Journal of Engineering and Applied Sciences 13 (23): 10059-10067, 2018
ISSN: 1816-949X

Reconstruction of Monotone Surface Using Rational Bi-Cubic Spline Interpolation

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Abstract: This study studies the reconstruction of monotone surface data with continuity by using partially blended rational cubic spline with twelve parameters. In order to preserve the shape preserving property, the sufficient condition for the monotonicity is obtained through mathematical derivation on all four boundary curves on each rectangular patch. Root Mean Square Error (RMSE) and the coefficient of determination R^2 is used to estimate the error of the proposed scheme.

Key words: Continuity, parameters, monotonicity, derivation, coefficient, scheme

INTRODUCTION

Curve and surface reconstruction is important in geometric modeling and scientific visualization. On top of that, the curve and surface must obey the geometric shape of the data. For instance, if the given data is monotone, then the resulting curve and surface also must be monotonic everywhere. Two common methods for data interpolation are cubic spline and cubic Hermite spline. But both schemes suffer from the fact that they are not capable to produce the desired shape preserving properties. There may be exist few unwanted flaws that will destroy the geometric properties of the data sets.

Monotonicity-Preserving (MP) is important in sciences and engineering. For instance, the surface of dose-response in biochemistry and pharmacology are monotone data (Beliakov, 2005). The production of the growth of economy is also always monotone surfaces data (Stewart, 2012). Furthermore, the approximation of copulas and quasi-copulas in statistics shows the behaviour of monotone function (Beliakov, 2005).

Carlson and Fristch (1985) constructed the bi-cubic Hermite spline surface for monotone data. But the main drawback is that, the first partial derivative need to be modified if the interpolating surfaces are not monotone on the interval. Furthermore, by using scheme of Carlson and Fristch (1985), we cannot change the shape of the curve and surface unless we change the interpolation data. Abbas (2012) and Abbas *et al*. (2012) studied the MP for surface data. From the numerical results, their methods work well. Perhaps the only weakness is that their scheme cannot be applied if the first partial derivatives are zero. Casciola and Romani (2003) have proposed the NURBS version of the rational interpolating spline with tension control for rectangular topology case. Costantini (1997) discussed the boundary-valued shape preserving with arbitrary constraints by satisfying the separable or non-separable boundary conditions. Duan *et al*. (2004, 2006) discussed various type of rational cubic splines together with bounded property and shape control for the bivariate spline on interpolating surface. But their schemes require true function values as well as the knots must be equally spaced. Hence, Duan *et al*. (2004, 2006) schemes may fail if the derivatives are given. Hussain and Hussain and Hussain (2007) and Hussain *et al*. (2012a, b) have discussed the monotonicity by using rational bi-cubic spline with eight parameters without any free parameters. Hussain *et al*. (2012a, b) discussed the monotone data visualization by using rational cubic spline (cubic numerator and quadratic denominator) with 8 parameters without any free parameters. This scheme also does not provide any extra degree of freedom to user in controlling the final shape of the interpolating surface. Hussain *et al*. (2014) studies the use of quadratic trigonometric spline with two parameters, Hussain *et al*. (2014) discussed shape-preserving trigonometric surfaces by using the quadratic trigonometric spline of Hussain *et al*. (2015). Ibraheem *et al*. (2012) have proposed the rational bi-cubic trigonometric spline with eight parameters. From graphical results, it can be seen clearly that, the resulting curve and surface does not smooth (on some intervals) as well as not very visually

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pleasing for scientific visualization purpose. Liu *et al.* (2014) studied the positivity and monotonicity preserving interpolation by utilizing the rational quartic Said-Ball with quadratic denominator. Karim and Kong (2014) has proposed rational cubic spline interpolation with three parameters for monotonicity preserving interpolation. Karim (2017) have extended the univariate spline to bivariate cases with 12 parameters where 8 of it are free parameters The object in this paper is to use the rational bi-cubic spline by Karim (2017) for monotone surface reconstruction. Some contributions are:

The proposed scheme has 12 parameters and 8 of it are free parameters meanwhile in Hussain and Hussain (2007) the bivariate spline has 8 parameters and without any free parameters.

The proposed scheme does not require any first partial derivative modification. But the bi-cubic spline of Carlson and Fristch (1985) as well as rational bi-quartic spline of Wang and Tan (2006) require the modification of the first partial derivative.

The proposed scheme is applicable for both equally or unequally space data meanwhile, the schemes by Duan *et al.* (2004, 2006) requires the data are equally spaced.

The scheme can also be used if the first partial derivative is given or not. Meanwhile, the research of Duan *et al.* (2004, 2006) requires the true function value without the first partial derivative value. Thus their method cannot applicable if the first partial derivative is given at the knots.

The proposed scheme is easy to use and not involving any trigonometric functions as appear in the work of Ibraheem *et al.* (2012) and Hussain *et al.* (2014). The proposed scheme gives more visually pleasing compare with (Hussain *et al.*, 2014), respectively.

MATERIALS AND METHODS

Partially blended rational bi-cubic spline interpolation: Abdul Karim *et al.* (2015) has constructed the partially blended rational bi-cubic spline function over each rectangular patch $[x_i, x_{i+1}]\times\lfloor y_j, y_{j+1}\rfloor$, i = 0, 1, ..., n-1; j = 0, 1, ..., m-1 and is defined as follows:

$$S(x,y) = -AFB^T \quad (1)$$

Where:

$$F = \begin{pmatrix} 0 & S(x,y_j) & S(x,y_{j+1}) \\ S(x_i,y) & S(x_i,y_j) & S(x_i,y_{j+1}) \\ S(x_{i+1},y) & S(x_{i+1},y_j) & S(x_i,y_{j+1}) \end{pmatrix}$$

