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Analytical Investigation of the Dynamic Response of a Timoshenko Thin-Walled Beam with Asymmetric Cross Section under Deterministic Loads

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Abstract: The objective of the present study is to analyze dynamic response of the Timoshenko thin-walled beam with coupled bending and torsional vibrations under deterministic loads. The governing differential equations is obtained by using Hamilton's principle. The Timoshenko beam theory is employed and the effects of shear deformations, rotary inertia and warping stiffness are included in the present formulations. Dynamic features of underlined beam are obtained using free vibration analysis. For this purpose, the dynamic stiffness matrix method is used. Applying exact dynamic stiffness matrix method on the movement differential equations leads to the issue of nonlinear eigenvalue problem that Wittrick-Williams algorithm is used to solve it. Orthogonality property of vibrational modes are extracted by applying differential equations of motion. The theoretical expressions for the displacement response of thin-walled beams subjected to concentrated or distributed loads are presented. The numerical results for dynamic flexural displacements, rotational displacements and torsional displacements are given. The proposed theory is fairly general and can be used for thin-walled beam assemblage of arbitrary boundary conditions subjected to various kinds of deterministic loads.

Key words: Thin-walled beams, Timoshenko beam theory, exact dynamic stiffness matrix method, Wittrick-Williams algorithm, deterministic dynamic loads, kinds

INTRODUCTION

Thin-walled elastic beams offer high performance in terms of minimum weight for a given strength, hence they are used widely in the structures and aerospace industries. However, the shear center and centroid of typical cross-sections for mono-symmetric and asymmetric thin-walled beams are not coincident. This leads to relatively complex structural behavior due to coupling between the bending and torsional modes and ultimately difficulties in establishing an accurate prediction of their dynamic characteristics.

The development of an exact dynamic stiffness matrix to model the coupled flexural-torsional motion of thin-walled beams has been of interest to researchers for a number of years. The method is deemed to be 'exact', since, it is derived from the closed form analytical solution of the governing differential equations and does not therefore rely on assumed shape functions, as in the finite element technique. Over the last few years, many studies have been performed in the field of formulation of Dynamic Stiffness Matrix (DSM) of beams. The dynamic stiffness matrix of a Timoshenko beam was investigated by Cheng (1970) for the first time. Howson and Williams (1973) considered the effect of axial load on the natural frequencies of Timoshenko beam. Banerjee (1989) studied a beam with a section having one axis of symmetry and derived some explicit terms for the stiffness matrix arrays regardless of axial load effect. Banerjee and Williams (1992) investigated dynamic stiffness matrix for coupled flexural-torsional vibration of Timoshenko beam. Banerjee *et al*. (1996) studied warping effect on the formulation of dynamic stiffness matrix. Bercin and Tanaka (1997) surveyed coupled flexural-torsional vibrations of uniform beam having single symmetric section, considering conventional support conditions. Li *et al*. (2004a, b) derived the free vibrations of thin-walled Timoshenko beam under axial load in which the effects of axial load, warping stiffness, shear deformation and rotational inertia were taken into consideration and it was used from continuous model. Rafezy and Howson (2006) derived the dynamic stiffness matrix of a 3-Dimensional (3D) shear-torsion beam with an asymmetric cross-section. The beam had the unusual theoretical property, so that, it allowed only for shear deformation but not bending deformation. Ghandi *et al*. (2012) replaced Euler-Bernoulli theory with Timoshenko theory when the external layer of thin-walled beam is modeled and they assumed that the

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thin-walled part of the beam could have either open or closed section shape and would create flexural, shear, warping and Saint-Venant rigidities. Ghandi *et al*. (2015) also derived the dynamic stiffness matrix of uniform beam with asymmetric cross section and elastic support under axial load. The mentioned beam was consisted of an external enclosed thin-walled layer that was combined with a shear resistant filled core. Ghandi and Shiri (2017) investigated the effect of the eccentricity of axial load on the natural frequencies of asymmetric thin-walled beams using exact dynamic stiffness matrix method. Many researches have been performed in the field of the response of beams with symmetric cross-section subjected to deterministic and random dynamic loads. Eslimi-Esfahani *et al*. (1996) analytically investigated the dynamic response of beam with coupled flexural-torsional vibration subjected to deterministic and stochastic dynamic loads for the first time. In another study, Eslimy-Isfahany and Banerjee (1996) analytically calculated dynamic response of beam with constant axial load with coupled flexural-torsional vibration under definitive and stochastic dynamic loads by using modal analysis method. Li *et al*. (2004a, b) derived an explicit term for dynamic response of single symmetric Timoshenko beam subjected to stochastic excitations. In the following of the previous research, Jun *et al*. (2004) derived the effects of axial load in the calculation of dynamic response of single symmetric Timoshenko beam against stochastic excitations.

In most of these researches, cross-section of the beam was mono-symmetric and Euler-Bernouli theory is used to model bending of beam. Moreover, to model beam bending, Euler-Bernouli theory is not capable of producing correct results when beams with large sections compared to their lengths or extraction of natural frequencies of higher modes are under study. In such conditions, Timoshenko beam theory in which shear deformation and rotary inertia parameters are considered should be employed. In this study, considering the effect of definitive dynamic load, the analytical dynamic response of 3D flexural-torsional beam with asymmetric cross-section will be investigated by the help of exact dynamic stiffness matrix and modal analysis methods.

MATERIALS AND METHODS

Theory: The cross-section of the intended beam is shown in Fig. 1. This beam is a uniform 3D beam with asymmetric cross-section. The Timoshenko beam theory is used for modeling the bending beam. This beam has flexural rigidities of EI_x and EI_y in x-z and y-z planes, torsional warping rigidity of EI_ω, torsional Saint-Venant rigidity of G_tJ_t and shear rigidities of G_tA_{xt} and G_tA_{yt} where G_t is shear modulus, J_t is the section torsional constant and A_{xt} and A_{yt} are equivalent shear section in x and y directions, respectively. In Fig. 1, the center of gravity is denoted by C, shear center is shown by O. The axes crossing the center of gravity and shear center are known as mass axis and bending axis, respectively. The origin of the coordinate system is placed at O, x and y-axis are in the direction of main axes of the cross-section and z-axis is coincided with the bending axis. The beam total mass is distributed along it's length as uniform distributed load and m is the beam mass per unit length. The external loads applied on the thin-walled beam are including unit length forces f_x (z,t) and f_y (z,t) which are applied on the bending axis, respectively in the directions of x and y-axis, unit length bending moment m_x (z, t) in the x-z plane around y-axis, unit length bending moment m_x (z, t) in the y-z plane around x-axis and also unit length torsion moment g(z, t) that is applied around the bending axis (Fig. 2).

