Journal of Engineering and Applied Sciences 12 (Special Issue 8): 8417-8420, 2017
ISSN: 1816-949X


# A Generalization of a Quasi-Homogeneous Polynomial Gradient System

Ana-Magnolia Marin-Ramirez, Ruben-Dario Ortiz-Ortiz and
Cristian-Fabian Baron-Pertuz
Faculty of Exact and Natural Sciences, University of Cartagena,
Cartagena de Indias, Bolivar, Colombia

**Abstract:** In this study, we find Dulac functions for a quasi-homogeneous polynomial gradient system and several systems. Using quasi-homogeneous polynomial system theory, we prove non-existence of periodic orbits at the first quadrant on the plane for a gradient system.

**Key words:** Poincare Bendixson theorem, gradient system, quasi-homogeneous polynomial, Dulac function, periodic orbits, polynomial

## INTRODUCTION

The characteristics of stationary and non-stationary gradient systems with strong Lyapunov functions were studied by Chen and Mei (2016). A gradient system and a skew-gradient system can be merged into a combined gradient system (Xiang-Wei *et al*., 2016). By Llibre and Salhi (2013) was characterized non-existence of periodic orbits for 2-dimensional Kolmogorov systems. By Wang *et al*. (2013), they used Dulac criterion to prove the non-existence of periodic orbits for predator-prey systems. Dulac functions for the Kolmogorov system and predator-prey system were found (Gasull and Giacomini, 2013). By Marin-Ramirez *et al*. (2014a) was studied a geometric method applied for proving the non-existence of periodic orbits in a dynamical system, Marin *et al*. (2013a, b) found a generalization of a gradient system, (Marin *et al*., 2014, 2013b) applied Dulac's criterium to a general quadratic system for finding a Dulac function Marin *et al*. (2014), studied a partial differential sinh-cosh-gordon equation and found solutions to this problem, Baron-Pertuz *et al*. (2014) solved several systems for finding a Dulac function, Marin-Ramirez *et al*. (2014b) studied the Duffing equation and found a generalization of this, Osuna *et al*. (2013), studied the existence of Dulac functions for planar differential systems. By Boukoucha and Bendjeddou (2015), they characterized the non-existence of limit cycles of cubic Kolmogorov systems. In this study, we look for Dulac functions for gradient systems and prove that there are not periodic orbits for generalized gradient systems with certain conditions.

## MATERIALS AND METHODS

We consider:

$$\begin{cases} x' = f_1(x, y) \\ y' = f_2(x, y) \end{cases} \quad (1)$$

where, $f_1, f_2 \in C^1(R^2)$.

**Definition 2.1:** We can take the following quasi-differential equation:

$$f_1 \frac{\partial v}{\partial x} + f_2 \frac{\partial v}{\partial y} = V\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right) \quad (2)$$

where, V is an inverse integrating factor of the system (Eq. 1) (Laura *et al*., 2011).

**Definition 2.2:** Let p, q, k, $l \in \aleph Z^+$. A real function f: $\mathbb{R}^2 \rightarrow R$ is called a p-q-quasi-homogeneous function of weighted degree k if $f(\alpha^k x_1, \alpha^q x_2) = \alpha^k f(x_1, x_2)$ for all $\alpha \in R/\{0\}$. A vector field:

$$F = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2}$$

is called a p-q quasi-homogeneous vector field of weighted degree 1 if $f_1$ and $f_2$ are p-q-quasi-homogeneous functions of weight degree p+1-1 and q+1-1, respectively. A p-q-quasi-homogeneous differential system of weighted degree 1 is determined by a p-q-quasi-homogeneous vector field (Laura *et al*., 2011).

**Corresponding Author:** Ana-Magnolia Marin-Ramirez, Faculty of Exact and Natural Sciences, University of Cartagena, Cartagena de Indias, Bolivar, Colombia

**Theorem 2.3:** Given a p-q-quasi-homogeneous vector field:

$$F = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2}$$

then $V = qx_2f_1 - px_1f_2$ is an inverse integrating factor of the system (Laura *et al*., 2011).