Where:

$$A = \begin{bmatrix} -1 & a_0(\theta) & a_1(\theta) \end{bmatrix},\ B = \begin{bmatrix} -1 & b_0(\phi) & b_1(\phi) \end{bmatrix}$$
$$a_0(\theta) = (1-\theta)^2(1+2\theta),\quad a_1(\theta) = \theta^2(3-2\theta),$$
$$b_0(\phi) = (1-\phi)^2(1+2\phi),\quad b_1(\phi) = \phi^2(3-2\phi)$$

$$\theta = \frac{x-x_i}{h_i},\quad \phi = \frac{y-y_j}{\hat{h}_j}$$

Thus, $\theta\epsilon[0, 1]$ and $\phi\epsilon[0, 1]$ and:

$$h_i = x_{i+1}-x_i,\quad \hat{h}_j = y_{j+1}-y$$

$S(x, y_j)$, $S(x, y_{j+1})$, $S(x_{i+1}, y)$ and $S(x_{i+1}, y)$ are rational cubic function defined on the boundary of rectangular patch $[x_i, x_{i+1}]\times\lfloor yj, yj+1\rfloor$, I = 0, 1, ..., n-1; j = 0, 1, ..., m-1 and defined as follows:

$$S(x, y_j) = \frac{\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i A_i}{q_1(\theta)}\upsilon \quad (2)$$

With:

$$A_0 = \alpha_{i,j}F_{i,j},$$
$$A_1 = (2\alpha_{i,j}\beta_{i,j}+\alpha_{i,j}+\gamma_{i,j})F_{i,j}+\alpha_{i,j}h_iF^x_{i,j},$$
$$A_2 = (2\alpha_{i,j}\beta_{i,j}+\beta_{i,j}+\gamma_{i,j})F_{i+1,j}-\beta_{i,j}h_iF^x_{i+1,j},$$
$$A_3 = \beta_{i,j}F_{i+1,j},$$
$$q_1(\theta) = (1-?)^2\alpha_{i,j}+(2\alpha_{i,j}\beta_{i,j}+\gamma_{i,j})\theta(1-\theta)+\theta^2\beta_{i,j}$$

$$S(x,y_{j+1}) = \frac{\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i B_i}{q_2(\theta)} \quad (3)$$

With:

$$B_0 = \alpha_{i,j+1}F_{i,j+1},$$
$$B_1 = (2\alpha_{i,j+1}\beta_{i,j+1}+\alpha_{i,j+1}+\gamma_{i,j+1})F_{i,j+1}+\alpha_{i,j+1}h_iF^x_{i,j+1},$$
$$B_2 = (2\alpha_{i,j+1}\beta_{i,j+1}+\beta_{i,j+1}+\gamma_{i,j+1})F_{i+1,j+1}-\beta_{i,j+1}h_iF^x_{i+1,j+1},$$
$$B_3 = \beta_{i,j}F_{i+1,j+1},$$
$$q_2(\theta) = (1-\theta)^2\alpha_{i,j+1}+(2\alpha_{i,j+1}\beta_{i,j+1}+\gamma_{i,j+1})\theta(1-\theta)+\theta^2\beta_{i,j+1}$$

$$S(x_i,y) = \frac{\sum_{i=0}^{3}(1-\phi)^{3-i}\phi^i C_i}{q_3(\phi)} \quad (4)$$

With:

$$C_0 = \hat{\alpha}_{i,j}F_{i,j},$$
$$C_1 = \left(2\hat{\alpha}_{i,j}\hat{\beta}_{i,j}+\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)F_{i,j}+\hat{\alpha}_{i,j}\hat{h}_jF^y_{i,j},$$
$$C_2 = \left(2\hat{\alpha}_{i,j}\hat{\beta}_{i,j}+\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)F_{i,j+1}-\hat{\beta}_{i,j}\hat{h}_jF^y_{i,j+1},$$
$$C_3 = \hat{\beta}_{i,j}F_{i,j+1},$$
$$q_3(\phi) = (1-\phi)^2\hat{\alpha}_{i,j}+\left(2\hat{\alpha}_{i,j}\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\varphi(1-\phi)+\phi^2\hat{\beta}$$

$$S(x_{i+1},y) = \frac{\sum_{i=0}^{3}(1-\phi)^{3-i}\phi^iD_i}{q_4(\phi)} \tag{5}$$

With

$$D_0 = \hat{\alpha}_{i+1,j}F_{i+1,j},$$
$$D_1 = \left(2\hat{\alpha}_{i+1,j}\hat{\beta}_{i+1,j}+\hat{\alpha}_{i+1,j}+\hat{\gamma}_{i+1,j}\right)F_{i+1,j}+\hat{\alpha}_{i+1,j}\hat{h}_jF^y_{i+1,j},$$
$$D_2 = \left(2\hat{\alpha}_{i,+1j}\hat{\beta}_{i+1,j}+\hat{\beta}_{i+1,j}+\hat{\gamma}_{i+1,j}\right)F_{i+1,j+1}-\hat{\beta}_{i+1,j}\hat{h}_jF^y_{i+1,j+1},$$
$$D_3 = \hat{\beta}_{i,j}F_{i+1,j+1},$$
$$q_4(\phi) = (1-\phi)^2\hat{\alpha}_{i+1,j}+\left(2\hat{\alpha}_{i+1,j}\hat{\beta}_{i+1,j}+\hat{\gamma}_{i+1,j}\right)\varphi(1-\phi)+\phi^2\hat{\beta}_{i+1,j}$$

Monotonicity preserving interpolation: Let $(x_i, y_i, F_{i,j})$ be monotone data defined over rectangular grid $[x_i, x_{i+1}]\times\lfloor y_i, y_{i+1}\rfloor$, i = 0, 1, ..., n-1; j = 0, 1, .., m-1 such that:

$$F_{i,j}<F_{i+1,j},\quad F_{i,j}<F_{i+1,j},\quad F_{i,j}<F_{i,j+1},$$
$$F^x_{i,j}>0,\quad F^y_{i,j}>0,\quad \Delta_{i,j}>0,\quad \hat{\Delta}_{i,j}>0$$

From Casciola and Romani (2003), the partially bi-cubic surface patch defined in Eq. 1 is monotone if each of the boundary curves network defined in Eq. 2-5 are monotone. Mathematically, this can be achieved if its first partial derivatives satisfy the inequalities:

$$\frac{\partial S(x, y_i)}{\partial x}>0,\ \frac{\partial S(x, y_{j+1})}{\partial x}>0,\ \frac{\partial S(x_i, y)}{\partial y}>0$$

