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Fixed Point of Set Valued Mappings

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Abstract: Establishing some fixed point theorems for set-valued maps in ordered D^*-metric spaces by using implicit-relation with example. These conditions are employed to demonstrate the existence of single, set-valued) mapping in metric space. Certainly, this type confirmed to have applicable for different equations.

Key words: Generalized metric, complete space, set-valued, theorems, demonstrate, different equations

INTRODUCTION

The hypothesis of implicit relation are researched by many researchers that associated with solving non-liner functional equation (Altun, 2011; Altun and Simsek, 2010; Beg and Butt, 2009). Dhage (1992) validatedthe existence of unique fixed point of the contractive mapping in bounded and complete D-metric space. Different researchers, Dhage (1992, 1999), Dhage *et al.* (2000) and Ahmad *et al.* (2001) have studied the fixed point theory in D-metric space. Sedghi *et al.* (2007) and Rhoades (1996) introduced the definition D^*-metric space.

MATERIALS AND METHODS

Definition 1.1; Rhoades (1996): Let X be a non-empty set and a function, ɱ: $X^3 \to [0\text{-}|, \infty)$, $\forall$ u, p, w, c in X satisfies the following:

$$ɱ(u,\ p,\ w) \leq 0$$

$$ɱ(u,\ p,\ w) = 0 \leftrightarrow u = p = w$$

$$ɱ(u,\ p,\ w) = ɱ(p\{u,\ p,\ w\})$$

ɱ(u, p, w) = ɱ(p{u, p, w}), (symmetry) where, p{x, p, w} is a permutation function:

$$ɱ(u,\ p,\ w) \leq ɱ(u,\ p,\ c) + ɱ(c,\ w,\ w)$$

Then (X, ɱ) is called a D^*-metric space or (generalized metric). Now, we provide examples for this definition are (Rhoades, 1996):

$$ɱ(u,\ p, w) = \max\{d(u,\ p),\ d(p,\ w),\ d(w,\ u)\}$$

$$ɱ(u,\ p,\ w) = d(u,\ z) + d(p,\ w) + (w,\ u)''$$

Remark 1.2; Rhoades (1996): Let (X, ɱ) be D^*-metric space then ɱ(w, w, c) = m(w, c, c) (symmetric).

Definition 1.3; Rhoades (1996): Let X be a D^*-metric space:

- A sequence $\{u_n\}$ in X converges to point u↔ɱ $(u_n, u_n, u) = ɱ(u_n, u_n, u) \to 0$ as $n \to \infty$, $\forall \epsilon > 0$ there exists n1∈N then ɱ $(u, u, u_n) < \epsilon$, for each $n \geq n_1$
- A sequence $\{u_n\}$ in X is called a Cauchy sequence if $\forall \epsilon > 0$, there exists$_1$ ∈N, since, ɱ $(u_n, u_n, u_ɱ) < \epsilon$, $\forall$n, $ɱ \geq n_1$
- The pair (X, ɱ) is called a completeif every Cauchy sequence is convergent to point

The class of all non-empty bounded subset of X denoted by X_b.

Definition 1.4: Let C, E, J∈X_b $\delta_ɱ$ define of the following:

$$\delta_ɱ(C,E,J) = \sup_{\substack{x \in C \\ y \in E \\ z \in J}} ɱ(x,y,z)$$

Remark 1.5:

$$\delta_ɱ(C,E,J) = 0 \leftrightarrow C = E = J = x$$

$$\delta_ɱ(C,E,K) \leq \delta ɱ(C,E,J) + \delta ɱ(J,K,K)$$

Proof: $\delta_ɱ(C,E,K) = \sup_{\substack{x \in C \\ y \in E \\ k \in K}} ɱ(x,y,k) \leq \sup_{\substack{x \in C \\ y \in E \\ z \in J}} ɱ(x,y,z) + \sup_{\substack{z \in J \\ k \in K}} ɱ(x,k,k)$ $\leq \delta_ɱ$ (C, E, J)+$\delta_ɱ$ (J, K, K) (by definition 1.1 and 1.4) ɱ (C, E, J)$\leq \delta_ɱ$ (C, E, J) for all C, E, J∈X_b. Now, we define implicit condition on D^*-metric: Let R_+ be the set non-negative real numbers and F: The set of continuous real numbers functions F: $R_{+4} \to R$ satisfying of the following: F_1: F(f_1, ..., f_4 is increasing in f_1 and decreasing in f_2, ..., f_4 F_2: $\exists A \in (0, 1)$ such that F (f, d, f, f+d)$\leq$0 or F(f, d, f+d)$\leq$0 implicit$\leq$Ad: F_3: F(f, 0, f, f)>0 and F(f, f, 0, f)>0, for all f>0.

