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Implicit Finite Difference Solution of One-Dimensional Porous Medium Equations Using Half-Sweep Newton-Explicit Group Iterative Method

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Abstract: This study considers the implicit finite difference solution of 1-Dimensional Porous Medium Equations (1D PMEs) using Half-Sweep Newton-Explicit Group (HSNEG) iterative method. The general finite difference approximation equation of 1D PME is formulated using the half-sweep implicit finite difference scheme. The generated nonlinear system is then solved using the proposed HSNEG iterative method. The comparative analysis is shown using two tested iterative methods namely Newton-Gauss-Seidel (NGS) and Newton-Explicit Group (NEG). The numerical results support the finding that the HSNEG is more superior than the NGS and the NEG in terms of total iterations and computation time. All three executed iterative methods showed good accuracy in solving 1D PME.

Key words: Porous medium equation, half-seep, implicit finite difference scheme, Newton, explicit group, NGS

INTRODUCTION

The porous medium equation or widely known as PME, gained interest from many researchers since the 70's and one of the earliest use is to describe the ideal gas flow in a homogeneous porous medium. PME also present in the fluid mechanics that deals with the filtration of an incompressible liquid in a porous medium and one of the mathematical problem are groundwater infiltration model. Moreover, PME is used to describe the heat radiation that occurs in the ionized gases at very high temperature. There are more interesting applications of the PME that can be found by Vazquez (2007).

Until today, PME is studied in order to solve both the theoretical and the mathematical challenges arising from the modeling of the non-linear fluid and heat transfer. For instance, Patel *et al*. (2016) studied the counter-current imbibition phenomenon that occurs during the secondary oil recovery process. They introduced the solution of the formulated 1-Dimensional or 1D PME Model using the homotopy analysis method. Meher *et al*. (2011, 2010) investigated the dispersion phenomenon that occurs in the oil reservoir and applied the Adomian decomposition method and the backlund transformation, respectively to solve the derived 1D PME problem.

Apart from the use of analytical approaches to obtain the approximate solution of 1D PME, numerical approaches have also been utilized for the approximate solution of 1D PME. For example, Pradhan *et al*. (2011) and Borana *et al*. (2014) applied the Galerkin finite element method and Crank-Nicolson finite difference method respectively in solving the 1D PME numerically.

Besides that, Chew and Sulaiman (2016a) initiated the investigation of an efficient numerical method for solving 1D PME with the use of Newton method and explicit group iterative method. Then, Chew and Sulaiman (2016b) introduced the Half-Sweep Newton-Gauss-Seidel iterative method and showed the effectiveness of the use of half-sweep in finite difference approximation equation for solving 1D PME problem.

The application of the half-sweep approximation equation in obtaining the numerical solution of 1D PME is found to be efficient in terms of total iterations and computation time than the standard implicit finite difference approximation equation.

The concept of half-sweep in finite difference method was introduced by Abdullah (1991). This technique capable to reduce the computational complexity of solving the large linear system by computing iteratively half of the total interior grid points. The remaining half of the total interior grid points are computed directly using the finite difference approximation equation. Since, then, the half-sweep has been extensively studied and applied in solving a number of mathematical problems which can be referred by Muthuvalu and Sulaiman (2009), Akhir *et al*. (2011), Aruchunan and Sulaiman (2011), Saudi and Sulaiman (2012) and Dahalan and Sulaiman (2016).

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This study aims to combine the technique of half-sweep and the explicit group iterative method for solving the 1D PME. The implicit finite difference solution of 1D PME is obtained using the proposed iterative method that is called the half-sweep Newton-Explicit Group (HSNEG) Method.

MATERIALS AND METHODS

Finite difference approximation equation: The considered mathematical problem:

$$\frac{\partial u}{\partial t} = c\frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right) \tag{1}$$

subjects to the following prescribed initial-boundary conditions:

$$u(x, 0) = u_0(x)u(0, t) = g_0(t)u(1,t) = g_1(t) \tag{2}$$

with c and m are real parameters. Let considers the solution function u = u (x_p, t_n) = u (ph, nk) with p = 1, 2, ... , M and n = 0, 1, ... , T is distributed evenly in the domain of $\Omega\in[0, 1]$ with a fixed size of h. Similarly, the time interval is set to be within $I\in[0, 1]$ with a fixed size of k. Both the spatial and temporal steps are denoted as h = 1/M and k = 1/T.