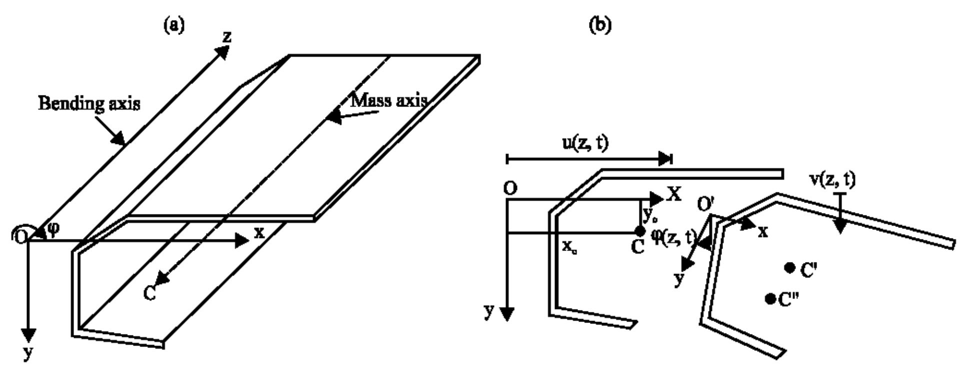

Fig. 1: a) 3D thin-walled beam with a length of L and asymmetric section and b) deformed shape of the cross-section after translational and torsional displacements

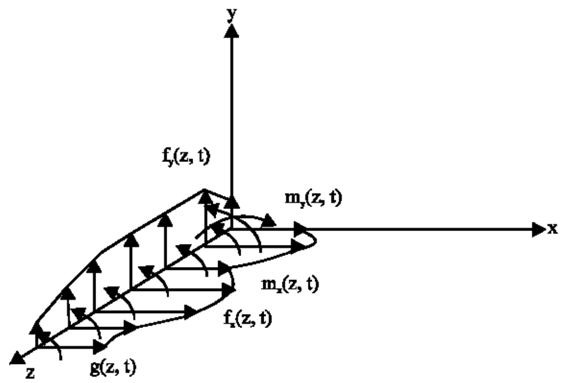

Fig. 2: Externally applied loads on a thin-walled beam

The differential equations governing the beam moment are as five coupled partial differential equations that are defined as bellow:

$$m\frac{\partial^2 u(z,t)}{\partial t^2}-my_c\frac{\partial^2\phi(z,t)}{\partial t^2}-G_tA_{xt}\frac{\partial^2 u(z,t)}{\partial z^2}+G_tA_{xt}\frac{\partial\theta_x(z,t)}{\partial z}=f_x(z,t) \quad (1)$$

$$m\frac{\partial^2 v(z,t)}{\partial t^2}+mx_c\frac{\partial^2\phi(z,t)}{\partial t^2}-G_tA_{yt}\frac{\partial^2 v(z,t)}{\partial z^2}+G_tA_{yt}\frac{\partial\theta_y(z,t)}{\partial z}=f_y(z,t) \quad (2)$$

$$\rho I_x\frac{\partial^2\theta_x(z,t)}{\partial t^2}-EI_x\frac{\partial^2\theta_x(z,t)}{\partial z^2}-G_tA_{xt}\frac{\partial u(z,t)}{\partial z}+G_tA_{xt}\theta_x(z,t)=m_x(z,t) \quad (3)$$

$$\rho I_y\frac{\partial^2\theta_y(z,t)}{\partial t^2}-EI_y\frac{\partial^2\theta_y(z,t)}{\partial z^2}-G_tA_{yt}\frac{\partial v(z,t)}{\partial z}+G_tA_{yt}\theta_y(z,t)=m_y(z,t) \quad (4)$$

$$-my_c\frac{\partial^2 u(z,t)}{\partial t^2}+mx_c\frac{\partial^2 v(z,t)}{\partial t^2}+mr_m^2\frac{\partial^2\phi(z,t)}{\partial t^2}-G_tJ_t\frac{\partial^2\phi(z,t)}{\partial z^2}+EI_\omega\frac{\partial^4\phi(z,t)}{\partial z^4}=g(z,t) \quad (5)$$

RESULTS AND DISCUSSION

Free vibration analysis: In order to determine natural frequencies and vibration modes, it is required to perform the undamped free vibration analysis of the system. For this purpose, the external applying forces are considered equal to zero and thus, the above equations could be written as follows:

$$m\frac{\partial^2 u(z,t)}{\partial t^2}-my_c\frac{\partial^2\phi(z,t)}{\partial t^2}-G_tA_{xt}\frac{\partial^2 u(z,t)}{\partial z^2}+G_tA_{xt}\frac{\partial\theta_x(z,t)}{\partial z}=0 \quad (6)$$

$$m\frac{\partial^2 v(z,t)}{\partial t^2}+mx_c\frac{\partial^2\phi(z,t)}{\partial t^2}-G_tA_{yt}\frac{\partial^2 v(z,t)}{\partial z^2}+G_tA_{yt}\frac{\partial\theta_y(z,t)}{\partial z}=0 \quad (7)$$

$$\rho I_x\frac{\partial^2\theta_x(z,t)}{\partial t^2}-EI_x\frac{\partial^2\theta_x(z,t)}{\partial z^2}-G_tA_{xt}\frac{\partial u(z,t)}{\partial z}+G_tA_{xt}\theta_x(z,t)=0 \quad (8)$$

$$\rho I_y\frac{\partial^2\theta_y(z,t)}{\partial t^2}-EI_y\frac{\partial^2\theta_y(z,t)}{\partial z^2}-G_tA_{yt}\frac{\partial v(z,t)}{\partial z}+G_tA_{yt}\theta_y(z,t)=0 \quad (9)$$

$$-my_c\frac{\partial^2 u(z,t)}{\partial t^2}+mx_c\frac{\partial^2 v(z,t)}{\partial t^2}+mr_m^2\frac{\partial^2\phi(z,t)}{\partial t^2}-G_tJ_t\frac{\partial^2\phi(z,t)}{\partial z^2}+EI_\omega\frac{\partial^4\phi(z,t)}{\partial z^4}=0 \quad (10)$$

In order to analyze the free vibration, the answers of u (z, t), v (z, t), θ_x (z, t), θ_y and ϕ (z, t) are written as follows:

$$u(z,t)=U_r(z)e^{i\omega_r t},\ v(z,t)=V_r(z)e^{i\omega_r t},\ \theta_x(z,t)=\Theta_{xr}(z)e^{i\omega_r t}$$
$$\theta_y(z,t)=\Theta_{yr}(z)e^{i\omega_r t},\ \phi(z,t)=\Phi_r(z)e^{i\omega_r t} \quad (11)$$

In the above relations, r = 1, 2, 3, …, represents the vibrational mode number. Substituting Eq. 3 into Eq. 2 gives:

$$-m\omega_r^2U_r(z)+my_c\omega_r^2\Phi_r(z)-G_tA_{xt}\frac{d^2U_r(z)}{dz^2}+G_tA_{xt}\frac{d\Theta_{xr}(z)}{dz}=0 \quad (12)$$

$$-m\omega_r^2V_r(z)-mx_c\omega_r^2\Phi_r(z)-G_tA_{yt}\frac{d^2V_r(z)}{dz^2}+G_tA_{yt}\frac{d\Theta_{yr}(z)}{dz}=0 \quad (13)$$

$$-\rho I_x\omega_r^2\Theta_{xr}(z)-EI_x\frac{d^2\Theta_{xr}(z)}{dz^2}-G_tA_{xt}\frac{dU_r(z)}{dz}+G_tA_{xt}\Theta_{xr}(z)=0 \quad (14)$$

$$-\rho I_y \omega_r^2 \Theta_{yr}(z) - EI_y \frac{d^2\Theta_{yr}(z)}{dz^2} - G_t A_{yt} \frac{dV_r(z)}{dz} + G_t A_{yt} \Theta_{yr}(z) = 0 \quad (15)$$

$$my_c \omega_r^2 U_r(z) - mx_c \omega_r^2 V_r(z) - m r_m^2 \omega_r^2 \Phi_r(z) - G_t J_t \frac{d^2\Phi_r(z)}{dz^2} + EI_\omega \frac{d^4\Phi_r(z)}{dz^4} = 0 \quad (16)$$

By applying the dynamic stiffness matrix method on above governing differential equations, we can obtain the natural frequencies and mode shapes. For this purpose refer to the Ghandi *et al*. (2012), Ghandi and Shiri (2017) , Howson and Williams (1973).