And:

$$\frac{\partial S(x_{i+1}, y)}{\partial y}>0$$

By simple algebraic manipulation, the first partial derivative for each of the boundary curves on the rectangular patch are given as follows:

$$\frac{\partial S(x, y_j)}{\partial x} = \frac{\sum_{i=0}^{4}(1-\theta)^{4-i}\theta^iL_i}{\left[q_1(\theta)\right]^2} \tag{6}$$

With:

$$L_0 = \alpha^2_{i,j}F^x_{i,j},$$
$$L_1 = 2\alpha_{i,j}\left(\beta_{i,j}\left[\left(1+2\alpha_{i,j}\right)\Delta_{i,j}-F^x_{i+1,j}\right]+\gamma_{i,j}\Delta_{i,j}\right),$$
$$L_2 = 2\alpha^2_{i,j}\beta_{i,j}\left(\left[1+2\beta_{i,j}\right]\Delta_{i,j}-F^x_{i,j}\right)+$$
$$\gamma_{i,j}\left(\gamma_{i,j}\Delta_{i,j}-\beta_{i,j}F^x_{i+1,j}+\beta_{i,j}\Delta_{i,j}\right)+$$
$$\alpha_{i,j}\begin{bmatrix}4\beta_{i,j}\left(1+\gamma_{i,j}\right)\Delta_{i,j}+\gamma_{i,j}\Delta_{i,j}-\gamma_{i,j}F^x_{i,j}-\beta_{i,j}F^x_{i+1,j}-\beta_{i,j}F^x_{i,j}\\+2\beta^2_{i,j}\left(\Delta_{i,j}-F^x_{i+1,j}\right)\end{bmatrix},$$
$$L_3 = 2\beta_{i,j}\left(\alpha_{i,j}\left[\left(1+2\beta_{i,j}\right)\Delta_{i,j}-F^x_{i,j}\right]+\gamma_{i,j}\Delta_{i,j}\right),$$
$$L_4 = \beta^2_{i,j}F^x_{i+1,j}.\ q_1(\theta) = (1-\theta)^2\alpha_{i,j}+\left(2\alpha_{i,j}\beta_{i,j}+\gamma_{i,j}\right)\theta(1-\theta)+\theta^2\beta_{i,j}$$

$$\frac{\partial S(x, y_{j+1})}{\partial x} = \frac{\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^iM_i}{\left(q_2(\theta)\right)^2} \tag{7}$$

With :

$$M_0 = \alpha^2_{i,j+1}F^x_{i,j+1},$$
$$M_1 = 2\alpha_{i,j+1}\begin{bmatrix}\beta_{i,j}\left[\left(1+2\alpha_{i,j}\right)\Delta_{i,j+1}-F^x_{i+1,j+1}\right]+\\\gamma_{i,j+1}\Delta_{i,j+1}\end{bmatrix},$$
$$M_2 = 2\alpha^2_{i,j+1}\beta_{i,j+1}\left(\left[1+2\beta_{i,j+1}\right]\Delta_{i,j+1}-F^x_{i,j+1}\right)+\gamma_{i,j+1}\begin{pmatrix}\gamma_{i,j+1}\Delta_{i,j+1}-\beta_{i,j+1}\\F^x_{i+1,j+1}+\beta_{i,j+1}\Delta_{i,j+1}\end{pmatrix}+$$
$$\alpha_{i,j+1}\begin{bmatrix}4\beta_{i,j+1}\left(1+\gamma_{i,j+1}\right)\Delta_{i,j+1}+\gamma_{i,j+1}\Delta_{i,j+1}-\gamma_{i,j+1}F^x_{i,j+1}-\\\beta_{i,j+1}F^x_{i+1,j+1}-\beta_{i,j+1}F^x_{i,j+1}+2\beta^2_{i,j+1}\left(\Delta_{i,j+1}-F^x_{i+1,j+1}\right)\end{bmatrix}$$
$$M_3 = 2\beta_{i,j+1}\left[\alpha_{i,j+1}\left[\left(1+2\beta_{i,j+1}\right)\Delta_{i,j+1}-F^x_{i,j+1}\right]+\gamma_{i,j+1}\Delta_{i,j+1}\right],$$
$$M_4 = \beta^2_{i,j+1}F^x_{i+1,j+1}$$
$$q_2(\theta) = (1-\theta)^2\alpha_{i,j+1}+\left(2\alpha_{i,j+1}\beta_{i,j+1}+\gamma_{i,j+1}\right)\theta(1-\theta)+\theta^2\beta_{i,j+1}$$

$$\frac{\partial S(x_i, y)}{\partial y} = \frac{\sum_{i=0}^{4}(1-\phi)^{4-i}\phi^iN_i}{\left(q_3(\phi)\right)^2} \tag{8}$$

With:

$$N_0 = \hat{\alpha}^2_{i,j}F^y_{i,j},$$
$$N_1 = 2\hat{\alpha}_{i,j}\left[\hat{\beta}_{i,j}\left[\left(1+2\hat{\alpha}_{i,j}\right)\hat{\Delta}_{i,j}-F^y_{i,j+1}\right]+\hat{\gamma}_{i,j}\hat{\Delta}_{i,j}\right],$$
$$N_2 = 2\hat{\alpha}^2_{i,j}\hat{\beta}_{i,j}\left(\left[1+2\hat{\beta}_{i,j}\right]\hat{\Delta}_{i,j}-F^y_{i,j}\right)+$$
$$\hat{\gamma}_{i,j}\left(\hat{\gamma}_{i,j}\hat{\Delta}_{i,j}-\hat{\beta}_{i,j}F^y_{i+1,j}+\hat{\beta}_{i,j}\hat{\Delta}_{i,j}\right)+$$
$$\hat{\alpha}_{i,j}\begin{bmatrix}4\hat{\beta}_{i,j}\left(1+\hat{\gamma}_{i,j}\right)\hat{\Delta}_{i,j}+\hat{\gamma}_{i,j}\hat{\Delta}_{i,j}-\hat{\gamma}_{i,j}F^y_{i,j}-\hat{\beta}_{i,j}F^y_{i,j+1}-\hat{\beta}_{i,j}F^y_{i,j}\\+2\hat{\beta}^2_{i,j}\left(\hat{\Delta}_{i,j}-F^y_{i,j+1}\right)\end{bmatrix},$$

