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The Radial Oscillation of Entire Solutions of a Class of Higher Order Complex Differential Equations

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Abstract: In this study, we shall study the radial oscillation of solutions of n-th order complex linear differential equations $f^{(n)} A_{n-1}(z) f^{(n-1)}+...+ A_1(z) f'+A_0(z)f = F(z)$ where, $A_j(z)$, j = 0, 1, ..., n-1 and F(z) are transcendental entire functions. We shall add some conditions on coefficients which show the relation between the Borel directions of solutions f and that of F(z), further we obtain some estimates of growth of solutions in the angular domains.

Key words: Borel direction, entire function, linear differential equation order, estimates, growth, angular

INTRODUCTION

In this study, the Nevanlinna theory in an angle is an important tool (Wu, 1994a, b; Zheng, 2011). In what follows, we also assume the reader is familiar with the classic Nevanlinna theory in the complex plan $\mathbb{C}$ and the standard notations such as Nevanlinna characteristic T(r, f), proximity function m(r, f) and the deficiency δ(a, f) with respect to a (Hayman, 1964; Laine, 1993). Let $0<\alpha<\beta<2\pi$, we let:

$$\Omega(\alpha, \beta) = \{z\in \mathbb{C} | \arg(z)\in(\alpha, \beta)\},$$
$$\Omega(\alpha, \beta, r) = \{Z | z\in\Omega(\alpha, \beta), |z|<r\}$$

and let $\bar{\Omega}(\alpha, \beta)$ denote the closure of $\Omega(\alpha, \beta)$.

Definition 1.1; Huang and Wang (2015): Let g(z) be a function analytic in $\Omega(\alpha, \beta)$. The order of g on $\Omega(\alpha, \beta)$ is defined by:

$$\rho_{\alpha, \beta}(g) = \lim_{r\to\infty} \sup \frac{\log^+\log^+ M(r, \Omega(\alpha, \beta), g)}{\log r}$$

where, $M(r, \Omega(\alpha, \beta), g) = \sup_{\alpha\le\theta\le\beta} |g(re^{i\theta})|$. If g(z) is an entire on $\mathbb{C}$, the order $\rho(g)$ of g satisfies $\rho(g)\ge\rho_{\alpha, \beta}(g)$.

Definition 1.2; Huang and Wang (2015): The sectorial order $\rho_{\theta, \epsilon}(g)$ and the radial order $\rho_\theta(g)$ are defined by:

$$\rho_{\theta, \epsilon}(g) = \lim_{r\to\infty} \sup \frac{\log^+\log^+ M(r, \Omega(\theta-\epsilon, \theta+\varepsilon, \theta+\epsilon), g)}{\log r}$$
$$\rho_\theta(g) = \lim_{\varepsilon\to\infty} \rho_{\theta, \epsilon}(g)$$

respectively. Similarly, the sectorial, resp. radial, exponent of convergence for zeros of g are defined by:

$$\lambda_{\theta, \epsilon}(g) = \lim_{r\to\infty} \sup \frac{\log^+ n(\Omega(\theta-\epsilon, \theta+\varepsilon, r), g=0}{\log r},$$
$$\lambda_\theta(g) = \lim_{\varepsilon\to\infty} \lambda_{\theta, \epsilon}(g)$$

where, $n(\Omega(\theta-\epsilon, \theta+\epsilon, r), g = 0)$ stands for the number of zeros of g in $\Omega(\theta-\epsilon, \theta+\epsilon, r)$ counting multiplicity.

Definition 1.3; Huang and Wang (2015): Let f(z) be a transcendental meromorphic function of order ρ. The ray arg (z) = θ is called a Borel direction of f if for any $\epsilon>0$, $\lambda_{\theta, \epsilon}(f-a) = \rho$ with at most two exceptional values $a\in\mathbb{C}\cup\{\infty\}$.

Definition 1.4; Huang and Wang (2015): Let f be a meromorphic function, then the lower order μ(f) of f is defined by:

$$\mu(f) = \lim_{x\to\infty} \inf \frac{\log^+ T(r, f)}{\log r}$$

We mention the Nevanlinna characteristic for an angle (Wu, 1994a, b).

