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Visible Sub-Modules of a Module X Over a Ring R is Introduced

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Abstract: The concept of visible submodule of a module X over a ring R is introduced (R is commutative ring with identity and X is unitary R-module) where is a new concept not previously presented. As well as the description of the visible radical submoule and many of the results own this concept has mad. Also, we have presented a concept of V closure operation. Through this study we have been able to obtain many of the results and characteristics that belong to those concepts above.

Key words: Visible submodule, visible radical of submodule, strongly cancellation module, pure submodule, vclosure operation, concepts

INTRODUCTION

In this study the concept of visible submodule has been presented as this concept is new and has not been addressed by anyone before us. A proper submodule K of a module X over a ring R is said to be visible, if K = UK for every a nonzero ideal U of R. Section 2 has been introduced a visible submodule and several properties with important characterization of such a submodules. Section 3 has been defined a visible radical of a submodule K and which is defined as the intersection of all visible submodule of X containing K and we dented by $Vrad_x(K)$. The definition of $Vrad_x(K)$ is gotten from the generalization of visible radical of an ideal G of R is denoted by $\sqrt{G}$.

The concept of V closure operation (for pithiness, V_{CL} operation) has also been provided in this study, where q:S→S, S is the set of all visible submodules of a module X over R is called V_{CL} operation if U⊑q()U, q(q(U)⊑q(U) U⊑K implies q(U)⊑q(K).

(V)Aq(U) = q(AU for all nonzero ideals A of R and submodules U, K of X. This concept is stranger than the concept of closure operation in Lu (1990), where we can make the 4th condition in the concept of V_{CL} operation to achieve equality rather than containment, thanks to the use of the concept of visible submodule. Resulted in this emergence of the concept of V_{CL} operations which a more general of the concept is located in Ali (2005).

In this study we have demonstrated a lot of important properties and characteristics, we have also provided several important and useful results in this search.

In our study, we need to the following fundamental concept. A module X is called faithful if ann(X) = {r∈R; rx = 0, x∈X is the zero ideal of . We call that a module X over is a multiplication module, if for every submodule U of X, then U is written as U = LX for some ideal L of R (Azizi and Jayaram, 2017).

According to Lu (1990) a proper submodule Uof X is said to be irreducible when $X_1 \cap X_2 = 2$, then X_1 = U or X_2 for every submodules X_1 and X_2 of X. If S is a multiplicative set of R and U is a submodule of X, then U(S) = {m∈X:j∈S such that jm∈U} is a submodule of X contain U.

A cancellation ideal of R is an ideal J of R such that XJ = YJ for all ideals X, Y, then X = Y (Ali, 2005) and a module X over R is called strongly cancellation module, if for each ideals X, Y of R such that XU = YU then X = Y for every submodule U of X (Elewi, 2016).

Visible submodules: In this study a new type of submoule was defined and named as visible submodule. Many essential properties and some characterizations a round this concept have been built (Anderson *et al.*, 2017).

Definition (2.1): A proper submodule K of an R-module X is said to be visible whenever K = AK for every a nonzero ideal A of. A proper ideal of a ring R is named visible ideal if A = JA for every a nonzero ideal J of R.

Remarks and examples (2.2):

- A zero submodule of any R is always visible
- Consider Z_4 as a Z-module. A submodule of $(\bar{2})$ is not visible. Since, for every a nonzero ideal A of Z, implies $(\bar{2}) \neq A(\bar{2})$.
- Two submodules $(\bar{2})$ and $(\bar{3})$ of the Z-module Z_6 are not visible for the same reason of No. 2

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- All a nonzero proper cyclic submodule of the module Q as a Z-module is not visible
- Let, L be a submodules of an R-module X such that K ≅ L. Then K is visible submodule ⇔L is visible submodule
- Let X_1 and X_2 be two R-module and ψ:X_1→X_2 be an R-homo. Then
- if K is a visible submodule of X_2, then ψ^{-1}(K) is also visible submodule of X_1
- If K is a visible submodule of X_1, then ψ(K) is visible submodule of X_2

Proof (4): Let L be a cyclic submodule of Q, generated by an element e/g, where e.g., are two nonzero element in Z. Let (s) be an ideal of Z, where s is a positive integer and x>1. Then (s)g = (g), that is (s)(e/g)≠(e/g). Therefore, L is not visible submodule (Atani, 2005).

Proof (5): Let ψ:K→L be an epimorphism. Then ψ(K) = L. Assume that K is a visible submodule which implies K = AK for every a nonzero ideal A of R. Therefore, L = ψ(K) = ψ(AK) = Aψ(K) = AL. Thus, L is visible submodule. Suppose that L is visible submodule. Let A be a nonzero ideal of ψ(K) = L = AL Aψ(L) = ψ(AL) but ψ is (1-1) then L = AL produce L is visible submodule.