Example 1.6: Let, $F(f_1, ..., f_4) = f_1 - \omega \max\{f_2, f_3, f_4/3\}$ where $0 \leq \omega \leq 1$.

Solution: F_1 is true; F_2: Let, f>0 then F(f, d, f, f+d) = f-ω max {f, d} ≤0→f≤ω max {f, d}. If f≥d, we get f≤ωf then ω≥1 this is contradiction, since, 0≤ω≤1. Thus, f<d and f≤ωd.

In the same method and let f>0→F (f, f, d, f+d)≤0, we have f≤ωd. If f = 0⇒f≤ωb. Therefore, F_2 is satisfied with A = ω<1.

F_3; F(f, 0, f, f) = f-ω max {0, f, f} = f-ωf = (1-ω) f>0 = F (f, f, 0, f) for all f>0.

Example 1.7: Let, $F(f_1, ..., F_4) = f_1-\rho \max\{f_2, f_4\}-\omega f_4$ where, ω, ρ>0 and ρ+2ω<1.

Solution: F_1 is true; F_2: let f>0, F(f, d, f, f+d) = f-ρ max {d, f}-ω(f+d)≤0 or f≤ρ max {d, f}+ω (f+d).

Then, f≤max {(ρ+ω) f+ωd, (ρ+ω) d+ωf}. If f≥d⇒f≤(ρ+w)f+ωd, we get ρ+2ω≥1 that is contraction. Then, f<d and f≤(ρ+ω/1-ω)d.

In the same method and let, f>0→F (f, f, d, f+d)≤0, we have f≤(ω+ρ/1-ρ)d. If f = 0⇒f≤(ρ+ω/1-ω)d. Therefor, F_2 is satisfied with A = (ρ+ω/1-ω)d<1. F_3: F(f, 0, f, f) = f-ρmax{0, f}-ωf = f-fρ-ωf = (1-ρ-ω)f>0 = F(f, f, 0, f), for all f>0.

RESULTS AND DISCUSSION

Theorem 2.1: Let (X, ɱ) be complete D^*-metric space and T, S: $X\to X_b$ satisfy the following conditions:

- If p∈Ku, then u≤p and if k∈Sx, then k≤u
- If u_n be converge sequence to x and u_n≤u
- F ($\delta_ɱ$(Ku, Sv, Sv), ɱ(u, Ku, Ku), ɱ(v, Sv, Sv)+ɱ(v, Ku, Ku))≤0 for all u, v be distinct comparable and F∈F for all u, v be distinct comparable and F

Then, u∈Tu∈Su.

Proof: Let, u_0 be any element in X and by condition (i) then there exists $x_1\in K_{u_0}$ such that $u_0\le u_2$ and $u_2\in S_{u_1}$ such that $u_0\le u_1$. Then by (iii) and $u_0\le u_1$, we have:

$$F(\delta_ɱ(K_{u_0}, S_{u_1}, S_{u_1}), ɱ(u_0, K_{u_0}, K_{u_0}), ɱ(u_1, S_{u_1}, S_{u_1}, S_{u_1})$$
$$ɱ(u_0, S_{u_1}, S_{u_1})+ɱ(u_1, K_{u_0}, K_{u_0}))\le 0$$

Since:

$$ɱ(u_1,u_2,u_2)\le\delta_ɱ(K_{u_0},S_{u_1},S_{u_1}),$$
$$ɱ(u_0,S_{u_1},S_{u_1})+ɱ(u_1,K_{u_0},K_{u_0}) = ɱ(u_0,u_2,u_2)+$$
$$ɱ(u_1,u_2,u_2)\le ɱ(u_0,u_2,u_2)$$
$$ɱ(u_0,u_2,u_2) = ɱ(u_0,u_0,u_2)\le ɱ(u_0,u_0,u_1)+ɱ(u_1,u_2,u_2)$$