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Definition 1.5 (Goldberg and Ostrovskii, 2008; Wu 1994a, b): Let $0<\beta-\alpha\leq 2\pi$, $k = \pi/\beta-\alpha$ and h(z) is meromorphic on the angular domain $\Omega(\alpha, \beta)$, we define:

$$A_{\alpha,\beta}(r, h) = \frac{k}{\pi}\int_1^r\left(\frac{1}{t^k}-\frac{t^k}{r^{2k}}\right)\left\{\log^+\left|h\left(te^{i\alpha}\right)\right|+\log^+\left|h\left(te^{i\beta}\right)\right|\right\}\frac{dt}{t}$$

$$B_{\alpha,\beta}(r, h) = \frac{2k}{\pi r^k}\int_\alpha^\beta \log^+\left|h\left(re^{i\theta}\right)\right|\sin k(\theta-\alpha)\, d\theta$$

$$C_{\alpha,\beta}(r, h) = 2\sum_{1<|b_v|<r}\left(\frac{1}{|b_v|^k}-\frac{|b_v|^k}{r^{2k}}\right)\sin k(\beta_v-\alpha)$$

$$D_{\alpha,\beta}(r,h) = A_{\alpha,\beta}(r, h)+B_{\alpha,\beta}(r, h)$$

$$S_{\alpha,\beta}(r, h) = D_{\alpha,\beta}(r, h)+C_{\alpha,\beta}(r, h)$$

where, $b_v = |b_v|e^{i\beta_v}$, (v = 1, 2, ...) are the poles of h(z) in $\bar{\Omega}(\alpha, \beta, r)$, counting multiplicities.

Definition 1.6; Huang and Wang (2012): For any $(a<\infty)\in C$, we write $C_{\alpha,\beta}(r, h = a) = C_{\alpha,\beta}(r, 1/(h-a))$ and for any $0<\epsilon<\pi/2k$ (Huang and Wang, 2012; Goldberg and Ostrovskii, 2008):

$$C_{\alpha,\beta}(r, h=a)\geq 2\left(1-\frac{1}{2^{2k}}\right)\sin(k\epsilon)\frac{1}{r^k}n\left(\Omega(\theta-\epsilon, \theta+\epsilon, r), h-a=0\right)$$

We use $\sigma_{\alpha,\beta}(h)$ to denote the order of $S_{\alpha,\beta}$ which is defined by:

$$\sigma_{\alpha,\beta}(r, h) = \limsup_{r\to\infty}\frac{\log S_{\alpha,\beta}(r, h)}{\log r}$$

Definition 1.7; Laine (1993): Set $B(z_n, r_n) = \{z:|z-z_n|<r_n\}$, the set $D = U^{\infty}_{n=1}$ is called a $\sum_{n=1}^{\infty} r_n<\infty$ if $z_n\to\infty$ and $r_n\to\infty$ as $z\to\infty$. Wu (2005) first studied the Borel directions of solutions of second order linear differential equations:

$$f''+A(z)f'+B(z)f = F(z) \tag{1.1}$$

where, A(z), B(z) and F(z) are entire functions. In this study, we shall consider the n-th order linear complex differential equation:

$$f^{(n)}+A_{n-1}(z)f^{(n-1)}+,...,+A_1(z)f'+A_0(z)f = F(z) \tag{1.2}$$

where, $A_j(z)$, j = 0, 1, ..., n-1 and F(z) are entire functions. We introduce the results obtained by Z. Huang and Wang (2015) for Eq. 1 as follows:

Theorem A; Huang and Wang (2015): Let A(z), B(z) be entire functions with finite order, let F(z) be transcendental entire and max $\{\rho(A), \rho(B)\}<\rho(F) = \infty$. If arg(z) is a Borel direction of F, then it is also a Borel direction of every non-trivial solution f(z) of Eq. 1.

Theorem B; Huang and Wang (2015): Let A(z), B(z) be entire functions with finite order, let F(z) be transcendental entire and max $\{\rho(A), \rho(B)\}<\rho<\infty$. Suppose that f(z) is a solution of Eq. 1. If arg(z) = θ is a Borel direction of F(z), then for any angular domain $\Omega(\alpha,\beta)$ contained the ray arg(z) = θ with $\beta-\alpha>\pi/\rho$, there exists a Borel direction of f in $\Omega(\alpha, \beta)$.

Theorem C; Huang and Wang (2015): Suppose that A(z), B(z) are entire functions with $\mu(A)>\rho(B)$. If f(z) is a non-trivial solution of equation:

$$f''+A(z)f'+B(z)f = 0$$

Then mesI(f)$\geq$min $\{2\pi, \pi/\mu(B)\}$ where, $I(f) = \{\theta\in[0.2\pi): \rho_\theta = \infty\}$. Wu (1994a, b) used the Nevanlinna theory in an angular region to study the growth of solutions of complex differential equations in an angular domain and some similar results about higher order differential equations were obtained by several researchers (Wu, 2013; Xu and Yi, 2009).

MATERIALS AND METHODS

Some needed lemmas: In this study, we give some lemmas which is used in proof of our results.