Extraction of the orthogonality properties: A most significant property of the mode shapes is that they form a set of orthogonal mathematical functions. To analyze the forced vibrations, the orthogonality condition would have to be used. The orthogonality conditions apply to any two different modes they do not apply to two modes having the same frequency. For discrete systems, the orthogonality conditions are available in all references related to the dynamics of structures. The studied beam in this study is a distributed properties system. The vibration mode shapes derived for beams with distributed properties have orthogonality relationships equivalent to those for the discrete parameter systems. Orthogonality conditions for three-dimensional asymmetric thin-walled Timoshenko beam is derived in this study as follows: Substituting Eq. 14 into 12 and also Eq. 15 into 13 gives:

$$-m\omega_r^2 U_r(z) + my_c \omega_r^2 \Phi_r(z) = -\left[\rho I_x \omega_r^2 \frac{d\Theta_{xr}(z)}{dz} + EI_x \frac{d^3\Theta_{xr}(z)}{dz^3}\right] \quad (17)$$

$$-m\omega_r^2 V_r(z) - mx_c \omega_r^2 \Phi_r(z) = -\left[\rho I_y \omega_r^2 \frac{d\Theta_{yr}(z)}{dz} + EI_y \frac{d^3\Theta_{yr}(z)}{dz^3}\right] \quad (18)$$

Multiplying Eq. 17 by U_s (z) (sth vibrational mode) and integrating with respect to Z gives:

$$-\omega_r^2 \int_0^L m U_r(z) U_s(z) dz + \omega_r^2 \int_0^L my_c(z) U_s(z) \phi_r(z) dz = -\int_0^L \left[\rho I_x \omega_r^2 \frac{d\Theta_{xr}(z)}{dz} + EI_x \frac{d^3\Theta_{xr}(z)}{dz^3}\right] U_s(z) dz \quad (19)$$

If the last integration in this equation is performed by parts, it is found to give:

$$-\left|\left[EI_x \frac{d^2\Theta_{xr}(z)}{dz^2} + \rho I_x \omega_r^2 \Theta_{xr}(z)\right] U_s(z)\right|_0^L + \int_0^L \left[EI_x \frac{d^2\Theta_{xr}(z)}{dz^2} + \rho I_x \omega_r^2 \Theta_{xr}(z)\right] \frac{dU_s(z)}{dz} dz \quad (20)$$

The integrated term is nil because thcontents of the square brackets is equal to the shear force which vanishes at the boundaries z = 0 and z = L. the integral may again be evaluated by parts to give:

$$\left|(EI_x \frac{d\Theta_{xr}(z)}{dz}) \frac{dU_s(z)}{dz}\right|_0^L - \int_0^L EI_x \frac{d\Theta_{xr}(z)}{dz} \frac{d^2U_s(z)}{dz^2} dz + \omega_r^2 \int_0^L \rho I_x \Theta_{xr}(z) \frac{dU_s(z)}{dz} dz \quad (21)$$

And again the integrated term vanishes since, the bracketed term is equal to bending moment which is zero at the extremities, so finally the Eq. 19 is expressed as follows:

$$-\omega_r^2 \int_0^L mU_r(z) U_s(z)\, dz + \omega_r^2 \int_0^L my_c \Phi_r(z) U_s(z) dz = -\int_0^L EI_x \frac{d\Theta_{xr}(z)}{dz} \frac{d^2U_s(z)}{dz^2} + \omega_r^2 \int_0^L \rho I_x \Theta_{xr}(z) \frac{dU_s(z)}{dz} dz \quad (22)$$

Similarly, the solving stages begin with the Eq. 17 rewrites for the sth mode and multiplied throughout by (rth vibrational mode) and by following the same steps it gives:

$$-\omega_s^2 \int_0^L m U_s(z) U_r(z) dz + \omega_s^2 \int_0^L my_c \Phi_s(z) U_r(z) dz = -\int_0^L EI_x \frac{d\Theta_{xs}(z)}{dz} \frac{d^2U_r(z)}{dz^2} dz + \omega_s^2 \int_0^L \rho I_x \Theta_{xs}(z) \frac{dU_r(z)}{dz} dz \quad (23)$$

If the last two equations are subtracted, it is now found that:

$$(\omega_r^2 - \omega_s^2) \int_0^L mU_r(z) U_s(z) dz - \omega_r^2 \int_0^L my_c \Phi_r(z) U_s(z) dz + \omega_s^2 \int_0^L my_c \Phi_s(z) U_r(z) dz = \int_0^L EI_x \left(\frac{d\Theta_{xr}(z)}{dz} \frac{d^2U_s(z)}{dz^2} - \frac{d\Theta_{xs}(z)}{dz} \frac{d^2U_r(z)}{dz^2}\right) dz - \omega_r^2 \int_0^L \rho I_x \Theta_{xr}(z) \frac{dU_s(z)}{dz} dz + \omega_s^2 \int_0^L \rho I_x \Theta_{xs}(z) \frac{dU_r(z)}{dz} dz \quad (24)$$

By performing similar operation for Eq. 18 the following result is finally obtained:

$$(\omega_r^2-\omega_s^2)\int_0^L mV_r(z)V_s(z)dz+\omega_r^2\int_0^L mx_c\Phi_r(z)V_s(z)dz-\omega_s^2\int_0^L mx_c\Phi_s(z)V_r(z)dz=\int_0^L EI_y\left(\frac{d\Theta_{yr}(z)}{dz}\frac{d^2V_s(z)}{dz^2}-\frac{d\Theta_{ys}(z)}{dz}\frac{d^2V_r(z)}{dz^2}\right)dz-\omega_r^2\int_0^L\rho I_y\Theta_{yr}(z)\frac{dV_s(z)}{dz}dz+\omega_s^2\int_0^L\rho I_y\Theta_{ys}(z)\frac{dV_r(z)}{dz}dz \quad (25)$$

By dividing Eq. 14 and 15 to find that:

$$\frac{dU_r(z)}{dz}=\Theta_{xr}(z)-\left[\left\{EI_x\frac{d^2\Theta_{xr}(z)}{dz^2}+\rho I_x\omega_r^2\Theta_{xr}(z)\right\}\Big/G_tA_{xt}\right] \quad (26)$$

$$\frac{dV_r(z)}{dz}=\Theta_{yr}(z)-\left[\left\{EI_y\frac{d^2\Theta_{yr}(z)}{dz^2}+\rho I_y\omega_r^2\Theta_{yr}(z)\right\}\Big/G_tA_{yt}\right] \quad (27)$$