$\le(u_1, u_2, u_2)+m(u_1+u_2, u_2)$ [by remark 1.2 and definition 1.1] From F_1, we get:

$$F(ɱ(u_1, u_2, u_2), ɱ(u_0, u_1, u_1),$$
$$ɱ(u_1, u_2, u_2), ɱ(u_1, u_2, u_2))\le 0$$

We implicit: F(f, d, f, f+d)≤, Since, $f = m(u_1, u_2, u_2)$, $d = ɱ(u_0, u_1, u_1)$. From f_2, there exists A∈(0, 1) such that:

$$ɱ(u_1,u_2,u_2)\le Aɱ(u_0,u_1,u_1)$$

Again, since, $u_1\le u_2$ for this u_2 and by condition (i), we get $u_3\in K$ such that $u_2\le u_3$. Therefore, by (iii), we have:

$$F(\delta_ɱ(T_{u_2}, S_{u_1}, S_{u_1}), ɱ(u_2, K_{u_0}, K_{u_2}), ɱ(u_2, S_{u_1}, S_{u_1})$$
$$ɱ(x_2, S_{x_1}, S_{u_1})+ɱ(u_1, K_{u_2}, K_{u_2}))\le 0$$

From F_1, we get:

$$F(ɱ(u_3, u_2, u_2), ɱ(u_2, u_3, u_3), ɱ(u_1, u_2, u_2),$$
$$ɱ(u_1, u_2, u_2)+ɱ(u_2, u_3, u_3))\le 0$$

Then, by remark 1.2:

$$F(ɱ(u_2, u_3, u_3), ɱ(u_2, u_3, u_3), ɱ(u_1, u_2, u_2),$$
$$ɱ(u_1, u_2, u_2)+ɱ(u_2, u_3, u_3)$$

That is F(f, f, d, f+d)≤, since, $f = ɱ(x_2, x_3, x_3)$, $d = ɱ(u_1, u_2, u_2)$. By using F2 and (1), $ɱ(u_1, u_2, u_2)\le Aɱ(u_0, u_1, u_1)$ (2) and Then, by continuous in this way, since, $u_{n+1}\in K_{u_n}$ and $u_{n+2}\in S_{u_{n+1}}$ we have:

$$F(\delta_ɱ\left(K_{u_n}, S_{u_{n+1}}, S_{u_{n+1}}\right), ɱ\left(u_n, K_{u_n}, K_{u_n}\right),$$
$$ɱ\left(u_{n+1}, S_{u_{n+1}}, S_{u_{n+1}}\right), ɱ\left(u_n, S_{u_{n+11}}, S_{u_{n+11}}\right)+$$
$$ɱ\left(u_{n+1}, K_{u_n}, K_{u_n}\right)\le 0$$

Which implicit that:

$$ɱ(u_{n+1}, u_{n+2}, u_{n+2})\le Aɱ(u_n, u_{n+1}, u_{n+1})$$

Therefore, we get:

$$ɱ(u_n,u_{n+1}, u_{n+1})\le A^n ɱ(u_0, u_1, u_1)$$

Next, let b>n then:

$$ɱ(u_n,u_b,u_b)\le ɱ(u_n,u_n,u_{n+1})+ɱ(u_{n+1},u_b,u_b)\le$$
$$ɱ(u_n,u_{n+1},u_{n+1})+ɱ(u_{n+1},u_b,u_b)\le$$
$$ɱ(u_n,u_{n+1},u_{n+1})+ɱ(u_{n+1}+u_{n+2}+u_{n+2})+...+ɱ(u_{b-1},u_b,u_b)\le$$
$$A^n\frac{1-A^{b-n}}{1-b}ɱ(u_0,u_1,u_1)<$$
$$\frac{A^n}{1-A}ɱ(u_0,u_1,u_1)[\text{since, } 1-A<1]$$