When Eq. 1 is discretized using the implicit finite difference scheme, the approximation equation can be written as:

$$\begin{aligned} &u_{p,n+1} - \alpha u^m_{p,n+1}u_{p+1,n+1} + 2\alpha u^{m+1}_{p,n+1} - \\ &\alpha u^m_{p,n+1}u_{p-1,n+1} - \beta m u^{m-1}_{p,n+1}u^2_{p+1,n+1} + \\ &2\beta m u^{m-1}_{p,n+1}u_{p+1,n+1}u_{p-1,n+1} - \\ &\beta m u^{m-1}_{p,n+1}u^2_{p-1,n+1} - u_{p,n} = f_{p,n+1} \end{aligned} \tag{3}$$

Where:
α = ck/h^2
β = $ck/4h^2$

Equation 3 is the common implicit finite difference approximation equation which is also known as the full-sweep approximation equation. Now, by implementing the half-sweep implicit finite difference scheme, Eq. 1 becomes:

$$\begin{aligned} &u_{p,n+1} - \alpha u^m_{p,n+1}u_{p+2,n+1} + 2\alpha u^{m+1}_{p,n+1} - \\ &\alpha u^m_{p,n+1}u_{p-2,n+1} - \beta m u^{m-1}_{p,n+1}u^2_{p+2,n+1} + \\ &2\beta m u^{m-1}_{p,n+1}u_{p+2,n+1}u_{p-2,n+1} - \\ &\beta m u^{m-1}_{p,n+1}u^2_{p-2,n+1} - u_{p,n} = f_{p,n+1} \end{aligned} \tag{4}$$

Where:
α = $ck/4h^2$
β = $ck/16h^2$

Both Eq. 3 and Eq. 4 can be generally represented as:

$$\begin{aligned} &u_{p,n+1} - \alpha u^m_{p,n+1}u_{p+s,n+1} + 2\alpha u^{m+1}_{p,n+1} - \\ &\alpha u^m_{p,n+1}u_{p-s,n+1} - \beta m u^{m-1}_{p,n+1}u^2_{p+s,n+1} + \\ &2\beta m u^{m-1}_{p,n+1}u_{p+s,n+1}u_{p-s,n+1} - \\ &\beta m u^{m-1}_{p,n+1}u^2_{p-s,n+1} - u_{p,n} = f_{p,n+1} \end{aligned} \tag{5}$$

Where:
α = $ck/(sh)^2$
β = $ck/4(sh)^2$

Based on Eq. 5, the Full-Sweep approximation equation is represented by s = 1 and the half-sweep approximation equation is by s = 2. The application of Eq. 5 into the interior grid points in the domain of Ω will result in a sparse and large-sized nonlinear system in the form of:

$$F(U) = 0 \tag{6}$$

Where:

$$F(U) = \left(f_{1,n+1}(U), f_{2,n+1}(U), \ldots, f_{M-1,n+1}(U)\right)$$

And:

$$U = \left(u_{1,n+1}, u_{2,n+1}, \ldots, u_{M-1,n+1}\right)$$

In this study, Newton method is applied in order to transform the nonlinear system into the corresponding linear system. The following procedure of Newton method for solving Eq. 6. First, the Jacobian matrix of Eq. 6 is obtained and written in the form of:

$$J_F = \begin{bmatrix} \frac{\partial f_{1,n+1}}{\partial u_{1,n+1}} & \frac{\partial f_{1,n+1}}{\partial u_{2,n+1}} & \cdots & \frac{\partial f_{1,n+1}}{\partial u_{M-1,n+1}} \\ \frac{\partial f_{2,n+1}}{\partial u_{1,n+1}} & \frac{\partial f_{2,n+1}}{\partial u_{2,n+1}} & \cdots & \frac{\partial f_{2,n+1}}{\partial u_{M-1,n+1}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_{M-1,n+1}}{\partial u_{1,n+1}} & \frac{\partial f_{M-1,n+1}}{\partial u_{2,n+1}} & \cdots & \frac{\partial f_{M-1,n+1}}{\partial u_{M-1,n+1}} \end{bmatrix} \tag{7}$$