Proof (6): For every ideal I of R and ≠0 we have IK = K where K is proper submodule of X_2. Then:

- Iψ^{-1}(K) = ψ^{-1}(IK) = ψ^{-1}(K)
- Iψ(K) = ψ(IK) = ψ(K). Therefore, ψ(K) is visible submodule of X_2

Proposition (2.3): Let D be a proper submodule of an R-module X. Then the coming are equivalent:

- D is visible submodule
- D = ID for each a nonzero finitely generated (briefly FG) ideal I of R.
- D = (a)D for each 0≠a∈I and 0≠I is any ideal of R

Proof:

- ⇒(2):Let D be a visible submodule of X. Consequently, ∀0 ≠I, I is an ideal of R, we have D = ID, we can take I is finitely generated ideal
- ⇒(3):Let D be a proper submodule of X and 0≠I be a FG ideal of R. Therefore, directly from (Eq. 2) we get D = (a)D where 0≠a∈I
- ⇒(3):Let 0 ≠a∈I and 0≠I be an ideal of R. Then a⊆I which implies that (a)D⊆ID. Therefore, by (Eq. 3) we get D⊆ID and so on ID⊆D. Thus, ID = D and hence, D is visible submodule

Proposition (2.4): Let X be an R-module and E be a visible submodule of X. If L is a submodule of E, then E/L is a visible submodule of X/L.

Proof: Let 0≠A be an ideal of R. Now, A(E/A) = AE/L. But AE = E (since, E is visible submule of X). Then (AE+L)/L = (E+L)/L. Therefore, E/L is visible submodule of M/L.

Proposition (2.5): Let x be an R-module and L be two submodules of X. If D, L are visible submodule, then D+L is visible (Dauns, 1980; Kasch, 1982).

Proof: Let A be a nonzero ideal of R and L be two submodules of X. Then A(D+L) (since, D and L are visible submodule). Therefore, D+L is visible submodule of X.

Remarke (2.6): As a generalization of proposition (2.5), we get: if $\{N_k\}^n_{k=1}$ is a finite collection of a submodule of an R-module X and N_k is visible submodule for all K, then the sum of all these submodules is visible submodule ofX.

Proposition (2.7): Every submodule of a visible submodule is also visible.

Proof: Let N be a visible submodule of an R-module X and let K be a proper submodule of that is K⊏N. Therefore, N = IN for every a nonzero ideal I of. Then K⊏IN which implies that:

$$IN+K = IN \quad (1)$$

Also, from the above inclusion, we get IK⊏IN. And hence:

$$IK+IN = IN \quad (2)$$

Form Eq. 1 and 2, we get IN+K = IN+K and hence, K = IK. Therefore, K is visible submodule.

Corollary (2.8): If either N_1 or N_2 is visible submodule of an-module, then $N_1 \cap N_2$ is also visible.

Proof: It is clearly that $N_1 \cap N_2 \sqsubset N_1$ and $N_1 \cap N_2 \sqsubset N_2$ but N_1 is visible, then by proposition (2.7), $N_1 \cap N_2$ is also visible. Similarly with N_2 is visible, we get $N_1 \cap N_2$ is visible submodule. As a directly result of corollary (2.8) , we give the following generalization.

Corollary (2.9): Let $\{N\}^n_{i=1}$ be a family of submodules of an R-module X such that at least one of them is visible, then $\cap^n_{i=1} N_i$ is visible submodule. The converse of proposition (2.7) need not to be true, for example:

The module Z_{12} as a Z_{36}-module. Since, $(\bar{0})$ is contains in any submodule of any R-module X and $(\bar{0})$ is visible submodule by remarks and examples (1). But a submodule $(\bar{6})$ of module Z_{12} is not visible, since, there exists $(\bar{2})$ is a nonzero ideal of Z_{36} such that $(\bar{6}) \neq (\bar{2})(\bar{6}) = (\bar{0})$.

Therefore, $(\bar{6})$ is not visible submodule in Z_{12}. However, under a certain condition the converse of proposition (2.7) holds: The module X over is named fully cancellation if for each submodules W, K and for each ideal C of we have CW = CK implies W = K (Ali, 2005). Next, we can use above concept to present the coming result.

Proposition (2.10): Let be a ring which all nonzero ideals are idempotent. Let D be a visible submodule of a fully cancellation R-modules. If K is a proper submodule of X containing, then K is a visible submodule of X.

Proof: Suppose that, I be anonzero ideal of R. To prove that K = K, we have D⊆K, then ID⊆IK which implies that:

$$IK = ID+IK \tag{3}$$

Also D⊆IK (since, D is visible submodule), then IK = ID+IK. Therefore,

$$IK = ID+I^2K \tag{4}$$

Now, form Eq. 1 and 2, we get ID+IK (since, D is visible submodule) and hence, IK = D+I^2. But X is fully cancellation module, then IK = K hence, K is visible submodule.