When n→∞, we get ɱ $(u_0, u_1, u_1)\to 0$ leads to u_n is Cauchy sequence. Then, $u_n\to u$[since, x is complete]:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{n+1} = u\in K_{u_n}$$
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{n+2} = u\in S_{u_{n+1}}$$

And by condition (ii):

$$F\big(\delta_{ɱ}(K_{u_n}, S_u, S_u), ɱ(u, u_n, u_{2n}), ɱ(u, K_u, K_u), ɱ(u_n, S_{u_n}, S_{u_n}), ɱ(u, S_{u_n}, S_{u_n})+ɱ(u_n, K_u, K_u)\big)\leq 0$$

When n→∞ and by f_3, we have:

$$F(ɱ(K_u, u, u), ɱ(K_u, u, u), 0, ɱ(K_u, u, u), 0, (k_u, u, u))\leq 0$$

That is F(f, f, 0, f)≤0 then, f = ɱ$(u, S_u, S_u) = 0\to u\in S_u$. That same away:

$$F\big(ɱ(K_u, S_{u_{n+1}}, S_u), ɱ(u, u_n, u_{2n}), ɱ(u, K_u, k_u), (u_n, S_{u_n}, S_{u_n}), ɱ(u, S_{u_n}, S_{u_n})+ɱ(u_n, K_u, K_u)\big)\leq 0$$

When n→∞, we get leads to u_n and by f_3, we have:

$$F(ɱ(K_u, u, u), ɱ(K_u, u, u), 0, ɱ(K_u, u, u))\leq 0$$

That is F(f, f, 0, f)≤then, f = ɱ$(K_x, u, u) = 0\to u\in K_u$. Then, $u\in K_u\in S_u$.

Corollary 2.2: Let (x, ɱ) be complete D^*-metric space and S: $X\to X_b$ satisfy the following conditions.

- There exist $u_0\in X$ such $u_{n+1}\in S_{u_n}$ that then, $u_n\leq u_{n+1}$, n = 0, 1, ...
- If $\langle u_n\rangle$ be any sequence in X, $u_n\to u$ and $u_n\leq u$
- F($\delta_{ɱ}$ (Su, Sv, Sv), ɱ(u, Su, Su), ɱ(u, Sv, Sv), ɱ(u, Sv, Sv)+ɱ(v, Su, Su))≤0, for all u, v be distinct comparable with F∈f. Then, u∈Su.

Proof: By using (iv), $u_1\in S_{u_0}$ then, $u_0\in u_1$ and $x_2\in S_{u_1}$ then, $u_1\leq u_2$. In the same way by (vi), we have:

$$F(\delta(S_{u_0}, S_{u_1}, S_{u_1}), ɱ(u_0, S_{u_0}, S_{u_0}), ɱ(u_0, S_{u1}, S_{u_1})+ɱ(u_1, S_{u_0}, S_{u_0}))\leq 0$$

Since:

$$ɱ(u_1, u_2, u_2)\leq \delta_{ɱ}(S_{u_0}, S_{u_1}, S_{u_1}),$$
$$ɱ(u_0, S_{u_1}, S_{u_1})+ɱ(u_1, S_{u_0}, S_{u_0}) = ɱ(u_0, u_2, u_2)+ɱ(u_1, u_1, u_1)$$
$$\leq ɱ(u_0, u_2, u_2)+ɱ(u_1, u_2, u_2) = ɱ(u_0, u_0, u_1)\leq$$
$$ɱ(u_0, u_0, u_1)+ɱ(u_1, u_2, u_2)\leq ɱ(u_0, u_1, u_1)+ɱ(u_1, u_2, u_2)$$

[by remark 1.2 and definition 1.1]. From F_1, we get:

$$F(ɱ(u_1, u_2, u_2), ɱ(u_0, u_1, u_1), ɱ(u_1, u_2, u_2), ɱ(u_0, u_1, u_1)+ɱ(x_1, u_2, u_2)\leq 0$$

We implicit: F(f, d, f, +d)≤0, since, f = ɱ(x_1, x_2, x_2). D = ɱ(u_0, u_1, u_1). From F_2, ∃A∈ such that:

$$F\begin{pmatrix}\delta_{ɱ}(u_3, u_2), ɱ(u_2, u_3, u_3),\\ ɱ(u_1, u_2, u_2), ɱ(u_1, u_2, u_2)+\\ ɱ(u_2, u_3, u_3)\end{pmatrix}\leq 0$$