$$N_3 = 2\hat{\beta}_{i,j}\left[\hat{\alpha}_{i,j}\left[\left(1+2\hat{\beta}_{i,j}\right)\hat{\Delta}_{i,j}-F^y_{i,j}\right]+\hat{\gamma}_{i,j}\hat{\Delta}_{i,j}\right],$$
$$N_4 = \hat{\beta}^2_{i,j}F^y_{i,j+1}$$
$$q_3(\phi) = (1-\phi)^2\hat{\alpha}_{i,j}+\left(2\hat{\alpha}_{i,j}\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\phi(1-\phi)+\phi^2\hat{\beta}_{i,j}$$

$$\frac{\partial S(x_{i+1}, y)}{\partial y} = \frac{\sum_{i=0}^{4}(1-\phi)^{4-i}\phi^i O_i}{\hat{h}_j\left(q_4(\phi)\right)^2} \tag{9}$$

With:

$$O_0 = \hat{\alpha}^2_{i+1,j}F^y_{i+1,j},$$
$$O_1 = 2\hat{\alpha}_{i+1,j}\left[\hat{\beta}_{i+1,j}\left[\left(1+2\hat{\alpha}_{i+1,j}\right)\hat{\Delta}_{i+1,j}-F^y_{i+1,}\right]+\hat{\gamma}_{i+1,j}\hat{\Delta}_{i+1,j}\right],$$
$$O_2 = 2\hat{\alpha}^2_{i+1,j}\hat{\beta}_{i+1,j}\left(\left[1+2\hat{\beta}_{i+1,j}\right]\hat{\Delta}_{i+1,j}-F^y_{i+1,j}\right)+$$
$$\hat{\gamma}_{i+1,j}\left(\hat{\gamma}_{i+1,j}\hat{\Delta}_{i+1,j}-\hat{\beta}_{i+1,j}F^y_{i+1,j+1}+\hat{\beta}_{i+1,j}\hat{\Delta}_{i+1,j}\right)+$$
$$\hat{\alpha}_{i+1,j}\begin{bmatrix}4\hat{\beta}_{i+1,j}\left(1+\hat{\gamma}_{i+1,j}\right)\hat{\Delta}_{i+1,j}+\hat{\gamma}_{i+1,j}\hat{\Delta}_{i+1,j}-\hat{\gamma}_{i+1,j}F^y_{i+1,j}\\ -\hat{\beta}_{i+1,j}F^y_{i+1,j+1}-\hat{\beta}_{i+1,j}F^y_{i+1,j}+2\hat{\beta}^2_{i+1,j}\left(\hat{\Delta}_{i+1,j}-F^y_{i+1,j+1}\right)\end{bmatrix}$$
$$O_3 = 2\hat{\beta}_{i+1,j}\left[\hat{\alpha}_{i+1,j}\left[\left(1+2\hat{\beta}_{i+1,j}\right)\hat{\Delta}_{i+1,j}-F^y_{i+1,j}\right]+\hat{\gamma}_{i+1,j}\hat{\Delta}_{i+1,j}\right],$$
$$O_4 = \hat{\beta}^2_{i+1,j}F^y_{i+1,j+1}.$$
$$q_4(\phi) = (1-\phi)^2\hat{\alpha}_{i+1,j}+\left(2\hat{\alpha}_{i+1,j}\hat{\beta}_{i+1,j}+\hat{\gamma}_{i+1,j}\right)\phi(1-\phi)+\phi^2\hat{\beta}_{i+1,j}$$

We have the following theorem for monotonicity preserving.

Theorem 1: The piecewise rational partially bi-cubic function $S(x, y)$ in (1) defined over the rectangular mesh $[x_i, x_{i+1}]\times[y_j, y_{j+1}]$, i = 0, 1, .., n-1; j = 0, 1, ..., m-1 preserves the monotone surface data if the parameters satisfies the following sufficient conditions:

$$\left.\begin{array}{l}\alpha_{i,j}>0,\ \alpha_{i,j+1}>0,\ \beta_{i,j}>0,\ \beta_{i,j+1}>0,\ \hat{\alpha}_{i,j}>0,\\ \hat{\alpha}_{i+1,j}>0,\ \hat{\beta}_{i,j}>0,\ \hat{\beta}_{i+1,j}>0,\\ \gamma_{i,j}>\text{Max}\left\{0,\alpha_{i,j}\left(\frac{F^x_{i,j}}{\Delta_{i,j}}-2\beta_{i,j}\right),\beta_{i,j}\left(\frac{F^x_{i+1,j}}{\Delta_{i,j}}-2\alpha_{i,j}\right)\right\},\\ \gamma_{i,j+1}>\text{Max}\left\{0,\alpha_{i,j+1}\left(\frac{F^x_{i,j+1}}{\Delta_{i,j+1}}-2\beta_{i,j+1}\right),\beta_{i,j+1}\left(\frac{F^x_{i+1,j+1}}{\Delta_{i,j+1}}-2\alpha_{i,j+1}\right)\right\}\\ \hat{\gamma}_{i,j}>\text{Max}\left\{0,\hat{\alpha}_{i,j}\left(\frac{F^y_{i,j}}{\hat{\Delta}_{i,j}}-2\hat{\beta}_{i,j}\right),\hat{\beta}_{i,j}\left(\frac{F^y_{i,j+1}}{\hat{\Delta}_{i,j}}-2\hat{\alpha}_{i,j}\right)\right\},\\ \hat{\gamma}_{i+1,j}>\text{Max}\left\{0,\hat{\alpha}_{i+1,j}\left(\frac{F^y_{i+1,j}}{\hat{\Delta}_{i+1,j}}-2\hat{\beta}_{i+1,j}\right),\hat{\beta}_{i+1j}\left(\frac{F^y_{i+1,j+1}}{\hat{\Delta}_{i+1,j}}-2\hat{\alpha}_{i+1,j}\right)\right\}\end{array}\right\} \tag{10}$$

