)f_k(\tau)d\tau)U_k(x,y)\end{aligned} \quad (5)$$

Proof: We look for a solution W (x, y, t) of the form (Eq. 15). We define the linear differential operator L:

$$LU = -\eta^2\Delta^2 U \quad (6)$$

where the function U is continuously differentiable and $\nabla^2$ is the Laplacian in dimension 2 defined by:

$$\Delta^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$$

The operator L is defined over some subset of the vector space $L^2$ (0, a)×(0, b) of the functions U (x, y) with (x, y)∈(0, a)×(0, b) such that the function $|U(x, y)|^2$ is integrable on(0, a)×(0, b). In other words, the domain of definition $D_L$ of the operator L consists of all functions U(x, y)∈$L^2$ (0, a)×(0, b) satisfying the boundary conditions:

$$\begin{aligned}\frac{\partial U(0,y)}{\partial x} &= \frac{\partial U(a,y)}{\partial x} = 0, 0\leq y\leq b\\ \frac{\partial U(x,0)}{\partial y} &= \frac{\partial U(x,b)}{\partial y} = 0, 0\leq x\leq a\end{aligned} \quad (7)$$

and whose images. LU∈$L^2$ [(0, a)×(0, b)]The eigenvalue problem is posed as follows: We have to find the values of the parameter Λ such that the equation:

$$LU = \Lambda U \quad (8)$$

has nontrivial solutions in the domain $_{DL}$. These functions are the eigenfunctions of L. Given the Eq. 6, we see the Eq. 8 is equivalent to:

$$\nabla^2 U + \frac{\Lambda U}{\eta^2} = 0$$

Now, let $V^2 = \Lambda/\eta^2$. So, the equation is written as :

$$\nabla^2 U + v^2 U = 0 \tag{9}$$

To solve Eq. 9, we use the method of separation of variables, we assume a nontrivial solution in the form U (x,y) = X (x) Y (y). Replacing this into Eq. 9, we get:

$$X''(x)Y(y) + X(x)Y''(y) + v^2 X(x)Y(y) = 0 \tag{10}$$

if we divide both sides by X (x) Y (y) it must be:

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} - v^2 = 0$$

in this way:

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} - v^2 = -\mu^2$$

Since, the left side depends only on x whereas the right side is independent of x, both sides must be equal to a constant. Therefore, we have:

$$X''(x) + \mu^2 X(x) = 0 \tag{11}$$

$$Y''(y) + \rho^2 Y(y) = 0 \tag{12}$$

Where, $\rho^2 = V^2 - \mu^2$. The corresponding solution to Eq. 11 is given by:

$$X(x) = \alpha_1 \cos\mu x + \alpha_2 \sin\mu x$$

Due to boundary conditions we obtain X' (0) Y (y) = 0 = X' (a) Y (y), $0 \le y \le b$. We must find a nontrivial solution of (Eq. 11), we see that X' (0) = 0, X' (a) = 0, respectively so that we have $\alpha_2 = 0$, $\alpha_1 \neq 0$ and $\sin \mu\alpha = 0$. The latter result gives $\mu = m\pi/\alpha$, m = 0, 1, 2, ... Note that $\mu = 0$ is also an eigenvalue. Accordingly:

$$X_m(x) = \alpha_m \cos\frac{m\pi x}{a}, m = 0,1,2...$$

Analogously, for a nontrivial solution Y taking $\rho^2 = V^2 - \mu^2$ we have:

$$y(y) = \beta_1 \cos\rho y + \beta_2 \sin\rho y$$

Applying the homogeneous boundary conditions, we have $\beta_2 = 0$, $\beta_1 \neq 0$ and $\sin \rho b = 0$. So, we see that $Y_n$ (y) $\beta_n \cos n\pi y/b$, n = 0, 1, 2, ... Therefore, the solutions of the Eq. 9 can be written as:

$$U_{mn}(x,y) = \gamma_{mn} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}, m,n = 0,1,2.... \tag{13}$$

