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A new Modulus of Smoothness of k-Monotone Functions in L_p Saces for p<1

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Abstract: The degree of best approximation utilizing classical modulus of smoothness isn't uniform. Additionally, we now and then need to enhance the degree of best approximation close to the end points. Subsequently, we have to enhance this traditional modulus of smoothness. Before, we characterize another modulus of smoothness to accomplish uniform degree of best estimate and a change of a level of such form of best approximation. Our modulus of smoothness is for k-monotone functions. Appraisals for utilizing our modulus of smoothness are presented. Applications for these estimates are likewise presented before.

Key words: Approximation, classical modulus, traditional, smoothness, k-monotone functions, Ditizian-Totik

INTRODUCTION

Uniform estimate of polynomial utilizing traditional moduli of smoothness is defective as we can't enhance the rate of best approximation close to the endpoints of the interval see for instance (Dzyadik, 1977). Consequently, we utilize the pointwise polynomial estimate in the uniform standard space. In the event that, we require correct uniform polynomial estimation in L_p spaces for $p<\infty$, we can utilize the new modulus of smoothness, (Ditzian and Totik, 1987), for example. The most vital modulus of smoothness that have the most consideration as of late is the Ditzian-Totik modulus of smoothness (Ditzian and Totik, 1987). Likewise the Ivanov modulus of smoothness (Ivanov, 1980). We introduced estimates as properties of kth Ditizian-Totik L_q moduli of smoothness of k-monotone function in $L_p[-1, 1]$ spaces for $0<q<p\leq 1$. Also, we relate our new modulus to kth Ditizian-Totik modulus of smoothness. We characterize a few kinds of moduli of smoothness of k-monotone in L_p, spaces for p<1. We introduce applications for these estimates. Let $L_p[a, b]$ p<0 denote the space of f: [a, b]→R:

$$\|f\|_p = \|f\|1_{p[a,b]} = \left[\int_a^b |f(x)|^p dx\right]^{1/p},\ 0<p<\infty$$

The kth divided difference (Bhaya, 2003) of functions defined on the interval [a, b] is:

$$f[x_0, x_1, ..., x_k] = \sum_{j=0}^{k} f(x_j)/\omega'(x_j)$$

Where:

$$\omega(x) = \prod_{j=0}^{k}(x\text{-}x_j) \text{ for } k\in\{0\}\cup N$$

The k-monotone function (Bhaya, 2003) defined on the interval [a, b] those function f: [a, b]→R is said to be k-monotone, $k\geq 1$ on [a, b] if and only if for all choices of k+1 distinct points $x_0, x_1, ..., x_k$ in [a, b] the inequality $f[x_0, x_1, ..., x_k]>0$ holds where:

$$f[x_0, x_1, \cdots, x_k] = \sum_{j=0}^{k} \frac{f(x_j)}{\omega'(x_j)}$$

Let us now define the new measure of smoothness. For $f\in L_p$ [a, b], we have:

$$\gamma_\delta(f, q, p) = \delta^{q\text{-}p}$$

The auxiliary results: In this study, we introduce our results that, we need them in our main theorems.

Lemma 2.1 (Kopotun, 2001): Let $0<\alpha<1$ and $-\alpha\leq\beta\leq\alpha$ then for any integrable function f, we have:

$$\int_{h\alpha} f(x+\beta\varphi(x))dx = \frac{1}{1+\beta^2}\int_{1+2\alpha(\alpha+\beta)/(1+\alpha^2}^{1-2\alpha(\alpha-\beta)/(1+\alpha^2} f(y)\left(1+\frac{\beta y}{\sqrt{1-y^2+\beta^2}}\right)dy$$

Where:

$$h_\alpha = \{x : x\pm\alpha\varphi(x)\in[-1, 1]\} = \{x : |x| \leq 1-\alpha^2/1+\alpha^2\}$$

Then:

$$\Delta^k_{h\varphi(x)}(f, x) = 0 \text{ if } x\in^{kh/2}$$

Corollary 2.2: Let, k = 1 or k = 2, $0<p<q\leq 1$ and let, $f\in\Delta^k\cap L_p$ be nonnegative. Then:

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$$\omega_k^{\varphi}(f, \delta)_p \le c(p, k)\|f\|_q \delta^{1-p}$$

Lemma 2.3: If f_1 and f_2 satisfying the condition, if $f\epsilon\Delta^k L_p$ then:

$$\omega_k^{\varphi}(f, g)_p \le c(p)\gamma_\delta(f, q, p)\|f\|_q$$

$$f_1(x), f_2(x) \ge 0,\ x \in [-1, 1]$$

Then, so as f_1+f_2.