Then, define W = ($W_{1,n+1}$, = $W_{2,n+1}$, ..., $W_{M-1,n+1}$)^T as a correction vector where $w_{p,n+1} = u^{(\ell+1)}_{p,n+1} - u^{(\ell)}_{p,n+1}$ with p = 1, 2, ..., M-1 and ℓ is the iterative index. Using Eq. 7, the corresponding linear system can be written as:

$$J_F W = -F(U) \tag{8}$$

Where:
J_F = (M-1)×(M-1) non-singular matrix
W and F (U) = The column matrices

The solution of Eq. 8 can be obtained using either direct or iterative method. For the case of handling very sparse and large-sized linear system, iterative methods are preferable. The next section will discuss the proposed iterative method that is used to solve Eq. 8.

Explicit group iterative method: In this study, the Newton-generated linear system that is shown in Eq. 8 has the form of a tridiagonal matrix. And then, it is the fact that when the total interior grid points to be computed increases, the computational complexity also increases. To reduce the computational complexity, Explicit Group (EG) iterative method is applied. The EG iterative method which was introduced by Evans (1985), uses small fixed-size groups of grid point strategy to reduce the computational complexity in the iteration process of solving the linear system. The iterative method that solves a block of several interior grid points at one time is believed to be much faster than the point iterative method. By taking a group of four points for instance, a submatrix of Eq. 8 can be written as:

$$\begin{bmatrix} D_1 & V_1 & & \\ L_2 & D_2 & V_2 & \\ & L_3 & D_3 & V_3 \\ & & L_4 & D_4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} R_1 - L_1 w_0 \\ R_2 \\ R_3 \\ R_4 - V_4 w_5 \end{bmatrix} \quad (9)$$

The inversion of Eq. 9 is then produces:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}^{(\ell+1)} = \begin{bmatrix} D_1 & V_1 & & \\ L_2 & D_2 & V_2 & \\ & L_3 & D_3 & V_3 \\ & & L_4 & D_4 \end{bmatrix}^{-1} \begin{bmatrix} R_1 - L_1 w_0 \\ R_2 \\ R_3 \\ R_4 - V_4 w_5 \end{bmatrix} \quad (10)$$

Thus, the HSNEG iterative method can be summarized in the following algorithm.

Algorithm 1; HSNEG iterative method:

i. Set the initial vector U = 1.000 and the tolerance error $\varepsilon = 10^{-10}$

ii. At the time level n

a) Initialize F (U) as well as J_F and set $W^{(0)} = 0$

b) For every group of four points, iterate Eq. 10,

c) Compute the remaining ungrouped points, see Evans (1985) for more details,

d) Check the convergence of:

$$\left|W^{(\ell+1)} - W^{(\ell)}\right| \le \varepsilon$$

e) Calculate, $U^{(\ell+1)} = U^{(\ell)} + W^{(\ell+1)}$

f) Check the convergence of:

$$\left|F\left(U^{(\ell+1)}\right) - F\left(U^{(\ell)}\right)\right| \le \varepsilon$$

iii. Go to n+1 if the solution converges

RESULTS AND DISCUSSION

Numerical experiments: For the comparative analysis between the HSNEG, NEG and NGS, three 1D PME examples are selected. Three criteria are considered namely the total iterations (ℓ_{total}), the computation time measured in seconds (sec.) and the maximum absolute error (Err.). The total number of grid points to be considered is 256, 512, 1024, 2048 and 4096. The following three 1D PME examples.

Example 1:

$$\frac{du}{dt} = \frac{d}{dx}\left(u\frac{du}{dx}\right) \quad (11)$$

with the exact solution u(x, t) = x+t (Polyanin and Zaitsev 2004).