Proof: To prove Theorem 1, we need to consider the first partial derivative for all boundary curves defined in Eq. 2 until Eq. 5. The boundary curves is monotone if and only if:

$$\frac{\partial S(x, y_i)}{\partial x}>0\ \frac{\partial S(x, y_{j+1})}{\partial x}>0,\ \frac{\partial S(x_i, y)}{\partial y}>0$$

And:

$$\frac{\partial S(x_{i+j}, y)}{\partial y}>0$$

respectively. Now:

$$\frac{\partial S(x, y_j)}{\partial x}>0$$

If:

$$\sum_{i=1}^{4}(1-\theta)^{4-i}\theta^i L_i>0$$

Since, for $\alpha_{i,j}>0$, $\beta_{i,j}>0$ the denominator $[q_1(\theta)]^2>0$. Clearly for $\alpha_{i,j}>0$, $\beta_{i,j}>0$, then, $L_0>0$, $L_4>0$. Now, $L_1>0$, $L_2>0$ and $L_3>0$ provide the following derivation:

$$\gamma_{i,j}>\frac{F^x_{i,j}}{\Delta_{i,j}}-2\beta_{i,j} \tag{11}$$

$$\gamma_{i,j}>\beta_{i,j}\left(\frac{F^x_{i+1,j}}{\Delta_{i,j}}-2\alpha_{i,j}\right) \tag{12}$$

Inequalities in Eq. 20 and 21 are combined to form the following sufficient conditions for:

$$\frac{\partial S(x, y_j)}{\partial x}$$

$$\gamma_{i,j}>\text{Max}\left\{0,\alpha_{i,j}\left(\frac{F^x_{i,j}}{\Delta_{i,j}}-2\beta_{i,j}\right),\beta_{i,j}\left(\frac{F^x_{i+1,j}}{\Delta_{i,j}}-2\alpha_{i,j}\right)\right\} \tag{13}$$

Likewise for the remaining three boundary curves $S(x, y_{j+1})$ and $S(x_{i+1}, y)$, the monotonicity will be preserved if and only if :

$$\frac{\partial S(x, y_{j+1})}{\partial x}>0,\ \frac{\partial S(x_i, y)}{\partial y}>0$$

And:

$$\frac{\partial S(x_{i+1}, y)}{\partial y}>0$$

respectively. For:

$$\alpha_{i,j+1}>0,\ \beta_{i,j+1}>0,\ \frac{\partial S(x, y_{j+1})}{\partial x}>0$$

If and only if:

$$\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^{i}M_i>0$$

Similarly for:

$$\hat{\alpha}_{i,j}>0\hat{\beta}_{i,j}>0,\ \frac{\partial S(x_i, y)}{\partial y}>0$$

If and only if:

$$\sum_{i=0}^{3}(1-\theta)^{3-1}\theta^{i}N_i>0$$

Finally, for:

$$\hat{\alpha}_{i+1,j}>0,\hat{\beta}_{i+1,j}>0\frac{\partial S(x_{i+1}, y)}{\partial y}>0$$

If and only if :

$$\sum_{i=0}^{3}(1-\theta)^{3-1}\theta^{i}O_i>0$$

Combining all conditions lead to:

$$\gamma_{i,j+1}>\alpha_{i,j+1}\left(\frac{F^{x}_{i,j+1}}{\Delta_{i,j+1}}-2\beta_{i,j+1}\right) \quad (14)$$

$$\gamma_{i,j+1}>\beta_{i,j+1}\left(\frac{F^{x}_{i+1,j+1}}{\Delta_{i,j+1}}-2\alpha_{i,j+1}\right) \quad (15)$$

$$\hat{\gamma}_{i,j}>\hat{\alpha}_{i,j}\left(\frac{F^{y}_{i,j}}{\hat{\Delta}_{i,j}}-2\hat{\beta}_{i,j}\right) \quad (16)$$

$$\hat{\gamma}_{i,j}>\hat{\beta}_{i,j}\left(\frac{F^{y}_{i,j+1}}{\hat{\Delta}_{i,j}}-2\hat{\alpha}_{i,j}\right) \quad (17)$$

$$\hat{\gamma}_{i+1,j}>\hat{\alpha}_{i+1,j}\left(\frac{F^{y}_{i+1,j}}{\hat{\Delta}_{i+1,j}}-2\hat{\beta}_{i+1,j}\right) \quad (18)$$

$$\hat{\gamma}_{i+1,j}>\hat{\beta}_{i+1j}\left(\frac{F^{y}_{i+1,j+1}}{\hat{\Delta}_{i+1,j}}-2\hat{\alpha}_{i+1,j}\right) \quad (19)$$

Combining inequalities Eq. 13 and 14 with Eq. 15 and 16 with Eq. 17 and 18 with Eq. 19 give the required sufficient conditions for monotonicity preserving:

$$\hat{\alpha}_{i+1,j}>0,\ \hat{\beta}_{i,j}>0,\ \hat{\beta}_{i+1,j}>0,$$

$$\gamma_{i,j}>\text{Max}\left\{0,\ \alpha_{i,j}\left(\frac{F^{x}_{i,j}}{\Delta_{i,j}}-2\beta_{i,j}\right),\beta_{i,j}\left(\frac{F^{x}_{i+1,j}}{\Delta_{i,j}}-2\alpha_{i,j}\right)\right\},$$

$$\gamma_{i,j+1}>\text{Max}\left\{0,\alpha_{i,j+1}\left(\frac{F^{x}_{i,j+1}}{\Delta_{i,j+1}}-2\beta_{i,j+1}\right),\beta_{i,j+1}\left(\frac{F^{x}_{i+1,j+1}}{\Delta_{i,j+1}}-2\alpha_{i,j+1}\right)\right\},$$

$$\hat{\gamma}_{i,j}>\text{Max}\left\{0,\hat{\alpha}_{i,j}\left(\frac{F^{y}_{i,j}}{\hat{\Delta}_{i,j}}-2\hat{\beta}_{i,j}\right),\hat{\beta}_{i,j}\left(\frac{F^{y}_{i,j+1}}{\hat{\Delta}_{i,j}}-2\hat{\alpha}_{i,j}\right)\right\},$$

$$\hat{\gamma}_{i+1,j}>\text{Max}\left\{0,\ \hat{\alpha}_{i+1,j}\left(\frac{F^{y}_{i+1,j}}{\hat{\Delta}_{i+1,j}}-2\hat{\beta}_{i+1,j}\right),\hat{\beta}_{i+1j}\left(\frac{F^{y}_{i+1,j+1}}{\hat{\Delta}_{i+1,j}}-2\hat{\alpha}_{i+1,j}\right)\right\}$$