Proof : Let $f\epsilon\Delta^k L_p$ satisfied, $0<p<q\le 1$:

$$f_1(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ f(x) & \text{if } 0 \le x \le 1 \end{cases},\ f_2(x) = \begin{cases} f(x) & \text{if } -1 \le x \le 0 \\ 0 & \text{if } 0 < x \le 1 \end{cases}$$

$$(f_1(x) f_2(x)) = 0$$

$$f_1+f_2 = f(x),\quad x \in [-1, 1]$$

$$\|f_1\|_p + \|f_2\|_p \le 2\|\max(|f_1|, |f_2|)\| \le 2\|\,|f_1| + |f_2|\,\|_p = 2\|f_1+f_2\|_p$$

We have :

$$\begin{aligned} &\omega_k^{\varphi}(f_1+f_2, \delta)_p \le \omega_k^{\varphi}(f_1, \delta)_p + \omega_k^{\varphi}(f_2, \delta)_p < \\ &c(p, k)\gamma_\delta(f, q, p)\|f_1\|_q + c(p, k)\gamma_\delta(f, q, p)\|f_2\|_q \le \\ &c(p, k)\gamma_\delta(f, q, p)(\|f_1\|_q + \|f_2\|_q) \le \\ &c(p, k)\gamma_\delta(f, q, p)\|f_{1q} + f_{2q}\| \end{aligned} \quad (2.1)$$

RESULTS AND DISCUSSION

The main results: Here we introduce our main results

Lemma 3.1:

$$\text{Let } k \in N,\ \left\|(1-y^2)^{-\frac{k}{2}}\right\|_{L_p[-1+k^2h^2/2,\,1-k^2h^2/2]} \le c(p, k)h^{-k+p}$$

Proof:

$$\left\|(1-y^2)^{-\frac{k}{2}}\right\|_{L_p[-1+k^2h^2/2,\ 1-k^2h^2/2]} \le c(p, k)\left\|(1-y^2)^{-\frac{k}{2}}\right\|_{L_p[-1+k^2h^2/2,\ 1-k^2h^2/2]} =$$

$$\left(\int_{-1+k^2h^2/2}^{1-k^2h^2/2} \left|(1-y^2)^{-\frac{k}{2}}\right|^p dy\right)^{1/p} =$$

$$\left(\int_{-1+k^2h^2/2}^{1-k^2h^2/2} (1-y^2)^{-\frac{kp}{2}} dy\right)^{1/p} =$$

$$\left(\int_{-1+k^2h^2/2}^{1-k^2h^2/2} \left(1-\frac{k^2h^2}{2}\right)^{-\frac{kp}{2}} dy\right)^{1/p} \le$$

$$\left(\left(1-\frac{k^2h^2}{2}\right)^{-\frac{kp}{2}} [y]_{-1+k^2h^2/2}^{1-k^2h^2/2}\right)^{1/p} \le$$

$$\left(\left(1-\frac{k^2h^2}{2}\right)^{-\frac{kp}{2}} \left[(1-k^2h^2/2)-(-1+k^2h^2/2)\right]\right)^{1/p} \le$$

$$c(p)\left(\left(\frac{k^2h^2}{2}\right)^{-\frac{kp}{2}} \left[(1-k^2h^2/2)-(-1+k^2h^2/2)\right]\right)^{1/p} \le$$

$$c(p)\left(\left(\frac{k^2h^2}{2}\right)^{-\frac{kp}{2}} \left[2-\frac{k^2h^2}{2}\right]\right)^{1/p} \le$$

$$c(p)\left(\left(\frac{k^{-kp}h^{-kp}}{2}\right)\left[2-k^2h^2\right]\right)^{1/p} \le$$

$$c(p)\left(k^{-kp}h^{-kp} - \frac{k^{-kp+2}h^{-kp+2}}{2}\right)^{1/p} \le$$

$$c(p)\left(k^{-kp}(h^{-kp}\frac{k^2h^{-kp+2}}{2}\right)^{\frac{1}{p}} \le$$

$$c(p, k)\left(h^{-kp} - k^2h^{-kp+2}/2\right)^{\frac{1}{p}} \le$$

$$c(p, k)\left(h^{-kp} - k^2h^{-kp+2}/2)\right)^{\frac{1}{p}}\ 0<p\le 1 \le$$

$$c(p, k)\left(h^{-kp+hP^2}\right))^{\frac{1}{p}} \le$$

$$c(p, k)\left(h^{-kp+P^2}\right))^{\frac{1}{p}} \le$$

$$c(p, k)h^{-k+p}$$

Corollary 3.2: Let $k\epsilon N$, $f\epsilon\Delta^k \cap L_p$, $0<p, q\le 1$, then:

$$\omega_k^{\varphi}(f, \delta)_p \le c(p, k)\|f\|_q \delta^{-k+p}$$

Lemma 3.3: Let $0<q<p\le 1$ and let $f\in L_q$ be nonnegative on [-1, 1]. Then:

$$\omega_1^{\varphi}(f,\delta)_p \le c(p)\omega_1^{\varphi}(f^q,\delta)_p \quad (3.1)$$

Proof:

$$(a+b)^q \le a^q+b^q \le 2^{p-q}(a+b)^q,\ a\ge 0,\ b\ge 0 \text{ and } q>0 \quad (3.2)$$

$$|a_1-a_2|^q \le c(q)|a_1^q-a_2^q|,\ a_1\ge 0,\ a_2\ge 0 \text{ and } q\ge 0$$

For any nonnegative function f, we have:

$$|\Delta_\mu^1(f,x)|^p \le c(p)|\Delta_\mu^1(f^q,x)|^p$$

$$\omega_k^{\varphi}(f,\delta)_p \le c(q)\omega_1^{\varphi}(f^q,\delta)_p$$

This implies:

$$\omega_1^{\varphi}(f,\delta)_q \le c(q)\omega_1^{\varphi}(f^q,\delta)_p$$

Lemma 3.4: Let $0<p<q\le 1$ and let $f\in\Delta^2\cap L_q$ be nonnegative on [-1, 1]. Then:

$$\omega_2^{\varphi}(f,\delta)_q \le c(p)\omega_2^{\varphi}(f^q,\delta)_p \quad (3.3)$$

Proof If $a_1\ge 0$, $a_2\ge 0$, $a_3\ge 0$ and $a_1-2a_2+a_3\ge 0$ and $q\ge 0$, then using (3.2.2) we have:

$$(a\text{-}2b+c)^q+(2b)^q \le c(q)(a+c)^p \le c(p)(a^q+c^q)$$

$$(a\text{-}2b+c)^q \le c(p)(a^q+b^q+c^q)$$

$$\Delta_\mu^2(f,x)^q \le c(p)\Delta_\mu^2(f^q,x)^p$$

$$\omega_k^{\varphi}(f,\delta)_q \le c(p)\omega_2^{\varphi}(f^q,\delta)_p$$

This implies:

$$\omega_2^{\varphi}(f,\delta)_p \le c(p)\omega_2^{\varphi}(f^q,\delta)_p$$

Lemma 3.5 (Kopotun, 2008): Let $k\in N$, $0\le p\le\infty$ and $f\in\Delta^k\cap L_p$. Denote by $T_{k1}(f,x)=\sum_{i=0}^{k-1}(i!)^{-1}f^i(0)x^i$ the Mclaurin polynomial of degree $\le$k-1 where $f^{k-i}(0)=f_-^{(k-i)}(0)$. Then, there exists a constant c = c(p, k) such that:

$$\|f-T_{k-1}(f,.)\|_p \le c\|f\|_p$$

Theorem 3.6: If $k\in N$, $0<q<p\le 1$, $f\in\Delta^k L_p$, then:

$$\omega_k^{\varphi}(f,\delta)_p \le c(p,k)\gamma_\delta(f,q,p)\|f\|_q$$

Proof : Let $f\in\Delta^k L_p$ satisfying $f^{(i)}(0)=0$, $0\le i\le k\text{-}2$ and $f^{(k-i)}(0)=0$. Then, as is easily shown by induction:

$$f\in\Delta^j[0,1] \text{ and } (-1)^{k-j}f\in\Delta^j[0,1]\ \forall j=0,\dots,k-1$$

Now let:

$$g(x)=\begin{cases}0 \text{ if } -1\le x\le 0\\ f(x) \text{ if } 0\le x\le 1\end{cases}$$

$$h(x)=\begin{cases}f(x) \text{ if } -1\le x\le 0\\ 0 \quad \text{if } 0<x\le 1\end{cases}$$