Where, $\gamma_{mn}$ for each of the corresponding eigenvalues:

$$v_{mn}^2 = \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\pi^2$$

for which Eq. 9 is expressed as:

$$\Lambda_{mn} = \eta^2 v_{mn}^2$$

thus, we define $\Lambda_{mn} \equiv \Lambda_k$, $U_{mn} \equiv U_k$ and $\gamma_{mn}$. Then:

$$LU_k = \Lambda_k U_k,\ U_k \in D_L,\ k = 1,2,...$$

These eigenfunctions of L can be chosen orthonormal with:

$$\gamma_k = \frac{2}{\sqrt{ab}} \tag{14}$$

therefore:

$$\langle U_k, U_l\rangle \int_0^a\int_0^b U_k(x,y)U_l(x,y)dydx$$
$$= \frac{4}{ab}\int_0^a\int_0^b \cos\frac{m_k\pi x}{a}\cos\frac{n_k\pi x}{b}\cos\frac{m_l\pi x}{a}\cos\frac{m_l\pi x}{b}dydx$$
$$= \delta_{kl}, m_k, m_l, n_k, n_l = 1,2,...$$

$\{U_k\}$ is a complete set of $L^2$ [(0, a)×(0, b)] and each function u (x, y)$\in D_L$ can be represented as a series:

$$u(x,y) = \sum_{k=1}^{\infty}\langle u, U_k\rangle U_k(x,y)$$

The solution of the problem in Eq. 1 that satisfies the initial conditions and boundary conditions can be written as:

$$W(x,y,t) = \sum_{k-1}^{\infty} U_k(x,y)T_k(t), T_k(t) = \langle W, U_k\rangle \tag{15}$$

for t>0. To find the fractional differential equation for functions $T_k(t)$, the solution (Eq. 15) is substituted into Eq. 2:

$$\sum_{l=1}^{\infty} U_l(x,y), \left[\left(\frac{d}{dt}\right)^{\alpha} T_l(t)\right]$$
$$\sum_{l=1}^{\infty} T_l(t)LU_l(x,y) + F(x,y,t)$$
$$\sum_{l=1}^{\infty} T_l(t)\Lambda_l U_l(x,y) + F(x,y,t)$$

Now, taking the scalar product of each side of this equation with an eigenfunction $U_k$ (x, y):

$$\sum_{l=1}^{\infty}\langle U_k, U_l\rangle\left(\frac{d}{dt}\right)^{\alpha} T_l(t) = -\sum_{l-1}^{\infty} T_l(t)\Lambda_l\langle U_k, U_l\rangle + \langle U_k, F\rangle$$

Now, using the orthonormality of eigenfunctions, we obtain Eq. 16:

$$\left(\frac{d}{dt}\right)^{\alpha} T_l(t)\Lambda_k T_k = f_k(t) \quad (16)$$

where $f_k$ (t) = ⟨$U_k$ (x, y), F (x, y, t)⟩, k = 1, 2, .... Now applying the initial conditions (Eq. 4) to Eq. 15:

$$W(x,y,0) = \phi(x,y)\sum_{k=1}^{\infty} U_k(x,y)T_k(0)$$

Where:

$$T_k(0) = \langle W(x,y,o), U_k(x,y)\rangle = \langle \phi(x,y), U_k(x,y)\rangle$$

$$W_t(x,y,0) = \varphi(x,y) = \sum_{k=1}^{\infty} U_k(x,y)T'_k(0)$$

Where:

$$T'_k(0) = \langle W_t(x,y,o), U_k(x,y)\rangle = \langle \varphi,(x,y)U_k(x,y)\rangle$$

For the initial conditions $T_k$ (0), $T_k'$ (0), note that the solution of the corresponding homogeneous problem (Eq. 2), (i,e, with F (x, y, t) ≡ 0) has the form:

$$W_H(x,y,t) = \sum_{k=1}^{\infty} U_k(x,y)T_{H,k}(t)$$

where $T_{H,k}(t) = T_{H,k}(0)\, E_{\alpha,1}(-\Lambda_k t^{\alpha}) + tT'_{H,k}(0)\, E_{\alpha,2}(-\Lambda_k t^{\alpha})$ is the general solution of the corresponding homogeneous equation to (Eq. 16) (since $f_k$ (t) if f ≡ 0) for each $\Lambda_k$ which is the same as for the nonhomogeneous equation:

$$W_H(x,y,0) = W(x,y,0) = \phi(x,y)$$
$$W_{t,H}(x,y,0) = W_t(x,y,0) = \varphi(x,y)$$

$$W_H(x,y,o) = \sum_{l=1}^{\infty} U_l(x,y)T_{H,l}(0) = W(x,y,o) = \phi(x,y) \quad (17)$$

$$W_{t,H}(x,y,o) = \sum_{l=1}^{\infty} U_l(x,y)T_{t,H,l}(0) = W_t(x,y,0) = \varphi(x,y) \quad (18)$$