Lemma 2.1; Milloux (1951): If h(z) is an entire function with $0<\rho(h)<\infty$, then a Borel directions of order ρ for $h'(z)$ is also a Borel direction of order ρ for h(z).

Lemma 2.2; Sun (1987): Let h(z) be an entire function of infinite order. Then the ray arg(z) = θ is a Borel direction of infinite order for h if and only if arg(z) = θ is a Borel direction of infinite order for h'. Let f(z) be meromorphic in the disc |z|<1, we define the order of f(z) in unit disc by $\limsup_{r\to 1}\log^+T(r, f)/(\log 1/1-r)$:

Lemma 2.3; Chuang (1999): Let f be meromorphic function of infinite order. Then, the ray arg(z) = θ is one Borel direction of order θ of f if and only if f satisfies the equality:

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\epsilon,\,\theta+\epsilon}(r, f)}{\log r} = \infty \tag{2.1}$$

For any $\epsilon(0<\epsilon<\pi/2)$.

Lemma 2.4; Wu (1994a, b) and Zhang (1993): Suppose that f(z) is a transcendental entire function with order

$\rho(f) = \rho(0<\rho<\infty)$ and $\Omega(\alpha,\beta)$ is an angular domain with $\beta-\alpha>\pi/\rho$. If there is no Borel direction of order ρ for f(z) in $\Omega(\alpha, \beta)$ then $\rho_{\alpha,\beta}(f)<\rho$.

Lemma 2.5; Huang and Wang (2012): Let $z = r\exp(i\psi)$, $r_0+1<r$ and $\alpha\leq\psi\leq\beta$ where, $0<\beta-\alpha\leq 2\pi$. Suppose that n (≥ 2) is an integer and h(z) is analytic in $\Omega(r_0, \alpha, \beta)$ with $\sigma_{\alpha,\beta}$. Then for every $\varepsilon_j\in(0, \beta_j-\alpha_j/2)$, (j = 1, 2, ..., n-1) outside a set of linear measure zero with $\alpha_j = \alpha+\sum_{s=1}^{j-1}\varepsilon_s$ and $\beta_j = \beta-\sum_{s=1}^{j-1}\varepsilon_s$ there exist K>0, M>0 only depending on h, $\epsilon_1, ..., \epsilon_{n-1}$ and $\Omega(\alpha_{n-1}, \beta_{n-1})$ and not depending on z such that:

$$\left|\frac{h^{(n)}(z)}{h(z)}\right| \leq Kr^M\left(\sin k(\psi-\alpha)\prod_{j=1}^{n-1}\sin k_{\varepsilon j}(\psi-\alpha_j)\right)^{-2}$$

for all $z\in\Omega(\alpha_{n-1}, \beta_{n-1})$ outside a R-set D where, $k = \pi/\beta-\alpha$ and $\sin k_{\epsilon j} = \pi(\beta_j-\alpha_j)^{-1}$, (j = 1, 2, ..., n-1). The definition of Polya peak for the Nevanlinna characteristic T (r, f) could be found by Zheng (2011) and Yang (1993).

Lemma 2.6; Baernstein (1973): Let f(z) be a transcendental meromorphic function of finite lower order μ and have one deficient value a. Let A(r) be a positive function with A(r) = o(T(r, f)) as $r\to\infty$. Then for any fixed sequence of Polya peaks $\{r_n\}$ of order μ, we have:

$$\liminf_{r\to\infty} \text{mes}\, D_\Lambda(r_n, a) \geq \min\left\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\delta(\alpha, f)}{2}}\right\} \quad (3.1)$$

where, $D_\Lambda(r, a)$ is defined by $D_\Lambda(r, \infty) = \{\theta\in[-\pi, \pi): |f(re^{i\theta})|> e^{\Lambda(r)T(r,f)}\}$ and for finite a, $D_\Lambda(r, a) = \{\theta\in[-\pi, \pi): |f(re^{i\theta})|< e^{\Lambda(r)T(r,f)}\}$.

Lemma 2.7; Zheng (2011) and Wu (2005): Let f(z) be meromorphic and of order λ $(0<\lambda<\infty)$ in the finite plane. If B: $\arg z = \theta_0$, $0\leq\theta_0<2\pi$ is a Borel direction of f(z), then there exists a sequence of disks:

$$\Gamma_j = \{z: |z-z_j|<\varepsilon_j|z_j|\},\quad z_j = |z_j|e^{i\theta_0}, (j = 1, 2, ...)$$

with $\lim_{j\to\infty}|z_j| = \infty$, $\lim_{j\to\infty}\epsilon_j = 0$ such that f(z) takes every complex number at least $n_j = |z_j|^{\lambda-\delta_j}$ times in Γ_j, except possibly for those numbers contained in two spherical disks each with radius e^{-nj} where $\lim_{j\to\infty}\delta_j = 0$.

