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Relatively Commuting Mapping and Symmetric Biderivations in Semiprime Rings

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Abstract: Let R be an associative ring. The major proposal of the present study is to introduce the concept of relatively commuting of R. Let S be a nonempty sub set of R, A mapping δ: S→R is called relatively commuting mapping on S if $[\delta (s), s]^n = 0$, for all ∈S and a fixed positive integer n≥2. We describe the relationship between the new concept and the concept of commuting mapping. Moreover, we looking for some necessary conditions under which these two concepts equivalence.

Key words: Relatively commuting mapping, commuting mapping, symmetric bi derivation, semiprime ring, mapping, equivalence

INTRODUCTION

In this research, R always denotes to an associative ring with the center Z (R). Given n≥2, a ring R is called is n-torsion free if for u∈R and nu = 0, implies that u = 0. Recall that in the prime Ring R the statement R is n-torsion free ring is equivalent to R of characteristic different from n (Herestein, 1976). For any u, v∈R the symbol [u, v] is refer to the commutator uv-vu and make use of the commutator identities [uω, v] = [u, v]ω+u [ω, v], [u, ωv] = [u, v]+v [u, ω] (Oukhtite *et al*., 2007). Recall that R is prime ring if uR v = (0) implies either u = 0 or v = 0 and is semiprime if whenever uRu = (0) implies u = 0 (Bresar, 1989). An element u of R is said be nilpotent if $u^n = 0$ for some positive integer n≥2. Let S be a nonempty sub set of R: A mapping φ:S→R is called Centralizing on S if [φ (s), s]∈Z (R), for all s∈S and called Commuting on S if [φ (s), s] = 0, for all s∈S (Bresar, 1993). The first classical result was introduce by Posner (1957) which state that the existence of non trivial centralizing derivation on a prime ring R forces R to be commutative. Another essential result in this direction was given by Awtar (1973), Awtar proved that for a 3, 2-torsion free prime ring R if φ: R→R is centralizing on a lie ideal L of R implies that L⊆Z (R). Mayne (1984) present a same conclusion for endomorphism. During the few decades, several researchers have extended these results of posner and Mayne. Vukman (1990) prove the following result (If a derivation d: R→R of prime ring R with char (R)≠2 such that [[d (u), u], u] = 0, for all u∈R then d = 0 or R is commutative. A biadditive mapping β:R×R→R is called Symmetric if β (u, v) =β (v, u) for all pairs u, v∈R. A Symmetric biadditive mapping D: R×R→R is called a symmetric biderivation if D (uω, v) = D (u, v) ω+u D (ω, v) holds for all u, v, ω∈R (Maksa, 1987). Note that any commuting mapping φ gives rise to a symmetric biderivation defined by D (u, v) = [φ (u), v], for all u, v∈R (Bresar, 1995). In this study we present the following new concept:

Definition 1.1: Let R be a ring and S be a nonempty sub set of R, A mapping δ:S→R is called relatively commuting mapping on S if $[\delta (s), s]^n = 0$, for all s∈S and a fixed positive integer n≥2.

It's clear that any commuting mapping is relatively commuting but the converse in general is not true as we see in the following example:

Example: Let Q be a commutative ring and:

$$R = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| a,b,c \in Q \right\}$$

Then R is a ring with respect to the usual operation of addition and multiplication of matrices. Define δ:R→R, such that:

$$\delta\left(\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, δ is relatively commuting on R but not commuting. Also, we have investigated the conditions that the two concepts are equivalence.

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MATERIALS AND METHODS

We give some lemmas and remakes which are essential for developing the proof of our result.

Lemma 2.1; Samman and Chaudhry (2008): Let R be a semiprime ring and U an ideal of R then U is a simeprime as subring of R.

Remark 2.2: It's easy to prove that every ring without zero devisor R has no nonzero nilpotent element and consequently every relatively commuting mapping δ: R→R is commuting.

Lemma 2.3: Let R be a semiprime ring, then R has no nonzero nilpotent element.

Proof: Suppose that there non zero element a∊R with $a^n = 0$. Let j be a two sided ideal of R defined by $j = a^{n-1}Ra^{n-1}$. It's clear that a^{n-1}∊j and for $a^{-(n-2)}$∊R, we have $a = a^{n-1}a^{-(n-2)}$∊j. Now, according to lemma 2.1 j can be considered as a semiprime as a sub ring, moreover:

$$aja = a\left(a^{n-1}Ra^{n-1}\right)a = a^nRa^n = 0$$

Using the semiprimenes of j, we arrive at a = 0 which contradicts with our hypothesis. Hence, R has no nonzero nilpotent element. According to our definition 1.1, we can rewrite lemma 2.3 as following.

Lemma 2.3: Every relatively commuting mapping on a semiprime ring R is commuting.

RESULTS AND DISCUSSION

Theorem 3.1: Let R be a semiprime ring of characteristic different from 2 and D:R×R→R is a symmetric biderivation satisfies that [[d (u), u], d (u)] = 0, for all u∊R where d is the trace of D, then D has a commuting trace.