Proposition (2.11): Let D be a visible submodule of a strongly cancellation R-module. Then nn(ID) = ann(I), for every a nonzero ideal I of R.

Proof: Let x∈ann_x(I). Then xI = 0 and hence, xID = 0 which implies that x∈ann(ID). Therefore, ann(I)⊆ann(ID). Now, let y∈ann(ID). Then ID = 0 but D is visible submodule, then yD = 0 and hence yD = 0D, we have X is strongly cancellation module. Then y = 0 thus, yI = 0 and hence, ∈ann(I), we obtain ann(ID)⊆ann(I). Therefore, ann(ID) = ann(I).

Proposition (2.12): Let D be a visible submodule of strongly cancellation R-module. Then every a nonzero ideal I of R is cancellation.

Proof: Let 0 ≠ I be an ideal of R s.t AI = BI where A, B are two ideals of let D be a submodule of X. Then AID = BID, but D is visible submodule which implies that AD = BD and hence A = B (since, D is strongly cancellation submodule).

Proposition (2.13): For each a nonzero ideal A of R and for each nonempty collection {W∝} of visible submodule of an R-module X. We have $A(\cap_\propto W_\propto) = \cap_\propto AW_\propto$.

Proof: It is known that for each $\cap_\propto W_\propto \subseteq W_\propto$ but $W_\propto$ is visible submodule for each ∝ and hence, $AW_\propto$ for each ∝ also by proposition (2.7), we get $\cap_\propto W_\propto$ is visible submodule of $\cap_\propto W_\propto$ of X.

Implies $\cap_\propto AW_\propto = \cap_\propto W_\propto = A(\cap_\propto W_\propto)$ (since, $W_\propto$ is visible submodule for each ∝).

Proposition (2.14): Let N be a visible submodule of an R-module X. Then, N is pure submodule of X.

Proof: Let N be a proper submodule of a module X. Then N = IN for every a nonzero ideal I of R. Since, ⊆X, then IN⊆IX. Therefore, N∩IX = N∩IX and hence, N∩IX = (N∩X) = IN by proposition (2.13). Which completes the proof.

Proposition (2.15): Let X be a multiplication cancellation R-module. Then every proper submodule N of X is visible submodule if and only if (N:X) is visible ideal of.

Proof: Suppose that (N:X) is visible ideal of X. Let x∈N. Then (x)⊆N and hence, ((x:X)⊆(N:X). Therefore, $((x):_R X)\subseteq(N:_R X)$ = I(N:X) and hence, $((x:_R X)X\subseteq 1(N:_R X)X$ which implies that (x)⊆IN (since, X is multiplication module).

Therefore, ∈IN and hence, N⊆IN also, it is known that IN⊆N. Thus, from two above inclusion, we have N = IN, that is N is visible submodule. Let N be a visible submodule to prove that (N:X) is visible ideal. Let $x\in(N:_R X)$. Then (x)X⊆N, implies (x)X⊆IN (since, N is visible submodu). Then (x)X⊆I(N:X). But X is cancellation module. Therefore, (x)X⊆I(N:X) and hence, (x)X∈I(N:X). Then (N:X)⊆I(N:X). Conversely, I(N:X)⊆(N:X). Therefore, (N:X)⊆I(N:X). This end the proof.

Corollary (2.16): Let N be a proper submodule of a (F.G) faithful multiplication R-module X. Then N is visible if and only if (N:X) is visible ideal of R.

Proof: From Ali (2005), we get X is cancellation and by proposition (2.15) we obtain the result.

Proposition (2.17): Let X be a FG faithful multiplication R-module and I be a proper ideal of R. Then the following hold:

- If I is visible ideal of R then IX is visible submodule of X
- If N is visible submodule of then ann(N)

Proof: Let I be a visible ideal of R. Then JI = I for each ideal J of R0 ≠J and hence, JIX = IX. Therefore, IX is visible submodule. Suppose that IX is visible submodule of X then JIX = IX (since, X is cancellation module because X is FG faithful multiplication module). Therefore, JI = I and hence, I is visible ideal of R let x∈ann(N:X). Then x(N:X) = 0 which implies xN = x(N:X)N = 0, therefore, x∈ann(N).

Now, let N be a visible submodule of X. Then N = IN for every ideal 0 ≠ I of and by proposition (2.14) , we have N is pure, from this fact, we write N = N∩IM for every ideal I of R. But N is visible, therefore, IN = N∩IM. Taking I = ann_R(N) and hence, nn(N)N = N∩ann(N). 0 = N∩ann(N)X. This lead us (0:X) = ((N∩ann(N)X:X) = (N:X)∩ann(NX:X) = (N:X)∩(IX:X) = (N:X)∩I (since, X is faithful FG and multipli. module) = (N:X)∩ann(N) = (N:X)ann(N) by proposition (2.15) and proposition (2.14) Then ann(X) = (N:X)ann(N). But X is faithful which implies that 0 = (N:X)ann(N). Therefore, ann(N)⊑ann(N:X). Which completes the proof.