taking the dot product of $\phi$ and $U_k$ and $\varphi$ and $U_k$:

$$T_k(0) = \langle U_k, \phi\rangle \quad (19)$$

$$T_k(0) = \langle U_k, \varphi\rangle \quad (20)$$

For $n_k$, $m_k \geq 1$:

$$T_k(0) = \sqrt{\frac{2}{ab}}\int_0^a\int_0^b \phi(x,y)\cos\frac{m_k\pi x}{a}\cos\frac{n_k\pi x}{b}dydx$$

$$T'_k 0) = \sqrt{\frac{2}{ab}}\int_0^a\int_0^b \varphi(x,y)\cos\frac{m_k\pi x}{a}\cos\frac{n_k\pi x}{b}dydx$$

To find the solution of the Cauchy problem for Eq. 16 with the initial conditions $T_k$ (0), $T_k'$ (0), we have to apply the Lemma 2.4, then:

$$\left(\frac{d}{dt}\right)^{\alpha} T_k(t) = I_{0+}^{2-\alpha}T''_k(t)$$

Substituting this result in Eq. 16, the following equation is obtained:

$$I_{0+}^{2-\alpha}T''_k(t) + \Lambda_k T_k(t) = f_k(t)$$

Applying the operator $I_{0+}{}^{\alpha}$ to this equation, we have the following Volterra integral equation of the second kind:

$$T_k(t) - T_k(0) - tT'_k(0) + \frac{\Lambda_k}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}T_k(\tau)d\tau = I_{0+}^{\alpha}f_k(t) \quad (21)$$

Using Eq. 21:

$$\frac{1}{\Gamma(\xi)}\int_0^z \tau^{\beta-1}E_{\alpha,\beta}(\varsigma\tau^{\alpha})(z-\tau)^{\xi-1}d\tau = z^{\beta+\xi-1}E_{\alpha,\beta+\xi}(\varsigma z^{\alpha})$$

$$\frac{1}{\Gamma(\beta)}zE_{\alpha,\alpha+\beta}(z) = E_{\alpha,\beta}(z)$$

and according to Theorem 2.5, it follows that:

$$T_k(t) = T_k(0)E_{\alpha,1}(-\Lambda_k t^{\alpha}) + tT'_k(0)E_{\alpha,1}(-\Lambda_k t^{\alpha}) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\Lambda_k(t-\tau)^{\alpha})f_k(\tau)d\tau \quad (22)$$

For $T_k$ (0) and $T_k'$ (0), we can expand the functions $\phi$ (x, y) and $\varphi$ (x, y) in the form of a Fourier series along with the functions $U_k$ (x, y), k = 1, 2, ...

$$\phi(x,y) = \sum_{k-1}^{\infty}\phi_k U_k(x,y),\ \varphi(x,y) = \sum_{k-1}^{\infty}\varphi_k U_k(x,y) \quad (23)$$

Where:

$$\phi_k = \int_0^a \int_0^b \phi(x,y) U_k(x,y) dydx$$

$$\varphi_k = \int_0^a \int_0^b \varphi(x,y) U_k(x,y) dydx$$

So, using Eq. 3, 20 and 22 we see that:

$$T_k(t) = E_{\alpha,1}(-\Lambda_k t^{\alpha})\phi_k + tE_{\alpha,2}(-\Lambda_k t^{\alpha})\varphi_k + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\Lambda_k (t-\tau)^{\alpha}) f_k(\tau) d\tau$$

Now, substituting this in Eq. 15, we obtain the formal solution of the problem given by Eq. 2 that satisfies the given initial and boundary conditions, that is:

$$W(x,y,t) = \sum_{k=1}^{\infty} (E_{\alpha,1}(-\Lambda_k t^{\alpha})\phi_k + tE_{\alpha,2}(-\Lambda_k t^{\alpha})\varphi_k + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\Lambda_k (t-\tau)^{\alpha}) f_k(\tau) d\tau) U_k(x,y)$$

Where:

$$U_k(x,y) = \frac{2}{\sqrt{ab}} \cos\frac{n\pi x}{a} \cos\frac{m\pi y}{b}$$

## CONCLUSION

We obtain a unique solution to the wave equation of time fractional order in the sense of Caputo with initial conditions, Neumann boundary conditions and force term.


## ACKNOWLEDGEMENT

The researchers express their deep gratitude to Universidad de Cartagena for partial financial support.


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