**Theorem 2.4:** If a non-zero p-q-quasi-homogeneous polynomial of weighted k is an inverse integrating factor of the system (Eq. 1), then it has no limit cycles (Laura *et al*., 2011). We have the following gradient system (Gasull and Giacomini, 2013):

$$\begin{cases} x' = -x \\ y' = -y \end{cases} \quad (3)$$

with x, y>0. This is a gradient system there exist a function $W(x, y) = (x^2+y^2)/2$ such that $-W_x = -x$ and $-W_y = -y$.

## RESULTS AND DISCUSSION

**Theorem 3.1:** The gradient system (Eq. 3) has a function defined by V(x, y) = xy and does not have periodic orbits.

**Proof:** $V(\alpha^p x, \alpha^q y) = \alpha^{(p+q)} V(x, y)$ then V(x, y) = xy is a p-q-quasi-homogeneous of degree p+q. From the gradient system (Eq. 3) and the differential equation (Eq. 2):

$$-x\frac{\partial v}{\partial x} + -y\frac{\partial v}{\partial y} = (-1-1)V = -2V \quad (4)$$

We can see that with V = xy it holds the equality. In fact:

$$-2xy = -x\frac{\partial v}{\partial x} + -y\frac{\partial v}{\partial y} = (-1-1)V = -2V = -2xy \quad (5)$$

Then, V is an inverse integral factor of the system, hence, the gradient system has no limit cycles.

**Theorem 3.2:** The system:

$$\begin{cases} x' = c_1(y)x \\ y' = c_2(x)y \end{cases} \quad (6)$$

**Proof:** As $f_2 = -y$, we can suppose that $\partial f_2/\partial x_2 = -1$. Then:

$$f_1\frac{\partial v}{\partial x} - x_2\frac{\partial v}{\partial x} = V\left(\frac{\partial f_1}{\partial x} - 1\right) \quad (7)$$

As:

$$V = xy, \frac{\partial v}{\partial x} = y \text{ and } \frac{\partial v}{\partial x} = x$$

$$f_1 - x = x\left(\frac{\partial f_1}{\partial x} - 1\right) \quad (8)$$

$$x\frac{\partial f_1}{\partial x} - f_1 = 0 \quad (9)$$

Then, $f_1 = c_1(y)x$. As $f_1 = -x$, we can suppose that $\partial f_1/\partial x = -1$. Then:

$$-x\frac{\partial v}{\partial x} + f_2\frac{\partial v}{\partial y} = V\left(\frac{\partial f_2}{\partial y} - 1\right) \quad (10)$$

As:

$$V = xy, \frac{\partial v}{\partial x} = y \text{ and } \frac{\partial v}{\partial x} = x$$

$$f_2 - y = y\left(\frac{\partial f_2}{\partial y} - 1\right) \quad (11)$$

$$y\frac{\partial f_2}{\partial y} - f_2 = 0 \quad (12)$$

Then, $f_2 = c_2(x)y$.

**Theorem 3.3:** The system Eq. 3 has a Dulac function defined by h = 1/y and does not have periodic orbits.

**Proof:** From the system (Eq. 3) and the differential equation (Eq. 2):

$$x(-1)\frac{\partial h}{\partial x} + y(-1)\frac{\partial h}{\partial y} = h\left[c - \left[-1 + (-1)\right]\right] \quad (13)$$

Taking:

$$c = (-1)$$

then c<0. If we substitute $\partial h/\partial x = 0$ and Eq. 14 into Eq. 13, we obtain:

$$y(-1)\frac{\partial h}{\partial y} = -h(-1) \quad (14)$$

Hence, $\partial h/\partial y = -h/y$ with solution:

$$h = 1/y > 0, \text{ if } y > 0 \quad (15)$$

We have that hc<0 and by the Poincare-Bendixson theorem this system does not have periodic orbits.

**Theorem 3.4:** This is a generalization of the system (Eq. 3):

$$\begin{cases} x' = x(-1) + c_1(y) \\ y' = y(c_2(x)) \end{cases}$$

with $c_1(y)$, $c_2(x)\in C^1(\mathbb{R})$ has a Dulac function defined by $h = 1/y$ and does not have periodic orbits.