Proposition (2.18): A visible submodule of an R-module X is an idempotent submodule.

Proof: N is visible submodule of X, then N = IN for every 0 ≠ I, I is an ideal of R thus, N is an idempotent (choose I = (N:$_R$ X)).

Proposition (2.19): Assume X is (F.G) faithful multiplication R-module and K is visible submodule of then $\cap_{k\in I}J_kK = (\cap_{k\in I}J_k)K$ for every a nonempty collection $J_k(k\in I)$ of visible ideal of R.

Proof: K is visible submodule of X, then by corollary (2.16), we have (K:X) is visible ideal of R. Suppose that $J_k(k\in I)$ is any collection of visible ideals of R. Now, $(\cap_{k\in I}J_k)K$ = K = (K:X) by proposition (2.18) which is equal (K:X) $(\cap_{k\in I}J_k)K = (\cap_{k\in I}J_k)$ (K:X)K = $(\cap_{k\in I}J_k)$ (K:X)AX for some ideal A of. (since, X is multiplication module), we want to show that $(\cap_{k\in I}J_kK:_R X) = \cap_{k\in I}J_k\,(K:_R X)$ obviously, $\cap_{k\in I}J_k(K:X)\sqsubseteq(\cap_{k\in I}J_kK:X)$. Conversely, let, $y\in(\cap_{k\in I}J_kK:X)$. Then $X\sqsubseteq\cap_{k\in I}J_kK = \cap_{k\in I}J_k(K:X)$ but we have X is cancellation module Therefore $y\in\cap_{k\in I}J_k(K:X)$.

$$\text{Now, } (\cap_{k\in I}J_k)(K:X)AX = (\cap_{k\in I}J_kK:XAX)$$

$$= A\left(\underset{k=1}{\cap}J_kK:X\right)M$$

$$= A\ k\ IJkK$$

But J_k is visible ideal for all k∈I, then by corollary (2.9), we get $\cap_{k\in I}J_k$ is visible ideal also by proposition (2.17) we obtain that $\cap_{k\in I}J_kK$ is visible, that is $(\cap_{k\in I}J_kK) = \cap_{k\in I}J_kK$ and hence, $(\cap_{k\in I}J_k)K = \cap_{k\in I}J_k$.

The visible radical of a submodule: During this study, the concept of visible radical of a submodule has been described. Also, we proved that the equality of the fourth condition of the concept of V_{CL} module is achieved with this type of module and without condition. Many properties and results of these concepts are given.

Definition (3.1): A visible radical of a submodule K of an R-module X, denoted by $Vrad_x(K)$ is defined as the intersection of all visible submodule of X which contain K. If there exists no visible submodule of X containing, we write $Vrad_x(K)$ = X. If X = and D is an ideal of R then $Vrad_x(D)$ is the intersection of all visible ideals of R containing D.

Definition (3.2): If D is an ideal of, then $\sqrt{D}$ is represent the intersection of all visible ideal containing D. The following results give some fundamental properties of visible radical.

Proposition (3.3): If θ:X→X be an epimorphism from an R-module X into R-module X, and H be a submodule of X with ker θ⊑K, then:

- $\theta(Vrad_xH) = Vrad_x\,\theta(H)$
- $\theta^{-1}(Vrad_xH) = Vrad_x\theta^{-1}(H)$, where H is a submodule of X

Proof: We have $(Vrad_xH)$ = ∩W where W is visible X with ⊑W, therefore, $\theta(Vrad_xH)$ = θ(∩W). Since, skerθ⊑H⊑W, and by Kasch (1982) we get $\theta(Vrad_xH)$ = ∩θ(W) where intersection over all visible submodule θW of X (the harmomorphic image of visible submodule is also visible. With θ(H)⊑θ(W) and hence, (i) is verified.

Let H be a submodule of X. Then $Vrad_x(H)$ = ∩W where ∩ is over all visible submodule W of X with H⊑W, then by proposition (2.14), $\theta^{-1}(Vrad_xH) = \theta^{-1}(\cap W) = \cap\theta^{-1}(W)$ where ∩ is over all visible submodule $\theta^{-1}(W)$ of X with $\theta^{-1}(H)\sqsubseteq\theta^{-1}(W)$. Hence, $\theta^{-1}(Vrad_xH) = Vrad_x(\theta^{-1}(H))$.