When Eq. 26 is differentiated with respect to z, multiplied by $Ei_x\, d\Theta_{xs}(z)/dz$ and then integrated with respect to z, finally it gives:

$$\int_0^L EI_x\frac{d\Theta_{xs}(z)}{dz}\frac{d^2U_r(z)}{dz^2}dz=\int_0^L EI_x\frac{d\Theta_{xs}(z)}{dz}\frac{d\Theta_{xr}(z)}{dz}dz+\int_0^L\frac{EI_x\frac{d^2\Theta_{xr}(z)}{dz^2}EI_x\frac{d^2\Theta_{xs}(z)}{dz^2}}{G_tA_{xt}}dz+\omega_r^2\int_0^L\frac{\rho I_x\,\Theta_{xr}(z)EI_x\frac{d^2\Theta_{xs}(z)}{dz^2}}{G_tA_{xt}}dz \quad (28)$$

When the suffices r and s in this last equation are interchanged it is found that:

$$\int_0^L EI_x\frac{d\Theta_{xr}(z)}{dz}\frac{d^2U_s(z)}{dz^2}dz=\int_0^L EI_x\frac{d\Theta_{xr}(z)}{dz}\frac{d\Theta_{xs}(z)}{dz}dz+\int_0^L\frac{EI_x\frac{d^2\Theta_{xs}(z)}{dz^2}EI_x\frac{d^2\Theta_{xr}(z)}{dz^2}}{G_tA_{xt}}dz+\omega_s^2\int_0^L\frac{\rho I_x\,\Theta_{xs}(z)EI_x\frac{d^2\Theta_{xr}(z)}{dz^2}}{G_tA_{xt}}dz \quad (29)$$

Subtraction now reveals that:

$$\int_0^L EI_x\left(\frac{d\Theta_{xr}(z)}{dz^2}\frac{d^2U_s(z)}{dz^2}-\frac{d\Theta_{xs}(z)}{dz^2}\frac{d^2U_r(z)}{dz^2}\right)dz=\omega_s^2\int_0^L\frac{\rho I_x\Theta_{xs}(z)EI_x\frac{d^2\Theta_{xr}(z)}{dz^2}}{G_tA_{xt}}dz-\omega_r^2\int_0^L\frac{\rho I_x\Theta_{xr}(z)EI_x\frac{d^2\Theta_{xs}(z)}{dz^2}}{G_tA_{xt}}dz \quad (30)$$

By performing similar operation for Eq. 27, the following result is finally obtained:

$$\int_0^L EI_y\left(\frac{d\Theta_{yr}(z)}{dz}\frac{d^2V_s(z)}{dz^2}-\frac{d\Theta_{ys}(z)}{dz}\frac{d^2V_r(z)}{dz^2}\right)dz=\omega_s^2\int_0^L\frac{\rho I_y\Theta_{ys}(z)EI_y\frac{d^2\Theta_{yr}(z)}{dz^2}}{G_tA_{yt}}dz-\omega_r^2\int_0^L\frac{\rho I_y\Theta_{yr}(z)EI_y\frac{d^2\Theta_{ys}(z)}{dz^2}}{G_tA_{yt}}dz \quad (31)$$

Multiply Eq. 26 throughout by $\omega_s^2\rho I_x\Theta_{xs}(z)$ and integrate with respect to z. This gives:

$$\omega_s^2\int_0^L\rho I_x\Theta_{xs}(z)\frac{dU_r(z)}{dz}=\omega_s^2\int_0^L\rho I_x\Theta_{xs}(z)\Theta_{xr}(z)dz-\omega_s^2\int_0^L\frac{EI_x\frac{d^2\Theta_{xr}(z)}{dz^2}\rho I_x\Theta_{xs}(z)}{G_tA_{xt}}dz-\omega_r^2\omega_s^2\int_0^L\frac{(\rho I_x)^2\,\Theta_{xr}(z)\Theta_{xs}(z)}{G_tA_{xt}}dz \quad (32)$$

Interchanging r and s gives:

$$\omega_r^2\int_0^L\rho I_x\Theta_{xr}(z)\frac{dU_s(z)}{dz}dz=\omega_r^2\int_0^L\rho I_x\Theta_{xr}(z)\Theta_{xs}(z)dz-\omega_r^2\int_0^L\frac{EI_x\frac{d^2\Theta_{xs}(z)}{dz^2}\rho I_x\Theta_{xr}(z)}{G_tA_{xt}}dz-\omega_s^2\omega_r^2\int_0^L\frac{(\rho I_x)^2\Theta_{xs}(z)\Theta_{xr}(z)}{G_tA_{xt}}dz \quad (33)$$

When these last two equations are subtracted they give:

$$\omega_s^2\int_0^L \rho I_x\Theta_{xs}(z)\frac{dU_r(z)}{dz}-\omega_r^2\int_0^L \rho I_x\Theta_{xr}(z)\frac{dU_s(z)}{dz}dz = (\omega_s^2-\omega_r^2)\int_0^L \rho I_x\Theta_{xr}(z)\Theta_{xs}(z)dz-\omega_s^2\int_0^L \frac{EI_x\frac{d^2\Theta_{xr}(z)}{dz^2}\rho I_x\Theta_{xs}(z)}{G_tA_{xt}}dz+\omega_r^2\int_0^L \frac{EI_x\frac{d^2\Theta_{xs}(z)}{dz^2}\rho I_x\Theta_{xr}(z)}{G_tA_{xt}}dz \quad (34)$$

Multiply Eq. 27 throughout by $\omega_s^2\rho I_y\Theta_{ys}(z)$ and integrate with respect to z. this gives:

$$\omega_s^2\int_0^L \rho I_y\Theta_{ys}(z)\frac{dV_r(z)}{dz}dz = \omega_s^2\int_0^L \rho I_y\Theta_{yr}(z)\Theta_{ys}(z)dz-\omega_s^2\int_0^L \frac{EI_y\frac{d^2\Theta_{yr}(z)}{dz^2}\rho I_y\Theta_{ys}(z)}{G_tA_{yt}}dz-\omega_r^2\omega_s^2\int_0^L \frac{(\rho I_y)^2\Theta_{yr}(z)\Theta_{ys}(z)}{G_tA_{yt}}dz \quad (35)$$

When the suffices r and s in this last equation are interchanged it is found that:

$$\omega_r^2\int_0^L \rho I_y\Theta_{yr}(z)\frac{dV_s(z)}{dz}dz = \omega_r^2\int_0^L \rho I_y\Theta_{ys}(z)\Theta_{yr}(z)dz-\omega_r^2\int_0^L \frac{EI_y\frac{d^2\Theta_{ys}(z)}{dz^2}\rho I_y\Theta_{yr}(z)}{G_tA_{yt}}dz-\omega_r^2\omega_s^2\int_0^L \frac{(\rho I_y)^2\Theta_{ys}(z)\Theta_{yr}(z)}{G_tA_{yt}}dz \quad (36)$$

Subtraction now reveals that:

$$\omega_s^2\int_0^L \rho I_y\Theta_{ys}(z)\frac{dV_r(z)}{dz}dz-\omega_r^2\int_0^L \rho I_y\Theta_{yr}(z)\frac{dV_s(z)}{dz}dz = (\omega_s^2-\omega_r^2)\int_0^L \rho I_y\Theta_{yr}(z)\Theta_{ys}(z)dz-\omega_s^2\int_0^L \frac{EI_y\frac{d^2\Theta_{yr}(z)}{dz^2}\rho I_y\Theta_{ys}(z)}{G_tA_{yt}}dz+\omega_r^2\int_0^L \frac{EI_y\frac{d^2\Theta_{ys}(z)}{dz^2}\rho I_y\Theta_{yr}(z)}{G_tA_{yt}}dz \quad (37)$$