Again, since, $u_1\leq u_2$ for this u_2 and by condition (iv), we get $u_3\in K$ such that $u_2\leq u_3$. Therefore, by (vi), we have:

$$F\begin{pmatrix}\delta_{ɱ}(S_{u_2}, S_{u_1}, S_{u_1}), ɱ(u_2, S_{u_0}, S_{u_2}), ɱ(u_1, S_{u_1}, S_{u_1}),\\ ɱ(u_2, S_{u_1}, S_{u_1})+ɱ(u_1, S_{u_2}, S_{u_2})\end{pmatrix}\leq 0$$

From F_1, we get:

$$F\begin{pmatrix}ɱ(u_3, u_2, u_2), ɱ(u_2, u_3, u_3)\ ɱ(u_1, u_2, u_2),\\ ɱ(u_1, u_2, u_2)+ɱ(u_2, u_3, u_3)\end{pmatrix}\leq 0$$

Then by remark 1.2:

$$F\begin{pmatrix}ɱ(u_2, u_3, u_3), ɱ(u_2, u_3, u_3)\ ɱ(u_1, u_2, u_2),\\ ɱ(u_1, u_2, u_2)+ɱ(u_2, u_3, u_3)\end{pmatrix}\leq 0$$

That is F(f, f, d, f+d)≤0, since, f = ɱ(u_2, u_3, u_3), d = ɱ(u_1, u_2, u_2). From (3) and F_2:

$$ɱ(u_1, u_2, u_2)\leq Aɱ(u_0, u_1, u_1)$$

Then by continuous in this way, since, $u_{n+1}\in S_{u_n}$ and $u_{n+2}\in S_{u_{n+1}}$ we have:

$$F\begin{pmatrix}\delta_{ɱ}(S_{u_n}, S_{u_{n+1}}, S_{u_{n+1}}), ɱ(u_n, S_{u_n}, S_{u_n}), ɱ(u_{n+1}, S_{u_{n+1}}, S_{u_{n+1}}),\\ ɱ(u_n, S_{u_{n+1}}, S_{u_{n+1}})+ɱ(u_{n+1}, S_{u_n}, S_{u_n})\end{pmatrix}\leq 0$$

From F_1, we get:

$$ɱ(u_{n+1},u_{n+2}) \le A\, ɱ(u_n, u_{n+1}, u_{n+1})$$

Therefore, we have:

$$ɱ(u_n, u_{n+1}, u_{n+1}) \le A^n ɱ(u_0, u_1, u)$$

Now, we prove that u_n be Cauchy sequence in X, let b>n then:

$$\begin{aligned}
&ɱ(u_n, u_b, u_b) \le ɱ(u_n, u_n, u_{n+1}) + ɱ(u_{n+1}, u_b, u_b)\\
&\le ɱ(u_n, u_{n+1}, u_{n+1}) + ɱ(u_{n+1}, u_b, u_b)\\
&\le ɱ(u_n, u_{n+1}, u_{n+1}) + ɱ(u_{n+1}, u_{n+2}, u_{n+2}) +, ..., + ɱ(u_{b-1}, u_b, u_b)\\
&\le (A^n + ... + A^{b-1}) ɱ(u_n, u_1, u_1)\\
&\le A^n \frac{1-A^{b-n}}{1-A} ɱ(u_0, u_1, u_1)\\
&< \frac{A^n}{1\text{-}A} ɱ(u_0, u_1, u_1)\ [\text{since, } 1\text{-}A<1]
\end{aligned}$$

when $n\to\infty$, we get $ɱ(u_0, u_1, u_1)\to 0$ leads to u_n Cauchy sequence. Then, u_n is converge to u [since, X is complete]:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} u_{n+2} = u \in S_{u_{n+1}}$$

And by condition (v):

$$F\begin{pmatrix} \delta ɱ(S_{u_n}, S_u, S_u),\ ɱ(u_n, S_{u_n}, S_{u_n}), \\ ɱ(u, S_u, S_u) + ɱ(u, S_{u_n}, S_{u_n}) \end{pmatrix} \le 0$$