RESULTS AND DISCUSSION

Theorem 3.1: Let $A_j(z)$, j = 0, 1, ..., n-1 be entire functions with finite order, let F(z) be transcendental entire and $\max_{1\leq j\leq n-1}\{\rho(A_j), \rho(A_0)\}<\rho(F)=\infty$. Let $f\neq 0$ be a solution of Eq. 1.2 and arg (z) = θ is a Borel direction of F, then arg (z) = θ is also a Borel direction of f.

Proof: Under the condition max $\{\rho(A_j), \rho(A_0)\}<\rho(F) = \infty$, it is easy to see that every solution f of Eq. 1.2 must also satisfy $\rho(f) = \infty$. Suppose that arg$z = \theta\in[0, 2\pi)$ is a Borel direction of F (z). By Lemma 2.5, for any sufficiently small ϵ:

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r, F)}{\log r} = \infty$$

It follows from (1.2) that:

$$S_{\theta-\varepsilon,\theta+\varepsilon}(r, F)\leq S_{\theta-\varepsilon,\theta+\varepsilon}(r, f^{(n)}+A_{n-1}(z)f^{(n-1)}+, ..., +A_1(z)f'+A_0(z)f)$$

which implies:

$$(1-o(1))\, S_{\theta-\varepsilon,\theta+\varepsilon}(r, F)\leq\sum_{j=0}^{n}S_{\theta-\varepsilon,\theta+\varepsilon}(r, f^{(j)})$$

It follows that there exists at least one such that:

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r, f^{(l)})}{\log r} = \infty$$

By using Lemma 2.3 again, we deduce that arg$z = \theta$ is also a Borel direction of $f^{(l)}$. Hence by Lemma 2.2, the ray arg$z = \theta$ is a Borel direction of f.

Theorem 3.2: Let $A_j(z)$, j = 0, 1, 2, ..., n-1, F(z) be as in theorem (3.1) and $\max_{1\leq j\leq n-1}\{\rho(A_j), \rho(A_0)\}<\rho(F)=\rho<\infty$. If arg (z) = θ is a Borel direction of F(z), then for any angular domain $\Omega(\alpha, \beta)$ contained arg(z) with $\beta-\alpha>\pi/\rho$, there exists a Borel direction of f in $\Omega(\alpha, \beta)$ where $f\neq 0$ is a solution of Eq. 1.2.

Proof: Suppose that arg$z = \theta\in[0, 2\pi)$ is a Borel direction of order ρ for F(z). By Lemma 2.7, there exists a sequence of disks $\Gamma_j = \{z:|z-z_j|<\epsilon_j|Z_j|\}$, (j = 1, 2, ...) with arg$z_j = \theta$, $\lim_{j\to\infty}|z_j| = \infty$ and $\lim_{j\to\infty}\varepsilon_j = 0$. For entire functions, ∞ is always a Picard value, so, ∞ is located in one of the two spherical disks in Lemma 2.7. Denote the spherical distance of z_1, z_2 by $|z_1, z_2|$ and we can find a point $b_j\in D_j$ such that:

$$|F(b_j), \infty| = \frac{1}{\left(1+|F(b_j)|^2\right)^{\frac{1}{2}}} = 2e^{-|z_j|\rho-\delta_j} \quad (3.2)$$

where, $\lim_{j\to\infty}\delta_j = 0$. Then, we can find a constant C independent of j such that for all sufficiently large j:

$$\left|F(b_j), \infty\right| > Ce\left|z_j\right|^{\rho-\delta_j}$$

Since, $|b_j| = (1+o(1))\,|z_j|$, we can conclude that for any given $\epsilon>0$:

$$\limsup_{r\to\infty} \frac{\log\log M(r, \Omega(\theta-\varepsilon, \theta+\varepsilon), F)}{\operatorname{Log} r} \geq \rho \quad (3.2)$$

We assume that there exists no Borel direction of f in $\Omega(\alpha, \beta)$. It is easy to see that every solution f of Eq. 1.2 must satisfy $0<\rho(f) = \rho<\infty$. By Lemma 2.1, $\Omega(\alpha, \beta)$ also does not contain any Borel direction of f^j, (j = 1, 2, ..., n-1). Thus, applying Lemma 2.6, we get:

$$\limsup_{r\to\infty} \frac{\log\log M(r, \Omega(r, \Omega(\alpha, \beta), F_j)}{\operatorname{Log} r} < \rho, \; j = 0, 1, ..., n \quad (3.3)$$

Since, $\max\{\rho(A_j), \rho(A_0)\}<\rho$, substituting Eq. 3.2 and Eq. 3.3 into 1.2 yields a contradiction. Therefore, there must have at least one Borel direction in $\Omega(\alpha, \beta)$.