If Eq. 30 and 34 are now added together, they give:

$$(\omega_s^2-\omega_r^2)\int_0^L \rho I_x\Theta_{xr}(z)\Theta_{xs}(z)dz = \int_0^L EI_x\left(\frac{d\Theta_{xr}(z)}{dz}\frac{d^2U_s(z)}{dz^2}-\frac{d\Theta_{xs}(z)}{dz}\frac{d^2U_r(z)}{dz^2}\right)dz+\omega_s^2\int_0^L \rho I_x\Theta_{xs}(z)\frac{dU_r(z)}{dz}d_z-\omega_r^2\int_0^L \rho I_x\Theta_{xr}(z)\frac{dU_s(z)}{dz}dz \quad (38)$$

Also, if Eq. 31 and 37 are now added together they give:

$$(\omega_s^2-\omega_r^2)\int_0^L \rho I_y\Theta_{yr}(z)\Theta_{ys}(z)dz = \int_0^L EI_y\left(\frac{d\Theta_{yr}(z)}{dz}\frac{d^2V_s(z)}{dz^2}-\frac{d\Theta_{ys}(z)}{dz}\frac{d^2V_r(z)}{dz^2}\right)dz+\omega_s^2\int_0^L \rho I_y\Theta_{ys}(z)\frac{dV_r(z)}{dz}dz-\omega_r^2\int_0^L \rho I_y\Theta_{yr}(z)\frac{dV_s(z)}{dz}dz \quad (39)$$

Comparison of Eq. 38 with that of Eq. 24 reveals that:

$$(\omega_s^2-\omega_r^2)\int_0^L \rho I_x\Theta_{xr}(z)\Theta_{xs}(z)dz = (\omega_r^2-\omega_s^2)\int_0^L mU_r(z)U_s(z)dz-\omega_r^2\int_0^L my_c\Phi_r(z)U_s(z)dz+\omega_s^2\int_0^L my_c\Phi_s(z)U_r(z)dz \quad (40)$$

Also, comparison of Eq. 39 with 25, found that:

$$(\omega_s^2-\omega_r^2)\int_0^L \rho I_y\Theta_{yr}(z)\Theta_{ys}(z)dz = (\omega_r^2-\omega_s^2)\int_0^L mV_r(z)V_s(z)dz+\omega_r^2\int_0^L mx_c\Phi_r(z)V_s(z)dz-\omega_s^2\int_0^L mx_c\Phi_s(z)V_r(z)dz \quad (41)$$

In this step, the Eq. 16 is written for the rth and sth modes, multiply these throughout $\Phi_s(z)$ and $\Phi_r(z)$ and integrate with respect to z. Finally it gives:

$$(\omega_r^2-\omega_s^2)\int_0^L m r_m^2\Phi_r(z)\Phi_s(z)dz+mx_c\left[\omega_r^2\int_0^L V_r(z)\Phi_s(z)dz-\omega_s^2\int_0^L V_s(z)\Phi_r(z)dz\right]-my_c\left[\omega_r^2\int_0^L U_r(z)\Phi_s(z)dz-\omega_s^2\int_0^L U_s(z)\Phi_r(z)dz\right]=0 \quad (42)$$

Equation 40-42 may be added together to give:

$$(\omega_r^2-\omega_s^2)\int_0^L\Big[\big(m r_m^2\Phi_r(z)\Phi_s(z)+mV_r(z)V_s(z)+ mU_r(z)U_s(z)+\rho I_x\Theta_{xr}(z)\Theta_{xs}(z)+\rho I_y\Theta_{yr}(z)\Theta_{ys}(z)\big)+ mx_c\big(V_s(z)\Phi_r(z)+V_r(z)\Phi_s(z)\big)-my_c\big(U_s(z)\Phi_r(z)+ U_r(z)\Phi_s(z)\big)\Big]dz=0 \quad (43)$$

We, therefore, reach orthogonality condition for different mode shapes of the thin-walled Timoshenko beams with asymmetric cross-section as follows:

$$(\omega_r^2-\omega_s^2)\int_0^L\Big[(mr_m^2\Phi_r\Phi_s+mV_rV_s+mU_rU_s)+\rho I_x\Theta_{xr}\Theta_{xs}+\rho I_y\Theta_{yr}\Theta_{ys}+ mx_c(V_s\Phi_r+V_r\Phi_s)-my_c(U_s\Phi_r+U_r\Phi_s)\Big]dz=\mu_r\delta_{sr} \quad (44)$$

Where μ_r the generalized mass in the rth mode is δ_{sr} is the Kronecker delta function.

With the free vibration natural frequencies, mode shapes and orthogonality condition described above, now we can calculate dynamic response of the Timoshenko thin-walled beam under deterministic loads.

Dynamic response analytical calculation: Now, the partial differential Eq. 1 are taken into consideration that are required to be solved for the applied external forces of $f_x(z, t)$, $f_y(z, t)$, $m_x(z, t)$ and $g(z, t)$. Assuming that the eigenvalue problem is solved for extracting natural frequencies and modes, the response against the applied loads is obtained from linear combination of the modes as follows:

$$u(z,t)=\sum_{r=1}^{\infty} q_r(t)\,U_r(z) \quad (45)$$

$$v(z,t)=\sum_{r=1}^{\infty} q_r(t)\,V_r(z) \quad (46)$$

$$\theta_x(z,t)=\sum_{r=1}^{\infty} q_r(t)\,\Theta_{xr}(z) \quad (47)$$

$$\theta_y(z,t)=\sum_{r=1}^{\infty} q_r(t)\Theta_{yr}(z) \quad (48)$$

$$\varphi(z,t)=\sum_{r=1}^{\infty} q_r(t)\Phi_r(z) \quad (49)$$

In the above equations, $q_r(t)$ is the modal coordinate (time coordinate) of the rth mode. As a result, the responses u (z, t), v (z, t), θ_x(z, t), ϕ (z, t) and are defined as the total participation effects of each mode. The rth term in the series of Eq. 31 represents the participation rate of the rth mode.