Proposition (3.4): Let, W be two submodule of R-module X Then:

- $k\sqsubseteq Vrad_xK$
- If ⊑W, then $Vrad_xK\sqsubseteq Vrad_xW$
- $Vrad_x(Vrad_xK) = Vrad_xK$
- $Vrad_xK\cap W\sqsubseteq Vrad_xK\cap Vrad_xW$
- $Vrad_xK+W = Vrad_x(Vrad_xK+Vrad_xW)$
- $Vrad_x(W) = Vrad_x(AW)$ for every visible submodule W of X and for every a nonzero ideal A of R
- $Vrad_x(W)$ for every a nonzero ideal A of R
- $Vrad_x(AW) = Vrad_x(A\ Vrad_xW)$

Proof: Since, $Vrad_xK = \cap P$, where the intersection is taken all visible submodule P of X with $K\sqsubseteq P$, also $K\sqsubseteq vradK$. Let P be a visible submodule of Xwith $\sqsubseteq P$ but we have $K\sqsubseteq W\sqsubseteq P$, therefore, $K\sqsubseteq P$ that is $Vrad_xK\sqsubseteq Vrad_xW$. Since, $Vrad_x(Vrad_xK) = \cap P$ where the intersection is taken on all visible submodule Pof X with $Vrad_xK\sqsubseteq P$ and from (Eq. 1), $K\sqsubseteq Vrad_xK$, then directly $Vrad_x(Vrad_xK)\sqsubseteq Vrad_xK$. Also by (Eq. 1) we obtain $Vrad_xK\sqsubseteq Vrad_x(Vrad_xK)$. Thus, the equality holds.

It is clear that $K\cap W\sqsubseteq W$ and $\cap W\sqsubseteq K$, then by (Eq. 2), we obtain $Vrad_x(K\cap W)\sqsubseteq Vrad\ K_x$ and $Vrad\ (K\cap W_x)$. Therefore, $Vrad_x(K\cap W)\sqsubseteq Vrad_xK\cap Vrad_xW$. We have $K\sqsubseteq vrad_xK$ and $W\sqsubseteq vrad_xW$. Then $K+W\sqsubseteq Vrad_xK+Vrad_xW$. Also by (Eq. 2), we get $Vrad_x(K+W)\sqsubseteq Vrad_x(Vrad_x+Vrad_xW)$.

Now, to prove another inclusion, let P be a visible submodule of X such that $K+W\sqsubseteq P$ from this step with $K\sqsubseteq P$ we get $W\sqsubseteq P$. Therefore, $Vrad_xK\sqsubseteq P$ and $Vrad_xW\sqsubseteq P$. Thus, $Vrad_xK+Vrad_xW\sqsubseteq P$ and consequently, $Vrad_x(Vrad_xK+Vrad_xW)\sqsubseteq P$. Thus, $Vrad_x(Vrad_xK+Vrad_xW)\sqsubseteq Vrad_x(K+W)$. Therefore, $Vrad_x(Vrad_xK+Vrad_xW) = Vrad_x(K+W)$.

It is clear that $W\sqsubseteq W$, then by using No. (Eq. 2), we get $Vrad_xAW\sqsubseteq Vrad_xW$. Another inclusion:let $Vrad_xW = \cap_{W\sqsubseteq P}$ where P is a visible submodule of X. Therefore, by proposition (2.7), we have also $\cap_{W\sqsubseteq P}$ is visible submodule of X implies W = AW for every a nonzero ideal A of R, therefore, $Aw\sqsubseteq P$ hence, the intersection over visible submodule of X containing AW which gives the visible radical of AW that is $Vrad_x(AW) = \cap_{W\sqsubseteq P}P$ and hence, $Vrad_x(W)\sqsubseteq Vrad_x(AW)$. Thus, $Vrad_x(W) = Vrad_x(AW)$. $Vrad_xW = \cap_{W\sqsubseteq P}P$ where P is visible submodule but W is also visible by proposition (2.7) and hence, W is pure submodule by proposition (4), we get $AW = W\cap AX$ for every ideal A of R. And hence, $Vrad_x(AW) = Vrad_x(W\cap AX)$. And form No. (6), we get $Vrad_x(AW) = Vrad_x(W\cap AX)$. By depending on (Eq. 1), we get $W\sqsubseteq Vrad_xW$, implies $AW\sqsubseteq A\ Vrad\ W_x$ and hence, $Vrad_x(AW)\sqsubseteq Vrad_x(A\ VradW)$.

Conversely: We have $AVrad_x\sqsubseteq Vrad_x(AW)$ (since, W is visible submodule this leads to use ( ). Therefore, $Vrad_x(AVrad_xW)\sqsubseteq Vrad_x(Vrad_x(AW))$. Thus, the equality holds. Immediate form proposition (3.4), we get the coming corollary.

Corollary (3.5): Let K be a submoduleof an R-module X. Then we have:

- $Vrad_xK\sqsubseteq Vrad_xK(S)$
- $Vrad_xK\sqsubseteq Vrad_x[K:_RI]$ for every ideal I of R

Proof: Since, K (S) is a submodule of X and $\sqsubseteq K(S)$ also for every ideal I of R we have $K\sqsubseteq[K:_RI]$. Then the result follows directly by proposition ((3.4), No. (Eq. 2)). In the following proposition we give a condition under it the equality f proposition ((3.4) (Eq. 4)) holds.