**Proof:** If $\partial f_1/\partial x = (x(-1))'$, then $f_1 = x(-1)+c(y)$. Substituting $\partial h/\partial x = 0$ and $\partial f_1/\partial x = \partial/\partial x\,(x(-1))$ into Eq. 2, we get:

$$f_2\frac{\partial h}{\partial y} = h\left(c-\left(\frac{\partial}{\partial x}\big(x(-1)\big)+\frac{\partial f_2}{\partial y}\right)\right)$$

From Eq. 14 and 15, we obtain an ordinary differential equation in $f_2$:

$$\frac{\partial f_2}{\partial y}-\frac{f_2}{y}=0$$

with the following solution:

$$f_2 = y\big(c_2(x)\big)$$

We have that hc>0 and by the Poincare-Bendixson this system does not have periodic orbits.

**Theorem 3.5:** The system (Eq. 3) has a Dulac function h defined by $h = 1/xy$ and does not have periodic orbits.

**Proof:** We can obtain another Dulac function and using Eq. 13:

$$x(-1)\frac{\partial h}{\partial x}+y(-1)\frac{\partial h}{\partial y} = -h(-1+-1)$$

Taking:

$$\frac{\partial h}{\partial x}=\frac{\partial h}{\partial z}\frac{\partial z}{\partial x},\ \frac{\partial h}{\partial y}=\frac{\partial h}{\partial z}\frac{\partial z}{\partial y}$$

$$\left[x\frac{\partial z}{\partial x}(-1)+y\frac{\partial z}{\partial y}(-1)\right]\frac{\partial h}{\partial z} = -h\big[(-1)+(-1)\big]$$

If $x\,\partial z/\partial x = -1$, $y\,\partial z/\partial y = -1$, then $z = -\ln xy$. If $h = \partial h/\partial z$, then $h = e^z$. Hence:

$$h=\frac{1}{xy}$$

Hence, hc>0 in consequence by the Poincare-Bendixson theorem this system does not have periodic orbits.

**Theorem 3.6:** The following generalization of the system (Eq. 3):

$$\begin{cases} x' = x\big(g_0(x)+g_1(x)y\big)\\ y' = y\big(h_0(x)+h_1(x)y\big)\end{cases}$$

has a Dulac function defined by $h = y^{-1}$ and does not have periodic orbits.

**Proof:** If $\partial h/\partial x = 0$ and using Eq. 2, we have:

$$y\big(h_0(x)+h_1(x)y\big)\frac{\partial h}{\partial y} =$$
$$h\Big(c-\big(g_0+g_1y+x\big(g_0+g_1{}'y+(h_0+h_1y)+y(h_1)\big)\big)\Big)$$

Where:

$$c = g_0(x)+xg_{0'}(x)+\big(g_1(x)+xg_{1'}(x)+h_1(x)\big)y$$

With:

$$g_1(x)+xg_{1'}(x)+h_1(x),\ g_0(x)+xg_{0'}(x)<0$$

Then, c<0. Hence:

$$y\big(h_0(x)+h_1(x)y\big)\frac{\partial h}{\partial y} = h\big(-(h_0(x)+h_1(x)y)\big)$$

Then, $y\,\partial h/\partial y = -h$ and $h = y^{-1}$ with y>0. Hence, hc<0 and by the Poincare-Bendixson theorem this system does not have periodic orbits.

**Theorem 3.7:** The generalization system (Eq. 3):

$$\begin{cases} x' = x\big(g_0(x)+g_1(x)y\big)\\ y' = y\big(h_0(x)+h_1(x)y\big)\end{cases}$$

has a Dulac function defined by $h = 1/xy$ and does not have periodic orbits.