Substituting Eq. 45 into Eq. 1 gives the equation as the following:

$$\sum_{r=1}^{\infty}\left[\begin{array}{l} mU_r\ddot{q}_r(t)-my_c\Phi_r\ddot{q}_r(t)-G_tA_{xt}\dfrac{d^2U_r}{dz^2}q_r(t)+\\ G_tA_{xt}\dfrac{d\Theta_{xr}}{dz}q_r(t)\end{array}\right]=f_x(\xi,t) \quad (50)$$

$$\sum_{r=1}^{\infty}\left[\begin{array}{l} mV_r\ddot{q}_r(t)+mx_c\Phi_r\ddot{q}_r(t)-G_tA_{yt}\dfrac{d^2V_r}{dz^2}q_r(t)+\\ G_tA_{yt}\dfrac{d\Theta_{yr}}{dz}q_r(t)\end{array}\right]=f_y(\xi,t) \quad (51)$$

$$\sum_{r=1}^{\infty}\left[\begin{array}{l} \rho I_x\Theta_{xr}\ddot{q}_r(t)-EI_x\dfrac{d^2\Theta_{xr}}{dz^2}q_r(t)-G_tA_{xt}\dfrac{dU_r}{dz}q_r(t)+\\ G_tA_{xt}\Theta_{xr}q_r(t)\end{array}\right]=m_x(\xi,t) \quad (52)$$

$$\sum_{r=1}^{\infty}\left[\begin{array}{l} \rho I_y\Theta_{yr}\ddot{q}_r(t)-EI_y\dfrac{d^2\Theta_{yr}}{dz^2}q_r(t)-G_tA_{yt}\dfrac{dV_r}{dz}q_r(t)+\\ G_tA_{yt}\Theta_{yr}q_r(t)\end{array}\right]=m_y(\xi,t) \quad (53)$$

$$\sum_{r=1}^{\infty}\left[\begin{array}{l} -my_cU_r\ddot{q}_r(t)+mx_cV_n\ddot{q}_r(t)+mr_m^2\Phi_r\ddot{q}_r(t)-\\ G_tJ_t\dfrac{d^2\Phi_r}{dz^2}q_r(t)+EI_w\dfrac{d^4\Phi_r}{dz^4}q_r(t)\end{array}\right]=g(\xi,t) \quad (54)$$

Substituting Eq. 12 into Eq. 50 gives:

$$\sum_{r=1}^{\infty}\left[m(U_r-y_c\Phi_r)\ddot{q}_r(t)+m\omega_r^2(U_r-y_c\Phi_r)q_r(t)\right]=f_x(\xi,t) \quad (55)$$

$$\sum_{r=1}^{\infty}\left[m(V_r+x_c\Phi_r)\ddot{q}_r(t)+m\omega_r^2(V_r+x_c\Phi_r)q_r(t)\right]=f_y(\xi,t) \quad (56)$$

$$\sum_{r=1}^{\infty}\left[\rho I_x\Theta_{xr}\ddot{q}_r(t)+\rho I_x\omega_r^2\Theta_{xr}q_r(t)\right]=m_x(\xi,t) \quad (57)$$

$$\sum_{r=1}^{\infty}\left[\rho I_y\Theta_{yr}\ddot{q}_r(t)+\rho I_y\omega_r^2\Theta_{yr}q_r(t)\right]=m_y(\xi,t) \quad (58)$$

$$\sum_{r=1}^{\infty}\left[\begin{array}{l} m\omega_r^2(x_cV_r-y_cU_r+r_m^2\Phi_r)q_r(t)+\\ m(x_cV_r-y_cU_r+r_m^2\Phi_r)\ddot{q}_r(t)\end{array}\right]=g(\xi,t) \quad (59)$$

In this step, each sentence of Eq. 55 in U_s, each sentence of Eq. 56 in V_s, each sentence of Eq. 57 in Θ_{xs}, each sentence of Eq. 58 in Θ_{ys} and each sentence of Eq. 59 is multiplied in Φ_s and integrate throughout the beam. After adding the obtained equations, gives the equation as following:

$$\begin{aligned}
&\sum_{r=1}^{\infty} q_r(t)\int_0^L m\omega_r^2 \begin{bmatrix} U_rU_s+V_rV_s+r_m^2\Phi_r\Phi_s-y_c(\Phi_rU_s+ \\ U_r\Phi_s)+x_c(\Phi_rV_s+V_r\Phi_s) \end{bmatrix} dz+ \\
&\sum_{r=1}^{\infty} \ddot{q}_r(t)\int_0^L m \begin{bmatrix} U_rU_s+V_rV_s+r_m^2\Phi_r\Phi_s-y_c(\Phi_rU_s+ \\ U_r\Phi_s)+x_c(\Phi_rV_s+V_r\Phi_s) \end{bmatrix} dz+ \\
&\sum_{r=1}^{\infty} q_r(t)\int_0^L \omega_r^2(\rho I_x\Theta_{xr}\Theta_{xs}+\rho I_y\Theta_{yr}\Theta_{ys})dz+ \\
&\sum_{r=1}^{\infty} \ddot{q}_r(t)\int_0^L (\rho I_x\Theta_{xr}\Theta_{xs}+\rho I_y\Theta_{yr}\Theta_{ys})dz= \\
&\int_0^L \begin{bmatrix} f_x(\xi,t)U_s+f_y(\xi,t)V_s+m_x(\xi,t)\Theta_{xs}+ \\ m_y(\xi,t)\Theta_{ys}+g(\xi,t)\Phi_s \end{bmatrix} dz
\end{aligned} \quad (60)$$

Equation 60 can be expressed in the following matrix form:

$$(\ddot{q}_r(t)+\omega_r^2q_r(t)) = \frac{1}{\mu_r}\int_0^L \begin{bmatrix} f_x(z,t)U_r+f_y(z,t)V_r+m_x(z,t)\Theta_{xr}+ \\ m_y(z,t)\Theta_{yr}+g(z,t)\Phi_r \end{bmatrix} dz \quad (61)$$

By introducing the following parameters:

$$\frac{1}{\mu_r}\int_0^L f_x(z,t)U_r(z)dz = F_{xr}(t) \quad (62)$$

$$\frac{1}{\mu_r}\int_0^L f_y(z,t)V_r(z)dz = F_{yr}(t) \quad (63)$$

$$\frac{1}{\mu_r}\int_0^L m_x(z,t)\Theta_{xr}(z)dz = M_{xr}(t) \quad (64)$$

$$\frac{1}{\mu_r}\int_0^L m_y(z,t)\Theta_{yr}(z)dz = M_{yr}(t) \quad (65)$$

$$\frac{1}{\mu_r}\int_0^L g(z,t)\Phi_r dz = G_r(t) \quad (66)$$

Now, Eq. 61 is rewritten as follows:

$$\ddot{q}_r(t)+\omega_r^2q_r(t) = F_{xr}(t)+F_{yr}(t)+M_{xr}(t)+M_{yr}(t)+G_r(t) \quad (67)$$

Therefore, there are an infinite number of equations similar to Eq. 67 and one equation for each mode. The partial differential Eq. 1 for unknown functions u(z, t), v(z, t), θ_x(z, t) and ϕ(z, t) are transferred into an infinite set of ordinary differential Eq. 67 in terms of q_r (t) unknowns.