Proposition (3.6): Let, W be two submoduleof an R-module X if every visible submodule P of P which contain $K\cap W$ is completely irreducible. Then $Vrad_x(K\cap W) = Vrad_xK\cap Vrad_xW$.

Proof: From proposition ((3.4) (Eq. 4)) we obtain $Vrad_x(K\cap W)\sqsubseteq Vrad_xK\cap Vrad_xW$. Now, to prove another side, if $Vrad_x(K\cap W) = X$, then $Vrad_xK = VradW = X$. If $Vrad_x(K\cap W)\neq X$, then $\exists$ a visible submodule P of X s.t $K\cap W$ but P is completely irreducible submodule, then either $K\sqsubseteq P$ or $W\sqsubseteq P$ and hence $Vrad_xK\sqsubseteq P$ or $Vrad_xW\sqsubseteq P$. Since every visible submodule containing $K\cap W$ is completely irreducible then $Vrad_xK\sqsubseteq Vrad_x(K\cap W)$ or $Vrad_xW\sqsubseteq Vrad_x(K\cap W)$ and hence, $Vrad_xK\cap Vrad_xW\sqsubseteq Vrad_x(K\cap W)$. Therefore, $Vrad_x(K\cap W) = Vrad_xK\cap Vrad_xW$.

Proposition (3.7): If X is a (F.G) faithful multiplication R-module and T is visible submodule of X, then $T = \sqrt{(T:X)}\,T$.

Proof: Let F be the set of all visible ideals P of R that contain (T:M). Therefore, (T:M). And hence by proposition (2.19), we get $\sqrt{(T:X)}\,T = (\cap_{P\in F}P)T = \cap_{P\in F}PT$. Now, for each visible ideal P of R we can write T = PT (since, P is visible) also for each $P\in F$, $T = (T:X)T\sqsubseteq PT\sqsubseteq T$. Therefore, $K = \cap_{P\in F}PT$ (since, $\cap_{P\in F}P$ is visible ideal of), then it is equal to $\sqrt{(T:X)}\,T$. Hence, $T = \sqrt{(T:X)}\,T$.

Proposition (3.8): If S is a visible ideal of a ring R, then $S = S\sqrt{(S)}$.

Proof: We have $S\sqsubseteq\sqrt{(S)}$, then $S.S = S\sqrt{(S)}$ but S is visible, then S is an idempotent. Therefore, $\sqsubseteq S\sqrt{(S)}$.

Conversely: $S\sqrt{(S)}\subseteq S\cap\sqrt{(S)} = S$ (since, $\sqsubseteq\sqrt{(S)}$) that is $S\sqrt{(S)}\sqsubseteq S$ and hence, $S = S\sqrt{(S)}$.

Proposition (3.9): Let T be a submodule of FG faithful multiplication R-module. Then $T = \sqrt{(T:X)}\,X = Vrad_xT$.

Proof: When $Vrad_xT = X$, the results is end. Otherwise, if P is any visible submodule of X which contains T, then $(T:X)\sqsubseteq(P:X)$ but P is visible submodule, then proposition (2.15), (P:M) is visible ideal of R and hence by proposition (2.7), we get (T:M) is visible ideal of R. Therefore, $(T:X) = \sqrt{(T:X)}(T:X)$ form proposition (3.8). Which implies that $(T:X)\sqrt{(T:X)}$ is equal to (T:M) which contains in (P:M). And hence, $(T:X)^2\sqrt{(T:X)}$ which inclsion in (P:X) (T:X), (since, every visible ideal is idempotent). Therefore, $\sqrt{(T:X)}(T:X)$ which contains in (P:X) (T:X)X. Since, (T:X)X is a submodule of X and by (Elewi, 2016) we get $\sqrt{(T:X)}\sqsubseteq$(P:X) and hence, $\sqrt{(T:X)}X$ which contains in (P:X)X = P. Since, P is any arbitrary visible submodule containing T, then we obtain $\sqrt{(T:X)}X \sqsubseteq Vrad_xT$.

Conversely: We have X is multiplication module, $Vrad_xT$ = ($Vrad_xT$:X)X. Since, T is visible submodule hence, by proposition (2.15) we have (T:X) is visible ideal of R. To show that ($Vrad_xT$:X)$\sqsubseteq\sqrt{(T:X)}$. Let P be any visible ideal such that (T:X)$\sqsubseteq$P. Look, P is visible ideal , then from proposition (2.17) PX is visible submodule of X containing T = (T:X)X. To prove this let x$\in$T. Then x$\in$(T:X)X. Therefor, (T:X)x$\sqsubseteq$(T:X)^{2}X = (TX)X. And hence, P(T:X)x$\sqsubseteq$(T:X)X = P^2(T:X)PX which implies that x$\in$PX (since, P(T:X) is an ideal of and X is fully cancellation module). That is T$\sqsubseteq$X. Thus, ($Vrad_xT$:X)X = $Vrad_xT\sqsubseteq$PX. Hence, ($Vrad_xTX$)$\sqsubseteq$(PX:X) = P (since, X is cancellation module). Consequently, ($Vrad_xTX$)$\sqsubseteq\sqrt{(T:X)}$. The result end.