Now the function g, $h\in[-1,1](g(x)h(x))=0\ \forall x$ and that f = g+h. If k = 1, then g(x) and -h(x) are both nonnegative function in Δ^1L_p and $k\ge 2$, then g(x) and $(-1)^k h(x)$ are both nonnegative functions in Δ^2L_p, therefore, by Corollary 3.2 and (3.1), we have:

$$\omega_k^{\varphi}(g+h,\delta)_p \le \omega_k^{\varphi}(g,\delta)_p+\omega_k^{\varphi}(h,\delta)_p \le$$
$$c(p,k)\delta^{1-p}\|g\|_q+c(p,k)\delta^{1-p}\|h\|_q \le$$
$$c(p,k)\delta^{1-p}\|g+h\|_q \le$$
$$c(p,k)\gamma_\delta(f,q,p)g+h_q$$

This implies that f = g+h then:

$$\omega_k^{\varphi}(f,\delta)_p \le c(p,k)\omega_k^{\varphi}(f,\delta)_p \text{ for } k\ge 2$$

Theorem 3.7: If $k\in N$, $f\in\Delta^k\cap L_p$ and $\delta\le 1/k$. If k is even, then:

$$\omega_k^{\varphi}(f,\delta)_p \le c(p,k)\left(\|f\|_{L_p[-1,-1+k^2h^2]}+\|f\|_{L_p[1-k^2h^2,1]}+\delta^{kp}\|f\|_{L_p}\right)$$

If is odd, then:

$$\omega_k^{\varphi}(f,d)_p \le c(p,k)\Bigg(\|f\|_{L_p[-1,-1+k^2\delta^2]}+\|f\|_{L_p[1-k^2\delta^2,1]}+$$
$$\sup_{0<h\le\delta}h^{kp}\left\||f(y)|^p\left|(1-y^2)^{-k/2}\right|^p\right\|_{L_p[-1+k^2h^2,1-k^2h^2]}\Bigg)$$

Proof: We have $\Delta_{h\varphi(x)}^k(f,x)\ge 0$ for any x and by Lemma 3.1 with $\alpha=kh/2$ and $(i\text{-}k/2)h$, $0\le i\le k$, we get:

$$\|\Delta_{h\varphi(x)}^k(f,x)\|_p^p=\int_{kh/2}|\Delta_{h\varphi(x)}^k(f,x)|^p dx=$$

$$\int_{h\frac{kh}{2}}\left|\sum_{i=0}^k\binom{k}{i}(-1)^{k-i}f(x+(i-k/2)h\varphi(x)\right|^p dx \le$$

$$\int_{-1+4kih^2/(4+k^2h^2)}^{1-\frac{4(k-i)h^2}{4+k^2h^2}}\sum_{i=0}^k\left|\binom{k}{i}(-1)^{k-i}\right|^p$$

$$\left|f(x+(i-k/2)\right|^p \left|h\varphi(x)\right|^p dy \le$$

$$\sum_{i=0}^{k}\left(\binom{k}{i}\right)^p \int_{-1+4kih^2/(4+k^2h^2)}^{1-4(k-i)h^2/(4+k^2h^2)} \left|f(y)\right|^p \left|S(i-k/2)h, y)\right|^p dy \le$$

$$\sum_{i=0}^{k}\left(\binom{k}{i}\right)^p \left(\int_{-1+4kih^2/(4+k^2h^2)}^{-1+4(k-i)h^2/(4+k^2h^2)} + \int_{-1+4k^2h^2/(4+k^2h^2)}^{1-4k^2h^2/(4+k^2h^2)} + \int_{1-4k^2h^2/(4+k^2h^2)}^{1-4k(k-i)h^2/(4+k^2h^2)}\right) \left|f(y)\right|^p \left|S(i-k/2)h, y)\right|^p dy \le$$

$$\sum_{i=0}^{k}\left(\binom{k}{i}\right)^p (S_1+S_2+S_3)$$

Since, $|S(i-k/2)h, y)|^p \le 2^p\ 0\le i\le k$, we have:

$$\sum_{i=0}^{k}\binom{k}{i}S_1+S_3 \le 2^p \sum_{i=0}^{k}\left(\binom{k}{i}\right)^p \left(\int_{-1+4kih^2/(4+k^2h^2)}^{-1+4(k-i)h^2/(4+k^2h^2)} + \int_{1-4k^2h^2/(4+k^2h^2)}^{1-4k(k-i)h^2/(4+k^2h^2)}\right)$$

$$\left|f(y)\right|^p dy \le 2^{(k+1)p}\left(\|f\|_{L_p[-1,-1+4k^2h^2/(4+k^2h^2)]} + \|f\|_{L_p[1-4k^2h^2/(4+k^2h^2),1]}\right) \le c(p,k)\left(\|f\|_{L_p[-1,-1+k^2h^2} + \|f\|_{L_p[1-k^2h^2,1}\right) \quad (3.4)$$