**Proof:** Using Eq. 2, we have:

$$x\big(g_0(x)+g_1(x)y\big)\frac{\partial h}{\partial x}+y\big(h_0(x)+h_1(x)y\big)\frac{\partial h}{\partial y} =$$
$$h\Big(c-\big(g_0+g_1y+x(g_0+g_1y)+(h_0+h_1y)+y(h_1)\big)\Big)$$

Where:

$$c = xg_{0'}(x)+\big(xg_{1'}(x)+h_1(x)\big)y$$

With:

$$xg_{1'}(x)+h_1(x),\big(xg_{0'}(x)\big)<0$$

This guaranties that c<0. Taking:

$$\frac{\partial h}{\partial x}=\frac{\partial h}{\partial t}\frac{\partial t}{\partial x},\ \frac{\partial h}{\partial y}=\frac{\partial h}{\partial t}\frac{\partial t}{\partial y}$$

Hence:

$$x(g_0+g_1y)\frac{\partial h}{\partial t}\frac{\partial t}{\partial x}+y(h_0+h_1y)\frac{\partial h}{\partial t}\frac{\partial t}{\partial y}=$$
$$h(-(g_0+g_1y+h_0+h_1y))$$

Then, if $x\partial t/\partial x = y\partial t/\partial y = -1$, $\partial h/\partial t = h$, then $t = -\ln(xy)$ and $h = e^t = (xy)^{-1}$ with $x, y>0$. Also, we have that $hc>0$ and by the Poincare-Bendixson theorem this system does not have periodic orbits.

## CONCLUSION

We generalized a quasi-homogeneous polynomial gradient system. And we obtain several generalizations of this gradient system. It is important in many fields as engineering to generalized dynamical systems. For a future work it is interesting to obtain a new generalized gradient system.

## ACKNOWLEDGEMENT

Researchers express their deep gratitude to Universidad de Cartagena for partial financial support.

## REFERENCES

Baron-Pertuz, C.F., C.D. Indias, A.M. Marin-Ramirez and R.D. Ortiz-Ortiz, 2014. An approximation to the benjamin-bona-mahony equation. Intl. J. Math. Anal., 8: 2757-2761.

Boukoucha, R. and A. Bendjeddou, 2015. On the non-existence of limit cycles for a cubic kolmogorov systems. Intl. J. Pure Appl. Math., 103: 227-233.

Chen, X. and F. Mei, 2016. Constrained mechanical systems and gradient systems with strong Lyapunov functions. Mech. Res. Commun., 76: 91-95.

Gasull, A. and H. Giacomini, 2013. Some Applications of the Extended Bendixson-Dulac Theorem. In: Progress and Challenges in Dynamical Systems, Springer, Berlin, Germany, pp: 233-252.

Laura, G.L., O. Osuna and V.G. Aguilar, 2011. Non-existence of limit cycles via inverse integrating factors. Electron. J. Differ. Equ., 2011: 1-6.

Llibre, J. and T. Salhi, 2013. On the dynamics of a class of Kolmogorov systems. Appl. Math. Comput., 225: 242-245.

Marin, A.M., R.D. Ortiz and J.A. Rodriguez, 2013b. A dulac function for a quadratic system. Theor. Math. Applic., 3: 49-54.

Marin, A.M., R.D. Ortiz and J.A. Rodriguez, 2013a. A generalization of a gradient system. Int. Math. Forum, 8: 803-806.

Marin, A.M., R.D. Ortiz and J.A. Rodriguez, 2014. On the nonexistence of periodic orbits of some quadractic systems. Far East J. Math. Sci., 89: 129-135.

Marin-Ramirez, A.M., C. de Indias, J.A. Rodriguez-Ceballos and R.D. Ortiz-Ortiz, 2014a. Quadratic systems without periodic orbits. Int. J. Math. Anal., 8: 2177-2181.

Marin-Ramirez, A.M., C.D. Indias, J.A. Ortega-Ramos and R.D. Ortiz-Ortiz, 2014b. A result of a duffing equation. Intl. J. Math. Anal., 8: 2173-2176.

Osuna, O., J. Rodriguez-Ceballos, C. Vargas-De-Leon and A.G. Villasenor, 2013. Dulac functions for transformed vector fields. Intl. J. Contemp. Math. Sci., 8: 291-297.

Wang, J., J. Shi and J. Wei, 2013. Nonexistence of periodic orbits for predator-prey system with strong Allee effect in prey populations. Electron. J. Differential Equations, 2013: 1-14.

Xiang-Wei, C., Z. Ye and M. Feng-Xiang, 2016. An application of a combined gradient system to stabilize a mechanical system. Chin. Phys. B , Vol. 25.