When $n\to\infty$ and by f_3, we have:

$$F(ɱu, S_u, S_u), 0, ɱ(u, S_u, S_u), ɱ(u, S_u, S_u) \le 0$$

That is $F(f, 0, f, f)\le 0$ then, $f = ɱ(u, S_u, S_u) = 0 \to u\in S_u$. That same away:

$$F\begin{pmatrix} ɱ(Su, S_{u_{n+1}}, S_u),\ ɱ(u, u_n, u_n),\ ɱ(u, S_u, S_u), \\ ɱ(u_n, S_{u_n}, S_{u_n}),\ ɱ(u, S_{u_n}, S_{u_n}) + ɱ(u_n, S_u, S_u) \end{pmatrix} \le 0$$

When $n\to\infty$ and by f_3, we have:

$$F(ɱ(S_u, u, u), ɱ(S_u, u, u), 0, ɱ(S_u, u, u)) \le 0$$

That is $F(f, f, 0, f)\le 0$ then, $f = ɱ(S_u, u, u) = 0 \to u\in S_u$. Then, $u\in S_u$.

Example 2.3: Let (X, ɱ) be complete metric space, X = {(0, 0), (0, -1/2), (-1/9)}$\subseteq R_2$ With defined usual order by the following:

$$(p,k) \le (u,v) \leftrightarrow p \le u, v \le k \text{ for } (p, k), (u, v) \in X$$

and let ɱ defined as: ɱ(u, v, v) = max{d(u, v), d(v,v), d(v,u)}. Where $d(u,v) = \max\{|u_1\text{-}v_1|, |u_2\text{-}v_2|\}$, for all u, $v,\subseteq R^2$. Define S:

$$X,\to X_b,\ S(x) = \begin{cases} \left(\frac{-1}{9}, \frac{1}{9}\right) & \text{if } x = \left(\frac{-1}{9}, \frac{1}{9}\right) \\ \left\{(0, 0), \left(\frac{-1}{9}, \frac{1}{9}\right)\right\} & \text{if } x = \left\{(0, 0), \left(0, \frac{-1}{2}\right)\right\} \end{cases}$$

Solution: for $(0, -1/2)\le(0, 0)$ then:

$$\begin{aligned}
&\delta_{ɱ}\left(S\left(0, \frac{-1}{2}\right), S(0,0), S(0,0)\right) = \\
&\delta_{ɱ}\left(\left\{(0,0), \left(\frac{-1}{9}, \frac{1}{9}\right)\right\}, \left\{(0,0), \left(\frac{-1}{9}, \frac{1}{9}\right)\right\}, \left\{(0,0), \left(\frac{-1}{9}, \frac{1}{9}\right)\right\}\right) = \\
&\frac{1}{9} \le \frac{1}{3}\times\frac{1}{2} = \frac{1}{3} ɱ\left(\left(0, \frac{-1}{2}\right), (0,0)(0,0)\right) = \\
&\frac{1}{3}\max\left\{d\left(0, \frac{-1}{2}, (0,0)\right), d((0,0),(0,0)), d\left((0,0), \left(0, \frac{-1}{2}\right)\right)\right\} = \\
&\frac{1}{3}\max\left\{0, \frac{1}{2}\right\} = \frac{1}{3}\times\frac{1}{2}
\end{aligned}$$

Then, for all $u\le v$, we get:

$$\delta_{ɱ}(Su, Sv, Sv) \le \frac{1}{3} m(u, v, v) \le \max$$
$$\left\{ɱ(u, Su, Su), ɱ(v, Sv, Sv), \frac{ɱ(u, Sv, Sv) + ɱ(v, Su, Su)}{2}\right\}$$

For all $u\le v$ then, $Sv\le Su$. So, all conditions of corollary 2.2 are satisfied then, we get:

$$S\left(\frac{-1}{9}, \frac{1}{9}\right) = \left(\frac{-1}{9}, \frac{1}{9}\right)$$

CONCLUSION

The aim of this research to define an implicit relation and prove the results of common fixed point for two set-valued mappings in partially order.

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