For the applied dynamic loads of f_x (z, t), f_y (z, t), m_x (z, t), m_y (z, t) and g (z, t) the unknown system functions u(z, t), v(z, t), θ_x(z, t), θ_y(z, t) and ϕ(z, t) could be determined through solving the modal equations in terms of q_r (t). The equation of each mode is independent to the other modes; thus, it could be solved separately. For solving Eq. 67 which is as the form of movement equation for Single Degree of Freedom (SDOF) system, it is used from Duhamel's integral. Therefore, the answer of Eq. 67 is defined as:

$$\begin{aligned}
&q_r(t) = (A_r\cos\omega_rt+B_r\sin\omega_rt)+\frac{1}{\omega_r}\int_0^t (G_r(\tau)+M_{yr}(\tau)+ \\
&M_{xr}(\tau)+F_{yr}(\tau)+F_{xr}(\tau))\sin\omega_r(t-\tau)d\tau
\end{aligned} \quad (68)$$

After determining q_r (t) by using Eq. 31 and 68 system response to arbitrary dynamic forces f_x(z, t), f_y(z, t), m_x(z, t) m_y(z, t) and g(z, t) is as follows:

$$\begin{aligned}
&u(z,t) = \sum_{r=1}^{\infty} U_r(z)\Big((A_r\cos\omega_rt+B_n\sin\omega_rt)+\frac{1}{\omega_r}\int_0^t (G_r(\tau)+ \\
&M_{yr}(\tau)+M_{xr}(\tau)+F_{yr}(\tau)+F_{xr}(\tau))\times\sin\omega_r(t-\tau)d\tau\Big)
\end{aligned} \quad (69)$$

$$\begin{aligned}
&v(z,t) = \sum_{r=1}^{\infty} V_r(z)\Big((A_r\cos\omega_rt+B_r\sin\omega_rt)+\frac{1}{\omega_r}\int_0^t (G_r(\tau)+ \\
&M_{yr}(\tau)+M_{xr}(\tau)+F_{yr}(\tau)+F_{xr}(\tau))\times\sin\omega_r(t-\tau)d\tau\Big)
\end{aligned} \quad (70)$$

$$\begin{aligned}
&\theta_x(z,t) = \sum_{r=1}^{\infty} \Theta_{xr}(z)\Big((A_r\cos\omega_rt+B_r\sin\omega_rt)+\frac{1}{\omega_r}\int_0^t (G_r(\tau)+ \\
&M_{yr}(\tau)+M_{xr}(\tau)+F_{yr}(\tau)+F_{xr}(\tau))\times\sin\omega_r(t-\tau)d\tau\Big)
\end{aligned} \quad (71)$$

$$\begin{aligned}
&\theta_y(z,t) = \sum_{r=1}^{\infty} \Theta_{yr}(z)\Big((A_r\cos\omega_rt+B_r\sin\omega_rt)+\frac{1}{\omega_r}\int_0^t (G_r(\tau)+ \\
&M_{yr}(\tau)+M_{xr}(\tau)+F_{yr}(\tau)+F_{xr}(\tau))\times\sin\omega_r(t-\tau)d\tau\Big)
\end{aligned} \quad (72)$$

$$\begin{aligned}
&\phi(z,t) = \sum_{r=1}^{\infty} \Phi_r(z)\Big((A_r\cos\omega_rt+B_r\sin\omega_rt)+\frac{1}{\omega_r}\int_0^t (G_r(\tau)+ \\
&M_{yr}(\tau)+M_{xr}(\tau)+F_{yr}(\tau)+F_{xr}(\tau))\times\sin\omega_r(t-\tau)d\tau\Big)
\end{aligned} \quad (73)$$

The obtained responses are useable for any arbitrary deterministic loading. In the following, the response against deterministic harmonic load is calculated using Eq. 69 for instance.

In this part, it is assumed that the centralized harmonic forces having F_{xi} amplitudes are applied in the direction of x axis in the points $z_i = a_i$, the forces with F_{yi} amplitudes are applied in the direction of y axis in the points z_i, bending moments having M_{xi} amplitudes are applied in x-z plane around y axis and in the points $z_i = c_i$, bending moments with M_{yi} amplitudes are applied in y-z plane around x axis and in the points $z_i = d_i$ and torsional moments with G_i amplitudes are applied around z axis and in the points $z_i = e_i$ where i = 1, 2, 3, ..., N. The mentioned applied loads are defined as follows:

$$f_{xi}(z,t) = F_{xi}\,\delta(z-a_i)\sin\omega_i t \quad (74)$$

$$f_{yi}(z,t) = F_{yi}\,\delta(z-b_i)\sin\omega_i t \quad (75)$$

$$m_{xi}(z,t) = M_{xi}\,\delta(z-c_i)\sin\omega_i t \quad (76)$$

$$m_{yi}(z,t) = M_{yi}\,\delta(z-d_i)\sin\omega_i t \quad (77)$$

$$g_i(z,t) = G_i\,\delta(z-e_i)\sin\omega_i t \quad (78)$$

where is the rotational frequency of the applied loads. In this state by using Eq. 60, the generalized loads functions become as follows:

$$F_{xir}(z,t) = \frac{1}{\mu_r}F_{xi}\,U_r(a_i)\sin\omega_i t \quad (79)$$

$$F_{yir}(z,t) = \frac{1}{\mu_r}F_{yi}\,V_r(b_i)\sin\omega_i t \quad (80)$$

$$M_{xir}(z,t) = \frac{1}{\mu_r}M_{xi}\,\Theta_{xr}(c_i)\sin\omega_i t \quad (81)$$

$$M_{yir}(z,t) = \frac{1}{\mu_r}M_{yi}\,\Theta_{yr}(d_i)\sin\omega_i t \quad (82)$$

$$G_{ir}(z,t) = \frac{1}{\mu_r}G_i\,\Phi_r(e_i)\sin\omega_i t \quad (83)$$

Finally, the response of u(z, t), v(z, t), θ_x(z, t), θ_y(z, t) and ϕ(z, t) to assumed applying loads is as follows:

$$u(z,t) = \sum_{r=1}^{\infty} U_r(z)(A_r\cos\omega_r t + B_r\sin\omega_r t + \sum_{r=1}^{\infty} U_r(z)\sum_{i=1}^{N}\frac{1}{\mu_r(\omega_r^2-\omega_i^2)}(F_{xi}U_r(a_i)+F_{yi}V_r(b_i)+ M_{xi}\Theta_{xr}(c_i)+M_{yi}\Theta_{yr}(d_i)+G_i\Phi_r(e_i))\sin\omega_i t \quad (84)$$

$$v(z,t) = \sum_{r=1}^{\infty} V_r(z)(A_r\cos\omega_r t + B_r\sin\omega_r t + \sum_{r=1}^{\infty} V_r(z) \sum_{i=1}^{N}\frac{1}{\mu_r(\omega_r^2-\omega_i^2)}(F_{xi}U_r(a_i)+F_{yi}V_r(b_i)+M_{xi}\Theta_{xr}(c_i)+ M_{yi}\Theta_{yr}(d_i)+G_i\Phi_r(e_i))\sin\omega_i t \quad (85)$$

$$\theta_x(z,t) = \sum_{r=1}^{\infty} \Theta_{xr}(z)(A_r\cos\omega_r t + B_r\sin\omega_r t + \sum_{r=1}^{\infty} \Theta_{xr}(z)\sum_{i=1}^{N}\frac{1}{\mu_r(\omega_r^2-\omega_i^2)}(F_{xi}U_r(a_i)+F_{yi}V_r(b_i)+ M_{xi}\Theta_{xr}(c_i)+M_{yi}\Theta_{yr}(d_i)+G_i\Phi_r(e_i))\sin\omega_i t \quad (86)$$

$$\theta_y(z,t) = \sum_{r=1}^{\infty} \Theta_{yr}(z)(A_r\cos\omega_r t + B_r\sin\omega_r t + \sum_{r=1}^{\infty} \Theta_{yr}(z)\sum_{i=1}^{N}\frac{1}{\mu_r(\omega_r^2-\omega_i^2)}(F_{xi}U_r(a_i)+F_{yi}V_r(b_i)+ M_{xi}\Theta_{xr}(c_i)+M_{yi}\Theta_{yr}(d_i)+G_i\Phi_r(e_i))\sin\omega_i t \quad (87)$$

$$\phi(z,t) = \sum_{r=1}^{\infty} \Phi_r(z)(A_r\cos\omega_r t + B_r\sin\omega_r t + \sum_{r=1}^{\infty} \Phi_r(z)\sum_{i=1}^{N}\frac{1}{\mu_r(\omega_r^2-\omega_i^2)}(F_{xi}U_r(a_i)+F_{yi}V_r(b_i)+ M_{xi}\Theta_{xr}(c_i)+M_{yi}\Theta_{yr}(d_i)+G_i\Phi_r(e_i))\sin\omega_i t \quad (88)$$

Numerical results: The following example is presented in order to validate the formulation proposed in the present paper.