Now:

$$\sum_{i=0}^{k}\binom{k}{i}S_2 \le \int_{-1+4k^2h^2/(4+k^2h^2)}^{1-4k^2h^2/(4+k^2h^2)} \left|f(y)\right|^p \left|A_k(y,h)\right|^p dy \quad (3.5)$$

Where:

$$\left|A_k(y,h)\right|^p \le \sum_{i=0}^{k}\left(\binom{k}{i}\right)^p \left|S((i-k/2)h, y)\right|^p =$$

$$\sum_{i=0}^{k}\left(\binom{k}{i}\right)^p \left|\frac{1}{1+(i-k/2)^2h^2}\frac{(i-k/2)hy}{\sqrt{1-y^2+(i-k/2)^2h^2}}\right|^p \quad (3.6)$$

If we assume j = k-I, we obtain:

$$\left|A_k(y,h)\right|^p \le \sum_{i=0}^{k}\left(\binom{k}{i}\right)^p \left|\frac{1}{1+(i-k/2)^2h^2}\frac{(i-k/2)hy}{\sqrt{1-y^2+(i-k/2)^2h^2}}\right|^p \quad (3.7)$$

Therefore, adding (3.6) and (3.7), we have:

$$\left|2A_k(y,h)\right|^p = \sum_{i=0}^{k}\left|\binom{k}{i}\frac{(-1)^i}{1+(i-k/2)^2h^2}\right|^p \left|1+\frac{(i-k/2)hy((-1)^k-1)}{\sqrt{1-y^2+(i-k/2)^2h^2}}\right|^p =$$

$$\sum_{i=0}^{k}\left(\binom{k}{i}\right)^p \left|1+\frac{(i-k/2)hy((-1)^k-1)}{\sqrt{1-y^2+(i-k/2)^2h^2}}\right|^p$$

In particular, if k is even, then:

$$\left|A_k(y,h)\right|^p \le \sum_{i=0}^{k}\left(\binom{k}{i}\right)^p \left|\frac{1}{1+(i-k/2)^2h^2}\right|^p$$

When, we suppose k is odd number, we have:

$$\left|A_k(y,h)\right|^p \le \sum_{i=0}^{k}\left(\binom{k}{i}\right)^p \left|\frac{(i-k/2)hy}{\sqrt{1-y^2+(i-k/2)^2h^2}}\right|^p$$

Now, let us consider the following cases.

Case 1: If, k is even, we have $g^{(m)}$ is continuous on [a, b] and if $x_0, x_1, ..., x_m$ are any m+1 distinct points in [a, b], then for some $\varepsilon\in(a, b)$, $[x_0, x_1, ..., x_m; f] = g^{(m)}_{(\varepsilon)}/m!$ (Kopotun, 2001). Since:

$$\left|\Delta_h^m(f,x)\right|^p = m!\left|h^m\right|^p \left[x-\frac{mh}{2}, x-\frac{mh}{2}+h, \cdots, x+\frac{mh}{2}; f\right]$$

Thus, if $g^{(m)}$ is continuous on $\left[x-\frac{mh}{2}, x+\frac{mh}{2}\right]$ /then for an $\varepsilon\in(x-mh/2, x+mh/2)$:

$$\left|\Delta_h^m(f,x)\right|^p = \left|h^m\right|^p \left|g^{(m)}(\varepsilon)\right|^p \quad (3.8)$$

$$\left|A_k(y,h)\right|^p \le \left|(-1)^k\Delta_h^k(g,0)\right|^p = \left|\Delta_h^k(g,0)\right|^p$$