This completes the proof. Condition in Eq. 13 further can be rewritten as Eq. 20:

$$\alpha_{i,j}>0,\ \alpha_{i,j+1}>0,\ \beta_{i,j}>0,\ \beta_{i,j+1}>0,\ \hat{\alpha}_{i,j}>0,$$

$$\hat{\alpha}_{i+1,j}>0,\ \hat{\beta}_{i,j}>0,\hat{\beta}_{i+1,j}>0,$$

$$\gamma_{i,j}=p_{i,j}+\text{Max}\left\{0,\alpha_{i,j}\left(\frac{F^{x}_{i,j}}{\Delta_{i,j}}-2\beta_{i,j}\right),\beta_{i,j}\left(\frac{F^{x}_{i+1,j}}{\Delta_{i,j}}-2\alpha_{i,j}\right)\right\},$$

$$\gamma_{i,j+1}=r_{i,j}+\text{Max}\left\{0,\alpha_{i,j+1}\left(\frac{F^{x}_{i,j+1}}{\Delta_{i,j+1}}-2\beta_{i,j+1}\right),\beta_{i,j+1}\left(\frac{F^{x}_{i+1,j+1}}{\Delta_{i,j+1}}-2\alpha_{i,j+1}\right)\right\},$$

$$\hat{\gamma}_{i,j}=s_{i,j}+\text{Max}\left\{0,\ \hat{\alpha}_{i,j}\left(\frac{F^{y}_{i,j}}{\hat{\Delta}_{i,j}}-2\hat{\beta}_{i,j}\right),\hat{\beta}_{i,j}\left(\frac{F^{y}_{i,j+1}}{\hat{\Delta}_{i,j}}-2\hat{\alpha}_{i,j}\right)\right\},$$

$$\hat{\gamma}_{i+1,j}=t_{i,j}+\text{Max}\left\{0,\hat{\alpha}_{i+1,j}\left(\frac{F^{y}_{i+1,j}}{\hat{\Delta}_{i+1,j}}-2\hat{\beta}_{i+1,j}\right),\ \hat{\beta}_{i+1,j}\left(\frac{F^{y}_{i+1,j+1}}{\hat{\Delta}_{i+1,j}}-2\hat{\alpha}_{i+1,j}\right)\right\} \quad (20)$$

Where:

$$p_{i,j},\ r_{i,j}\ s_{i,j}\ t_{i,j}\in(0,\ 0.25)$$

Theorem 2: The partially blended rational bi-cubic spline S(x, y) that satisfies the sufficient condition for monotonicity in Eq. 19 is C^1 continuous everywhere.

Proof: Since, all four boundary curves are C^1 continuity, from Casciola and Romani (2003), the resulting monotonicity preserving surface interpolation also has C^1 continuity.

RESULTS AND DISCUSSION

In this study, a numerical example for monotonicity by using partially blended rational bi-cubic spline interpolation are discussed. Two monotone data sets taken from Abbas (2012) and Hussain and Hussain (2007). We reconstruct the monotone surface by using the

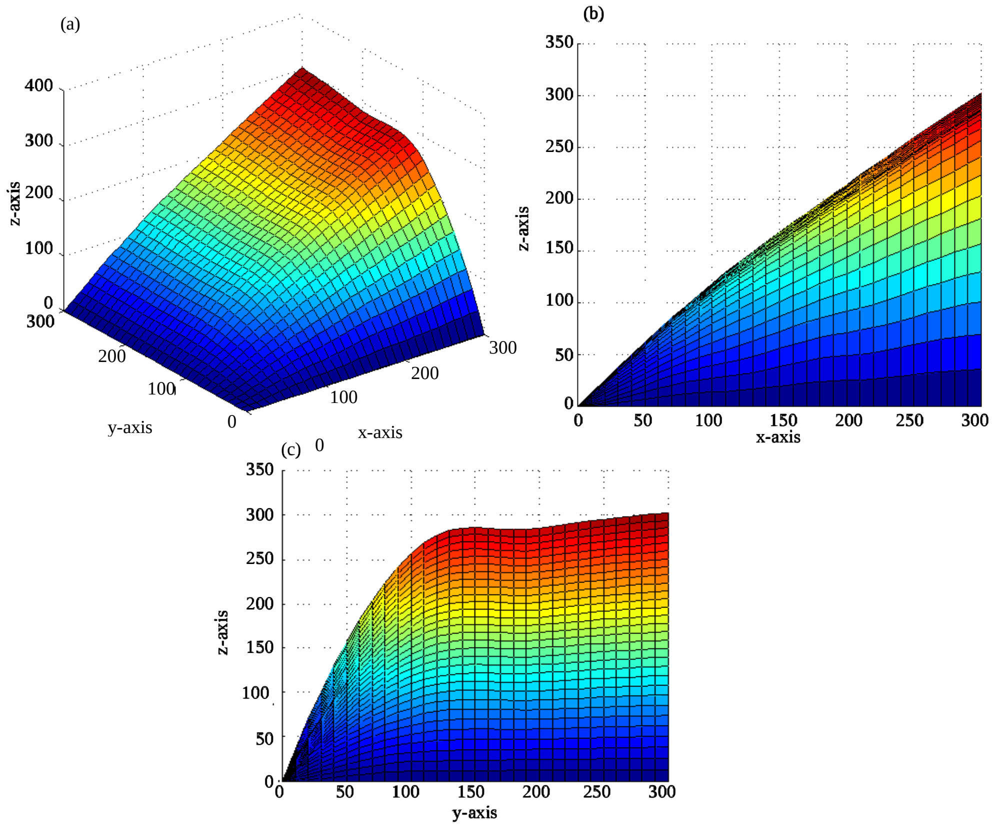

Fig. 1: Bi-cubic Hermite surface; a) Bi-cubic Hermite surface; b) xz-view and c) yz-view

sufficient condition given in Eq. 20. RMSE value and the coefficient of determination is used to calculate the error of the reconstruction surface.