Example 2:

$$\frac{du}{dt} = \frac{d}{dx}\left(u^2\frac{du}{dx}\right) \quad (12)$$

Equation 12 is known as Boussinesq equation that is used in the field of buoyancy-driven flow (Vazquez, 2007). The exact solution is given by $u(x, t) = (x+1)/2(4-t)^{1/2}$ (Wazwaz, 2007).

Example 3:

$$\frac{du}{dt} = \frac{1}{2}\frac{d}{dx}\left(\frac{1}{u^2}\frac{du}{dx}\right) \quad (13)$$

The exact solution is $u(x, t) = (0.7x-0.1225t+1.35)^{-½}$ (Wazwaz, 2007).

From the implementation of the HSNEG, NEG and NGS iterative methods, numerical results are obtained and tabulated in Table 1-3.

Table 1: The numerical result of example 1

M	Method	ℓ_{total}	Secound	Error
256	NGS	48395	19.31	5.33E-07
	NEG	13799	6.58	1.10E-07
	HSNEG	3899	1.93	2.64E-08
512	NGS	169693	133.84	2.10E-06
	NEG	48666	45.24	4.99E-07
	HSNEG	13799	8.31	1.10E-07
1024	NGS	587031	919.22	7.62E-06
	NEG	170300	312.23	2.08E-06
	HSNEG	48666	49.85	4.99E-07
2048	NGS	1993096	6208.25	2.67E-05
	NEG	589214	1153.89	7.63E-06
	HSNEG	170300	344.92	2.08E-06
4096	NGS	6612931	40998.73	9.66E-05
	NEG	2002264	14666.26	2.67E-05
	HSNEG	589214	2382.03	7.63E-06

Table 2: The numerical result of example 2

M	Method	ℓ_{total}	Second	Error
256	NGS	17308	10.88	8.39E-05
	NEG	4836	3.76	8.39E-05
	HSNEG	1361	1.56	8.39E-05
512	NGS	61658	76.91	8.40E-05
	NEG	17333	25.54	8.39E-05
	HSNEG	4836	5.16	8.39E-05
1024	NGS	218147	557.97	8.43E-05
	NEG	61779	180.40	8.40E-05
	HSNEG	17333	28.39	8.39E-05
2048	NGS	763998	3839.51	8.55E-05
	NEG	218686	1287.90	8.43E-05
	HSNEG	61779	195.56	8.40E-05
4096	NGS	2630914	26497.28	8.99E-05
	NEG	766144	9115.08	8.55E-05
	HSNEG	218686	1398.62	8.43E-05

Table 3: The numerical result of example 3

M	Method	ℓ_{total}	Second	Error
256	NGS	24325	16.70	2.71E-06
	NEG	7007	5.74	2.92E-06
	HSNEG	2033	3.86	2.94E-06
512	NGS	81729	113.26	1.86E-06
	NEG	23769	39.00	2.73E-06
	HSNEG	7007	6.44	2.92E-06
1024	NGS	265698	767.23	3.33E-06
	NEG	79057	267.51	1.89E-06
	HSNEG	23769	42.30	2.73E-06
2048	NGS	882282	5164.64	1.66E-05
	NEG	273259	1847.28	3.26E-06
	HSNEG	79057	289.70	1.89E-06
4096	NGS	2853985	33726.52	6.10E-05
	NEG	911564	12906.59	1.64E-05
	HSNEG	273259	1990.15	3.26E-06

CONCLUSION

Based on the numerical results obtained, the efficiency of the HSNEG in solving 1D PME examples has been demonstrated. Numerical results, that are presented showed that the HSNEG is more superior than the NGS and the NEG in terms of total iterations and computation time. HSNEG has reduced the total iterations approximately 90.43-92.16% against the NGS and 69.93-72.10% against the NEG. And then, the computation time required by the HSNEG is shorter about 76.89%-94.49% against the NGS and 32.75%-84.03% against the NEG. All three executed iterative methods showed good accuracy in solving 1D PME.

In this study, we showed that the application of EG iterative method together with the half-sweep implicit finite difference approximation equation gives a promising decrement in the computational complexity for solving the 1D PME. For the future study, the use of Successive Over-Relaxation (SOR) iterative method by Young (1954) in improving the rate of convergence for the numerical solution will be examined.

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