Proof: For any u∊R, we have:

$$\left[\left[d(u),u\right],d(u)\right] = 0 \tag{1}$$

Linearization Eq. 1 with respect to u gives:

$$\begin{aligned}&\left[[d(u),u],d(\omega)\right]+2\left[[d(u),u],D(u,\omega)\right]+\left[[d(u),\omega],d(u)\right]+\left[[d(u),\omega],d(\omega)\right]+\\&2\left[[d(u),\omega],D(u,\omega)\right]+\left[[d(\omega),u],d(u)\right]+\left[[d(\omega),u],d(\omega)\right]+2\left[[d(\omega),u],D(u,\omega)\right]+\\&\left[[d(\omega),\omega],d(u)\right]+2\left[[d(\omega),\omega],D(u,\omega)\right]+2\left[[D(u,\omega),u],d(u)\right]+2\left[[D(u,\omega),u],d(\omega)\right]+\\&4\left[[D(u,\omega),u],D(u,\omega)\right]+2\left[[D(u,\omega),u],d(u)\right]+2\left[[D(u,\omega),\omega],d(\omega)\right]+4\left[[D(u,\omega),\omega],D(u,\omega)\right] \text{ for all } u,\omega\in R\end{aligned} \tag{2}$$

Replacing ω by -ω in Eq. 2 and comparing the relation so, obtained with Eq. 2, we arrive because of char (R)≠2 at:

$$\begin{aligned}&2\left[[d(u),u],D(u,\omega)\right]+\left[[d(u),\omega],d(u)\right]+\left[[d(u),\omega],d(\omega)\right]+2\left[[d(\omega),u],D(u,\omega)\right]+\left[[d(\omega),\omega],d(u)\right]+\\&2\left[[D(u,\omega),u],d(u)\right]+2\left[[D(u,\omega),u],d(\omega)\right]+4\left[[D(u,\omega),\omega],D(u,\omega)\right] \text{ for all } u,\omega\in R\end{aligned} \tag{3}$$

Putting 2ω instead of ω in Eq. 3, comparing the new relation with Eq. 3 then using the characteristic property of R, we obtain:

$$\left[[d(u),\omega],d(u)\right]+2\left[[d(u),u],D(u,\omega)\right]+2\left[[D(u,\omega),u],d(u)\right]=0 \text{ for all } u,\omega\in R \tag{4}$$

Substituting uω for ω in Eq. 4 gives:

$$\begin{aligned}&5\left[[d(u),u],d(u)\right]\omega+4[d(u),u][\omega,d(u)]+u\left[[d(u),\omega],d(u)\right]+2d(u)\left[[d(u),u],\omega\right]+\\&2\left[[d(u),u],u\right]D(u,\omega)+2u\left[[d(u),u],D(u,\omega)\right]+2d(u)\left[[\omega,u],d(u)\right]+\\&2[u,d(u)][D(u,\omega),u]+2u\left[[D(u,\omega),u],d(u)\right]=0 \text{ for all } u,\omega\in R\end{aligned}$$

According to Eq. 1 and 4, the above relation becomes:

$$\begin{aligned}&4[d(u),u][\omega,d(u)]+2\left[[d(u),u],u\right]D(u,\omega)+2d(u)\left[[d(u),u],\omega\right]+2d(u)\left[[\omega,u],d(u)\right]+2[u,d(u)][D(u,\omega),u] =\\&0 \text{ for all } u,\omega\in R\end{aligned} \tag{5}$$

But in view of Eq. 4 and by using the commutator identities [uω, v] = [u, v] ω+u [ω, v], [u, ωv] = [u, v]+v [u, ω], we get:

$$\begin{aligned}&2\big[[d(u),u],u\big]D(u,\omega)+2[u,d(u)][D(u,\omega),u]=\\&2\big[[d(u),u],uD(u,\omega)\big]-2u\big[[d(u),u],D(u,\omega)\big]+2\big[u[D(u,\omega),u],d(u)\big]-2u\big[[D(u,\omega),u],d(u)\big]=\\&\Big(2\big[[d(u),u],uD(u,\omega)\big]+2\big[u[D(u,\omega),u],d(u)\big]\Big)-u\Big(2\big[[d(u),u],D(u,\omega)\big]+2\big[[D(u,\omega),u],d(u)\big]\Big)=\\&-\big[u[d(u),\omega],d(u)\big]+u\big[[d(u),\omega],d(u)\big]=\\&-[u,d(u)][d(u),\omega]=\\&-[d(u),u][\omega,d(u)]\text{ for all }u,\omega\in R\end{aligned}$$

According to the above conclusion, the relation (Eq. 5) reduces to:

$$3[d(u),u][\omega,d(u)]+2d(u)\big[[d(u),u],\omega\big]+2d(u)\big[[\omega,u],d(u)\big]=0\text{ for all }u,\omega\in R \tag{6}$$

Replacing ω by ωd (u) in Eq. 6, to get:

$$\begin{aligned}&3[d(u),u][\omega,d(u)]d(u)+2d(u)\big[[d(u),u],\omega\big]d(u)+2d(u)\omega\big[[d(u),u],d(u)\big]+2d(u)[\omega,d(u)]\\&[d(u),u]+2d(u)\omega\big[[d(u),u],d(u)\big]+2d(u)\big[[\omega,u],d(u)\big]d(u)=0\text{ for all }u,\omega\in R\end{aligned}$$