Example 1: A monotone data from the following function is truncated to two decimal places (Abbas, 2012). The function is a Cobb-Douglas production to model the growth of the American economy during the period 1899-1922 (Stewart, 2012):

$$F_1(x, y) = 1.01x^{0.85}y^{0.15}, \ 0 \le x, y \le 300 \qquad (21)$$

Figure 1 shows the default bi-cubic Hermite spline (Farin, 2002) for the monotonicity data given in Table 1 above. Figure 1b shows the xz-view and Fig. 1c shows the yz-view for Fig. 2, respectively. The bi-cubic Hermite cannot preserves the monotonicity of the monotone data as shown clearly from Fig. 1b. Figure 2a shows the monotonicity preserving by using the proposed rational bi-cubic spline with $\alpha_{i,j} = \beta_{i,j} = 3, \hat{\alpha}_{i,j} = \hat{\beta}_{i,j} = 3$. Meanwhile, Fig. 2b, c show the xz-view and yz-view for Fig. 2a, respectively. Figure 2a shows that the monotonicity of the data is preserved with continuity.

Table 1: Monotone surface data from function

y/x	0	100	200	300
0	0	0	0	0
100	0	101.0000	112.07	119.09
200	0	182.05	202	214.67
300	0	256.97	285.12	303.00

Table 2: Positive surface data from function

y/x	1	100	200	300
1	0.6931	9.2104	10.5967	11.4076
100	9.2104	9.9035	10.8198	11.5129
200	10.5967	10.8198	11.2898	11.7753
300	11.4076	11.5129	11.7753	12.1007

Example 2: A monotone data from the following function is truncated to five decimal places from Hussain and Hussain (2007):

$$F_2(x, y) = \ln\left(x^2+y^2\right), \ 1 \le x, y \le 300$$

Figure 3 shows the default bi-cubic Hermite spline (Farin, 2002) for the monotonicity data given in Table 2 with xz-view and yz-view shown in Fig. 3b, c. Figure 4 shows the monotonicity preserving by using the proposed rational bi-cubic spline with $\alpha_{i,j} = \beta_{i,j} = 6.5, \hat{\alpha}_{i,j} = \hat{\beta}_{i,j} = 6.5$. The xz-view and yz-view is given in Fig. 4b, c, resepctively.

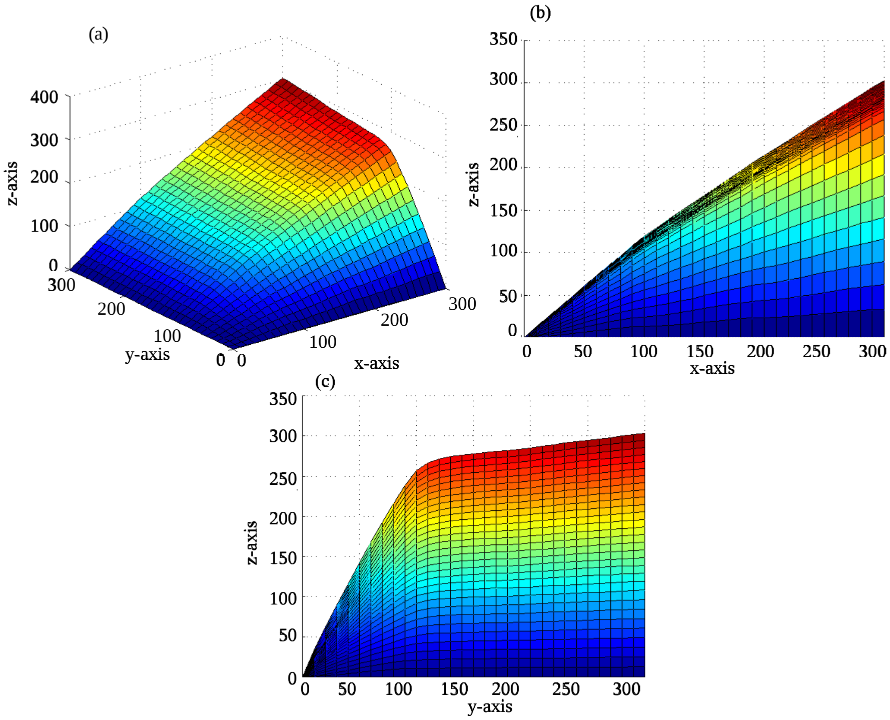

Fig. 2: Monotonicity preserving; a) The proposed scheme; b) xz-view and c) yz-view