In this example, a thin-walled cantilever beam having semicircular section with one axis of symmetry is investigated. The physical and geometrical specifications of the studied section are as follows (Fig. 3):

- $E = 68.9\times10^9 Nm^{-2}$
- $I_y = 9.26\times10^{-8}\ m^4$
- $I_\omega = 1.52\times10^{-12} m^6$
- $x_c = 0.0155 m$
- $y_c = 0.0\ m$
- $r^2_m = 5.998\times10^{-4} m^2$
- $m = 0.835\ kg\times m^{-1}$
- L = 0.82 m
- $G = 26.5\times10^9 Nm^{-2}$
- $J = 1.64\times10^{-9} m^4$

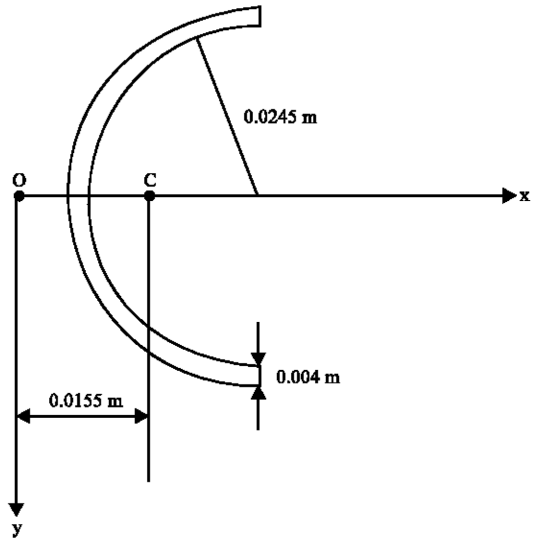

Fig. 3: Beam cross-section used in numerical example

Table 1: The first five natural frequencies

Frequency order	ω(rad/sec)	f(HZ)
1	399.0379	63.5089
2	863.4747	137.4263
3	1733.2600	275.8580
4	3023.2400	481.1625
5	4020.2500	639.8936

In this example, it is assumed that unit harmonic forces with unit amplitude are applied to the tip of the cantilever beam and the transitional bending displacement, rotational and torsional angles are calculated at the cantilever beam tip under the applied harmonic load with different frequencies. For calculating the response, it is used from the first five modes of vibration. Therefore, the first five bending-torsion coupled natural frequencies and vibrating modes are first calculated by the help of the dynamic stiffness matrix. The calculated natural frequencies are presented in Table 1.

Then, for each mode are calculated. For the first five modes, the is as follows:

$$\mu_1 = 4.5944,\ \mu_2 = 4.5453,\ \mu_3 = 2.5601,\ \mu_4 = 3.6127,\ \mu_5 = 4.7357$$

By using Eq. 84 and considering $F_x = 0$, $M_x = 0$, $M_y = 0$, $G = 0$ and $F_y = 1$ as well as the location of the point, b = 0.82 the intended responses were calculated at the cantilever edge point and plotted in the semi-logarithmic diagrams of Fig. 4-6. In the mentioned figures, the absolute magnitudes of the obtained values are considered on the vertical axis which is a logarithmic axis.

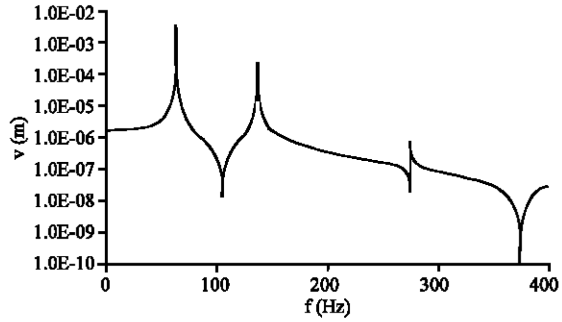

Fig. 4: Dynamic transitional bending displacement of thin-walled beam at its tip for different frequencies of the applied load

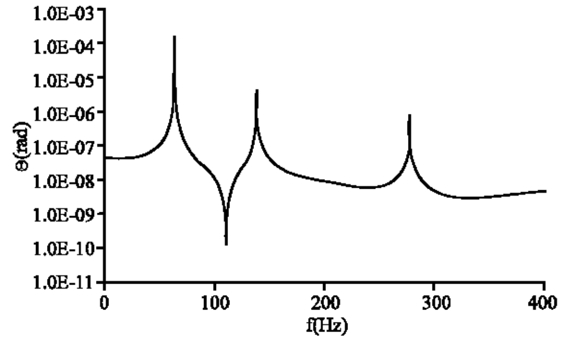

Fig. 5: Dynamic rotational bending displacement of thin-walled beam at its tip for different frequencies of the applied load

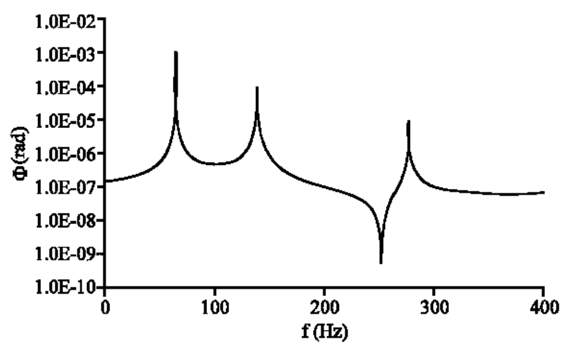

Fig. 6: Dynamic torsional displacement of thin-walled beam at its tip for different frequencies of the applied load

CONCLUSION

In the present study, an analytical method is proposed for determining the dynamic response of asymmetric thin-walled beams subjected to different types of centralized and distributed definitive dynamic loads. In order to solve vibration problems, it has been used from accurate dynamic stiffness matrix. Since the dynamic stiffness matrix are derived from analytical solution of the differential equations of movement, it makes it possible to calculate natural frequencies and vibrating modes, accurately and without any lost in the precision. The natural frequencies and mode shapes are obta ined by using Wittrick-Williams algorithm.

Due to considering the general shape of the beam (i.e., a perfect asymmetric section), the mass and shear centers are not coincident and thus, flexural and torsional vibrations are dependent to each other. Accordingly, determining the analytical response of the 3D thin-walled beam with asymmetric section against dynamic deterministic loads is considered as a very complicated problem. Using the introduced dynamic stiffness matrix in combination with modal analysis method, it could be overcome this complexity. By using the formulation presented in this study, it could be derived the dynamic response of members with arbitrary asymmetric section and also with different boundary conditions under any arbitrary applied definitive dynamic loading at various points.

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