Appling Eq. 1 and 6 on the last relation, since, char (R)≠2 yields that:

$$d(u)[\omega,d(u)][d(u),u]=0\text{ for all }u,\omega\in R$$

That is:

$$d(u)\omega d(u)[d(u),u]-d(u)^2\omega[d(u),u]=0\text{ for all }u,\omega\in R \tag{7}$$

Now in relation (Eq. 7), the substitution uω instead of ω once and left multiplication by u in another implies that:

$$d(u)u\omega d(u)[d(u),u]-d(u)^2u\omega[d(u),u]=0\text{ for all }u,\omega\in R \tag{8}$$

$$ud(u)\omega d(u)[d(u),u]-ud(u)^2\omega[d(u),u]=0\text{ for all }u,\omega\in R \tag{9}$$

Subtracting Eq. 9 from Eq. 8 yields that:

$$[d(u),u]\omega d(u)[d(u),u]-\big[d(u)^2,u\big]\omega[d(u),u]=0\text{ for all }u,\omega\in R \tag{10}$$

Now in view of Eq. 1, one can replace d (u) [d (u), u] by [d (u), u]d (u) that is [d $(u)^2$, u] = 0. So, the statement (Eq. 10) reduces to:

$$[d(u),u]\omega d(u)[d(u),u]=0\text{ for all }u,\omega\in R \tag{11}$$

Right multiplication of Eq. 11 by d (u) gives:

$$d(u)[d(u),u]\omega d(u)[d(u),u]=0\text{ for all }u,\omega\in R$$

The semiprimeness of R leads to:

$$d(u)[d(u),u]=0\text{ for all }u\in R \tag{12}$$

Consequently:

$$[d(u),u]d(u)=0\text{ forr all }u\in R \tag{13}$$

Using same processes on Eq. 12 as used to get Eq. 3 from Eq. 1, we arrive at:

$$\begin{aligned}&d(u)[d(u),\omega]+2d(u)[D(u,\omega),u]+\\&2D(u,\omega)[d(u),u]=0\text{ for all }u,\omega\in R\end{aligned} \tag{14}$$

The substituting ωu for ω in Eq. 14 gives:

$$\begin{aligned}&d(u)[d(u),\omega]u+3d(u)\omega[d(u),u]+\\&2d(u)[D(u,\omega),u]u+2d(u)[\omega,u]d(u)+\\&2D(u,\omega)u[d(u),u]+2\omega d(u)[d(u),u]=\\&0\text{ for all }u,\omega\in R\end{aligned}$$

In view of Eq. 12 and 14 the last relation reduces to:

$$\begin{aligned}&3d(u)\omega[d(u),u]+2d(u)[\omega,u]d(u)-\\&2D(u,\omega)\big[[d(u),u],u\big]=0\text{ for all }u,\omega\in R\end{aligned} \tag{15}$$

Replacing ω by uω in Eq. 7 once and left multiplication by u in another yields that:

$$3d(u)u\omega[d(u),u]+2d(u)u[\omega,u]d(u)-2d(u)\omega[[d(u),u],u]-2uD(u,\omega)[[d(u),u],u]=0 \text{ for all } u,\omega\in R \quad (16)$$

$$3ud(u)\omega[d(u),u]+2ud(u)[\omega,u]d(u)-2uD(u,\omega)[[d(u),u],u]=0 \text{ for all } u,\omega\in R \quad (17)$$

Subtracting Eq. 17 from Eq. 16 leads to:

$$3[d(u),u]\omega[d(u),u]+2[d(u),u][\omega,u]d(u)-2d(u)\omega[[d(u),u],u]=0 \text{ for all } u,\omega\in R$$

That is:

$$[d(u),u]\omega[d(u),u]+2[d(u),u][\omega d(u),u]-2d(u)\omega[[d(u),u],u]=0 \text{ for all } u,\omega\in R \quad (18)$$

Putting ω [d (u), u] instead of ω in Eq. 18 yields because of Eq. 13 that:

$$[d(u),u]\omega[d(u),u]^2-2d(u)\omega[d(u),u][[d(u),u],u]=0 \text{ for all } u,\omega\in R \quad (19)$$

Left multiplication of Eq. 19 by [d(u), u] gives because of Eq. 13 that:

$$[d(u),u]^2\omega[d(u),u]^2=0 \text{ for all } u,\omega\in R$$

Using the semiprimeness of R leads to:

$$[d(u),u]^2=0 \text{ for all } u\in R$$

Hence, d is relatively commuting mapping on R. Finally, using lemma 2.3, we conclude that D has a commuting trace.

CONCLUSION

Associated with any symmetric biadditive mapping T:R×R→R there exist a mapping t:R→R defined by t (u) = T (u, u), for all u∈R called the Trace of T (Maksa, 1987). It follows that for any u, v∈R the Trace t satisfies t (u+v) = t (u)+2T (u,v)+t(v).

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