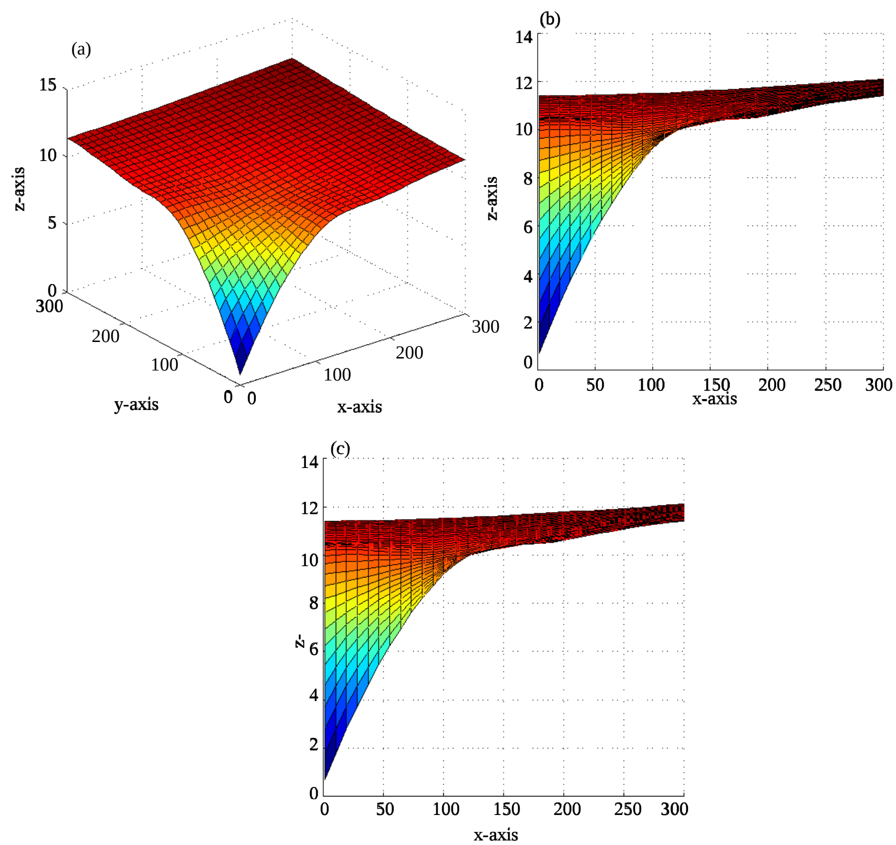

Fig. 3: Bi-cubic Hermite surface; a) Bi-cubic Hermite surface; b) xz-view and c) yz-view

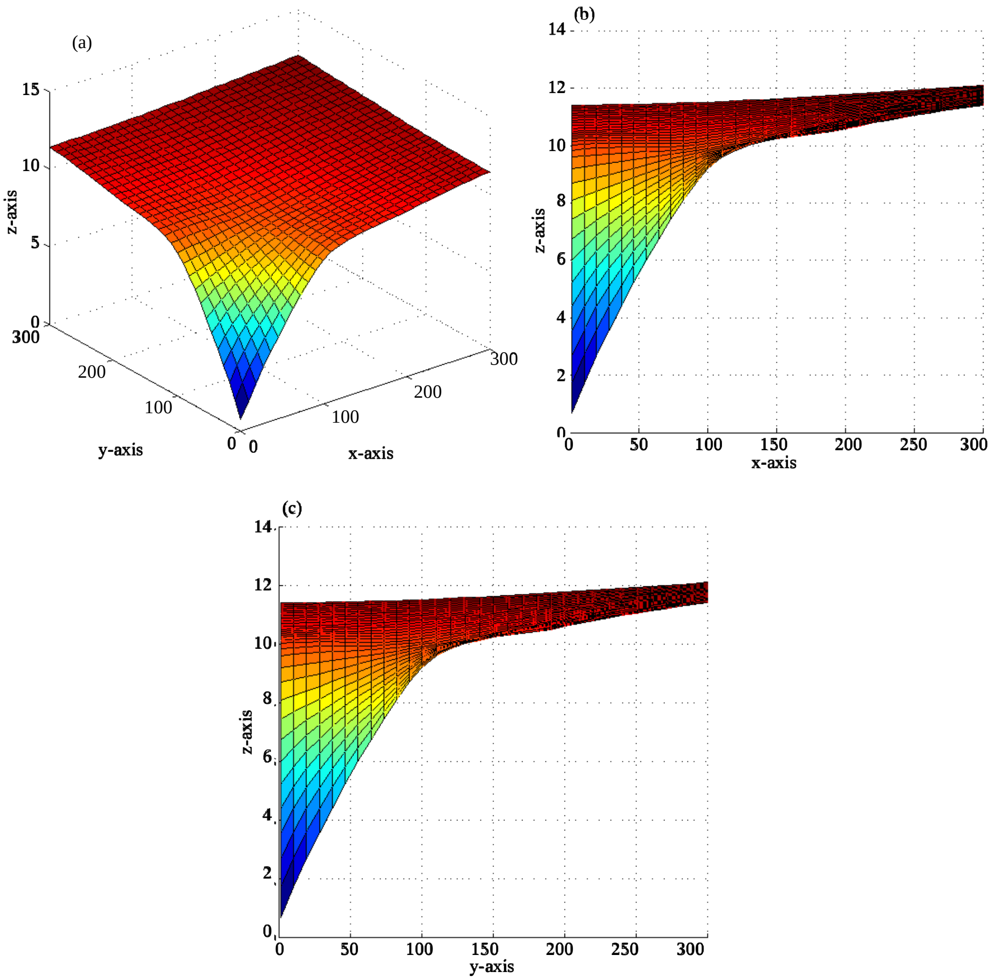

Fig. 4: Monotonicity preserving surface; a) The proposed scheme; b) xz-view and c) yz-view

Table 3: RMSE estimation and R^2 value

Parameters	Example 1	Example 2
RMSE	0.0325	0.02099
R^2	0.9200	0.9000

From numerical results, clearly the rational bi-cubic spline by Abdul Karim *et al*. (2015) and Karim (2017) preserves the monotonicity of the surface data with degree smoothness attained is C^1. The proposed scheme is applicable even with the first partial derivative $F^x_{i,j}$ and $F^y_{i,j}$ is zero as shown in Table 1. In contrast, the rational bi-cubic spline from Abbas (2012) and Abbas *et al*. (2012) could only be used if the first partial derivatives are not equal to zero. Otherwise, their schemes fail. Comparing with the research of Hussain and Hussain (2007), the proposed scheme has eight free parameters, meanwhile in (Hussain and Hussain, 2007) there is no free parameter for shape modification.

Table 3 summarized RMSE value and the coefficient of determination for surface reconstruction by using the presented schemes. Overal the proposed scheme is capable to reconstruct the surface with the coefficient of determination R^2 more than 0.90.

CONCLUSION

The partially blended rational bi-cubic spline from Abdul Karim *et al*. (2015) is use to reconstruct monotonic surface. The sufficient condition for the monotonicity are derived on four parameters meanwhile the remaining 8 parameters can be used to changes the shape of the interpolating surface without the need to change the data points. One possible extension to the presented scheme is the application in real life situation such as in medical image interpolation and image resizing.

ACKNOWLEDGEMENT

This research is fully supported by Universiti Teknologi Petronas (UTP) through STIRF No.

0153AA-D91 and YUTP No. 0153AA-H24 grants including Mathematica and MATLAB Software.

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