Proposition (3.10): Let X be a (F. G) faithful multiplication-module. T be a visible submodule of X Then:

- $T = \sqrt{(T:X)}T$
- (T:X)$Vrad_xT$ = T = ($Vrad_x$ T:X)T
- If (T:X) is (F.G) (principle ideal generated by idempotent element), then $Vrad_xT$ is a visible submodule of X and moreover, T = VradT

Proof: K is submodule of then by proposition (2.18), we get that, T is an idempotent ideal of R, therefore, T = (T:X)T, hence, $\sqrt{(T:X)}T = \sqrt{(T:X)}(T:X)T$. And by proposition (3.7) we obtain $\sqrt{(T:X)}T = (T:X)T = T$. It follows from No. (Eq. 1) and proposition (3.7) we get $T = \sqrt{(T:X)}:T$ is equal to $\sqrt{(T:X)}(T:X)X = (T:X)\sqrt{(T:X)}X$ is equal to (T:X)VradT. Suppose that (T:X) is (F.G) ideal of R.

Therefore, $(T:X)\sqrt{(T:X)}$ by" [on radicals of submodules of F.G modules]", hence, $(T:X)X = \sqrt{(T:X)}X = VradT$. Now, we will introduce the concept of Vclosure operation (for short V_{CL} operation). Let X be an R-module and S be the set of all visible submodules of q:S→S we call H a V_{CL} operation if:

- q$\sqsubseteq$q(G)
- q(q(G$\sqsubseteq$q(G)
- G$\sqsubseteq$K, implies q(G)$\sqsubseteq$q(K)
- Aq(G) = q(AG)

For all nonzero ideals A of R and submodules G, K of X. Next, we give a characterization for V_{CL} operation.

Proposition (3.11): A mapping q:S→S is a V_{CL} operation if and only if q(X):q(B) for all X, B$\in$S.

Proof: Suppose that q is V_{CL} operation. Since, $\sqsubseteq$q(B), then q(X):q(B)$\sqsubseteq$q(X):B for all X, B$\in$S. Another inclusion q(X)$\sqsupseteq$(q(X):B):B)$\sqsupseteq$((q(X):B):q(B)). Thus, (q(X):q(B))$\sqsupseteq$((q(X):B). Therefore, (q(X):q(B)) = (q(X):B). On the opposite side: for all, B$\in$S we have (q(X):h(B)) = (q(X):B). To prove q is V_{CL} operation. Put = B, then (q(X):q(X)) = R. Therefore, X$\sqsubseteq$q(X) for all X$\in$S.

Now, put = q(X), then (q(X):q(q(X)) = (q(X):q(X)) Therefore, q(q(X)) = q(X) for all X$\in$S. Next if $\sqsubseteq$X, then (q(X):q(B)) = (q(X):(q(X):B)$\sqsupseteq$(X:B) = R and hence, q(X)$\sqsupseteq$q(B). In the last, we have X$\sqsubseteq$q(X) but X is visible submodule, then IX = q(IX) for each a nonzero ideal I of R. Therefore, (q(IX):q(X)) = (q(IX):X) = (q(X:X)$\sqsupseteq$(X:X) = R (since, X is visible submodule) form (Eq. 1), thus, (q(IX):q(X) = R. And hence, q(X)$\sqsubseteq$q(IX)(X is visible submodule, then q(X) is also visible submodule). This lead to Iq(X)$\sqsubseteq$q(IX).

Conversely: From (Eq. 1), we get (X)$\sqsubseteq$q(X). Then I(X)$\sqsubseteq$q(X). For each a nonzero ideal I of R. And hence, q(IX)$\sqsubseteq$q(q(X)) = q(X). Therefore, q(IX)$\sqsubseteq$Iq(X) (since (q(X) is visible submodule). Thus, we obtain a(IX). Finally, we get h is V_{CL} operation.

Proposition (3.12): Let h_λ:S→S where ($\lambda\in\wedge$) be a family of V_{CL} operation and h(W) = $\cap_{\lambda\in\wedge}h_\lambda(W)$ for all W$\in$S. Then h:S→S is a V_{CL} operation.