Using (3.8), we get that for some $\varepsilon\in(-kh/2, kh/2)$:

$$\left|A_k(y,h)\right|^p \le h^{kp}\left|g^{(k)}(\varepsilon)\right|^p$$

Because $g\in C^\infty[-1, 1]$, we get $|g^{(k)}(\varepsilon)|^p \le c(p, k)$ and therefore:

$$\left|A_k(y,h)\right|^p \le c(p,k)h^{kp}$$

Using (3.5), we have:

$$\sum_{i=0}^{k}\binom{k}{i}S_2 \le c(p,k)h^{kp}\int_{-1+4k^2h^2/(4+k^2h^2)}^{1-4k^2h^2/(4+k^2h^2)}\left|f(y)\right|^p dy \le c(p,k)h^{kp}\|f\|_{L_p}$$

Hence:

$$\left\|\Delta^k_{h\varphi(x)}(f, x)\right\|_p^p \le c(p,k)\left(\|f\|_{L_p[-1,-1+k^2h^2]} + \|f\|_{L_p[1-k^2h^2,1]} + h^{kp}\|f\|_{L_p}\right)$$

If, k is even number, we obtain:

$$\omega_k^\varphi(f, \delta)_p \le (c(p,k)\left(\|f\|_{L_p[-1,-1+k^2h^2]} + \|f\|_{L_p[1-k^2h^2,1]} + \delta^{kp}\|f\|_{L_p}\right)$$

Case 2: If k is odd number. Assume $y\epsilon[-1+4k^2h^2/4+k^2h^2, 1-4k^2h^2, 1-4k^2h^2/(4+k^2h^2)]$ and let $\gamma = \sqrt{1-y^2}$ and $\tilde{g}(t)/(1+t^2)\sqrt{1-y^2}$, therefore:

$$\left|A_k(y,h)\right|^p \le \left|y\Delta_h^k(\tilde{g}, 0)\right|^p$$

Using (3.8), we get:

$$\left|A_k(y,h)\right|^p \le |y|^p h^{kp}\left|\tilde{g}^k(e)\right|^p,\ \varepsilon\in\left(-\frac{kh}{2}, \frac{kh}{2}\right)$$

Now, let us estimate the kth derivative of $\tilde{g}(t)$ when $t\in[-hh/2, kh/2]$. We have $\gamma\epsilon kh/2$ and therefore, $|t|\le\gamma$. Notice that $\tilde{g}(t) = G(t/\gamma)$ where:

$$G(x) = \frac{1}{1+\gamma^2x^2}\frac{x}{\sqrt{1+x^2}}$$

It is easy to get that $|G^{(k)}|\le c(k)$:

$$\left|\tilde{g}^k(\varepsilon)\right|^p \le \left|\gamma^{-k}G^{(k)}(t/\gamma)\right|^p \le c(p,k)\gamma^{-kp},\ \forall|t|\le\gamma \quad (3.9)$$

Hence:

$$\left|A_k(y,h)\right|^p \le c(p,k)h^{kp}(1-y^2)^{-k/2}$$

Equation 3.5, implies that:

$$\sum_{i=0}^k \binom{k}{i} S_2 \le c(p,k)h^{kp}\int_{-1+4k^2h^2/(4+k^2h^2)}^{1-4k^2h^2/(4+k^2h^2)} |f(y)|^p\left|(1-y^2)^{-k/2}\right|^p dy \le c(p,k)h^{kp}\left\||f(y)|^p\left|(1-y^2)^{-k/2}\right|^p\right\|_{L_p[-1+k^2h^2/2,\ 1-k^2h^2/2]}$$

Therefore, recalling (3.5), we have:

$$\left\|\Delta^k_{h\varphi(x)}(f, x)\right\|_p^p \le c(p,k)(\|f\|_{L_p[-1,-1+k^2h^2]} + \|f\|_{L_p[1-k^2h^2,1]} + h^{kp}\left\||f(y)|^p\left|(1-y^2)^{-k/2}\right|^p\right\|_{L_p[-1+k^2h^2,1-k^2h^2]}$$

$$\omega_k^\varphi(f, d)_p \le c(p,k)\left(\|f\|_{L_p[-1,-1+k^2\delta^2]} + \|f\|_{L_p[1-k^2\delta^2,1]} + \sup_{0<h\le\delta}h^{kp}\left\||f(y)|^p\left|(1-y^2)^{-k/2}\right|^p\right\|_{L_p[-1+k^2h^2,1-k^2h^2]}\right)$$

CONCLUSION

In this study, we obtain estimates as properties of kth Ditizian-Totik L_q moduli of smoothness of k-monotone function in $L_p[-1, 1]$ space for $0<q<p\le1$. Also, we relate our new modulus to kth Ditizian-Totik modulus of smoothness.

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