Proof: We have W$\sqsubseteq h_\lambda$(W) for all, then W = $\cap_{\lambda\in\wedge}h_\lambda(W)$ and hence, W$\sqsubseteq$(W). In particular h(W)$\sqsubseteq$h(h(W). And the opposite:

$$h_\lambda(W) = h_\lambda(h_\lambda(A) \supseteq h_\lambda(\bigcap_{\lambda\in\wedge} h_\lambda) =$$
$$h_\lambda(h(W) \supseteq \bigcap_{\lambda\in\wedge} h_\lambda(W) = h(h(W)$$

Therefore, h(h(W)$\sqsubseteq$h(W). And hence, h(h(W = h(W). Now, if $\sqsubseteq$K, then $h_\lambda(W)\sqsupseteq h_\lambda(K)$ implies, h(W)$\sqsupseteq$h(K). In the end Ih(A) = I$\cap_{\lambda\in\wedge}h_\lambda(A)$ = $\cap I_{\lambda\in\wedge}h_\lambda(A)$ by proposition (3.12) but $Ih_\lambda(A)$ = h_λ(IA)(h_λ(A) is V_{CL} operation. Therefore, Ih(A) = $\cap_{\lambda\in\wedge}h_\lambda(IA)$ = h(IA). This complete the proof.

Proposition (3.13): Let h:S→S be a V_{CL} operation. Then:

- $h(\cap_{\lambda\in\wedge}A_\lambda)\sqsubseteq(\cap_{\lambda\in\wedge}h(A_\lambda) = h(\cap_{\lambda\in\wedge}h(A_\lambda))$
- $\sum_\lambda h(A_\lambda)\sqsubseteq h(\sum_\lambda A_\lambda) = h(\sum_\lambda h(A_\lambda))$
- h(A:I)$\sqsubseteq$h(A):I = h(h(A):I)

Proof: Since, $\cap W_\lambda$ for all, so, $h(\cap_\lambda W_\lambda)$ for all λ and $h(\cap_\lambda W_\lambda)\sqsubseteq\cap_\lambda h(W_\lambda)\sqsubseteq h(\cap_\lambda h(W_\lambda)$. Then $h(\cap_\lambda W_\lambda)\sqsubseteq\cap_\lambda h(W_\lambda))$ = $\cap_\lambda h(W_\lambda)$. $W_\lambda\sqsubseteq\sum W_\lambda$, so, $W_\lambda\sqsubseteq h(W_\lambda)\sqsubseteq h(\sum_\lambda W_\lambda)$ for all λ.

And $\sum_{\lambda} W_{\lambda} \sqsubseteq \sum_{\lambda} h(W_{\lambda}) \sqsubseteq h(\sum_{\lambda} W_{\lambda})$. Therefore, $h(\sum_{\lambda} W_{\lambda}) \sqsubseteq h(\sum_{\lambda} h(W_{\lambda}) \sqsubseteq h(h(\sum_{\lambda} W_{\lambda}) = h(\sum_{\lambda} W_{\lambda})$. Since, $\sqsupseteq I(A:I)$, then $h(A) \sqsupseteq h(I(A:I))$. Now, $h(A:I) \sqsubseteq (h(A:I) \sqsubseteq h(h(A):I \sqsubseteq (h(h(A)):I) = (h(A):I$. Therefore, $(h(A):I) \sqsubseteq (h(A):I)$ and $(h(A):I) = (h(A):I)$ and (Eq. 3) follows.

Proposition (3.14): Let X be an R-module and $h:S \rightarrow S$ such that $h(N) = VradN$ for every $N \in S$ and N is visible radical submodule of X. Then h is V_{CL} operator.

Proof: Form proposition (3.4), we get (Eq. 1-3) which are conditions of definition of closure operation. It remains to achieve the last condition we have $Vrad_x N = Vrad_x(AN)$ for every a nonzero ideal A of R but N is visible radical submodule that is VradN = N. Then $Vrad_x(AN) = Vrad_x N = N = AN = AVrad_x N$. Therefore, h is V_{CL} operator.

Corollary (3.15): X is a module over R and h is defined in proposition (3.14). Let, L be submodule of X and A is a nonzero ideal of. Then:

- $(Vrad_x N: Vrad_x L) = (Vrad_x N:L)$
- $Vrad_x(N:A) \sqsubseteq Vrad_x N:A =$

Proof: We have $h(N) = Vrad_x N$. Then $(Vrad_x N: Vrad_x L) = h(N):h(L)$ but from proposition (3.13), $h(N):h(L) = h(N):L$. Therefore, $(Vrad_x N: Vrad_x L) = h(N):L = = (Vrad_x N:L)$. $Vrad_x(N:A) = h(N:A) \sqsubseteq h(N):A = Vrad_x N:A$. And $h(N):A = h(h(N):A) = Vrad_x(Vrad_x(N):A)$. Therefore, (Eq. 2) holds.

CONCLUSION

During this study, we are dealing with commutative rings that contain an identity element as well as all the modules here are unitary.

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