Theorem 3.3: Suppose that $A_j(z)$, j = 0, 1, ..., n-1 are entire functions with $(A_0)>\rho(A_j)$, j = 1, 2, ..., n-1. If $f\neq 0$ is a solution of equation:

$$f^{(n)}+A_{n-1}(z) f^{(n-1)}+, ..., +A_1(z) f'+A_0(z) f = 0 \quad (3.1)$$

Then, $\operatorname{mesI}(f)\geq\min\{2\pi, \pi/\mu(A_0)\}$ where $I(f) = \{\theta\epsilon[0, 2\pi): \rho_\theta(f) = \infty\}$.

Proof: Suppose that $\operatorname{mesI}(f)<\sigma: = \min\{2\pi/\mu(A_0)$, so, $\zeta: = \sigma\text{-mesI}(f)>0$. Since, I(f) is closed, clearly $S = (0, 2\pi)/I(f)$ is open, so, it consists of at most countably many open intervals. We can choose finitely many open intervals $I_i = (\alpha_i, \beta_i)$ (i = 1, 2, ..., m) satisfying $[\alpha_i, \beta_i]\subset S$ and $\operatorname{mess}(S/\cup^m_{i=1} I_i)<\zeta/4$. For the angular domain $\Omega(\alpha_i\, \beta_i)$, it is easy to see:

$$\Omega(\alpha_i, \beta_i)\cap I(f) = \varnothing$$

for sufficiently large r. This implies that for each i = 1, 2, ..., k, $\rho_{\alpha i, \beta i}<+0$ and from the definition of $S_{\alpha i, \beta i}(f)<\infty$. Therefore, by Lemma 2.7 for sufficiently small $\varepsilon>0$, there exist two constants M>0 and K>0 such that:

$$\left|\frac{f^{(s)}(z)}{f(z)}\right| \leq Kr^M, \; s = 1, 2, ..., n \quad (3.4)$$

for all $z\in\cup^k_{i=1}\Omega(\alpha_i+2\epsilon, \beta_i-2\epsilon)$, outside a R-set H. Applying Lemma 2.6 to $A_0(z)$ gives the existence of the Polya peak $\{r_n\}$ of order $\mu(A_0)$ such that $r_n\notin\{|z|, z\in H\}$ and for sufficiently large n:

$$\operatorname{mes}(D_A(r_n, \infty)) \geq \sigma = \min\left\{2\pi, \frac{\pi}{\mu(A_0)}\right\} \quad (3.5)$$

where, we take the function A(r) as:

$$A(r) = \max\left\{\sqrt{\frac{\log r}{T(r, A_0)}}, \sqrt{\frac{T(r, A_j)}{T(r, A_0)}}\right\}, \; j = 1, 2, ..., n-1$$

Without loss of generality, we assume Eq. 3.5 holds for all the n and set $D(r_n) = D_N(r_n, \infty)$ for brevity. Clearly:

$$\operatorname{mes}\left(D(r_n)\cap S\right) = \frac{\operatorname{mes}(D(r_n))}{(I(f)\cap D(r_n))} \geq \operatorname{mes}\left(D(r_n)\right)\text{-mesI}(f) > \frac{3\varsigma}{4} \quad (3.6)$$

Then for each n we have:

$$\operatorname{mes}\left(\left(\bigcup_{i=1}^{k} I_i\right)\cap D(r_n)\right) = \operatorname{mes}(S\cap D(r_n))\text{-mes}\left(\left(\frac{S}{\bigcup_{i=1}^{k} I_i}\right)\cap D(r_n)\right) > \frac{3\varsigma}{4}-\frac{\varsigma}{4} = \frac{\varsigma}{2'} \quad (3.7)$$

It implies that there exists at least one open interval $I_{i0} = (\alpha, \beta)$ of I_i (I = 1, 2, ..., k) such that for infinitely many j:

$$\operatorname{mes}(D(r_j)\cap(\alpha, \beta)) > \frac{\varsigma}{2k} > 0 \quad (3.8)$$

Set $F_j = (D(r_j)\cap(\alpha+2\varepsilon, \beta-2\varepsilon))$ it is easy to see:

$$\int_{F_j} \log^+\left|A_0\left(r_j e^{i\theta}\right)\right| d\theta \geq \left(\operatorname{mes}\left(F_j\right)\right)\, \Lambda(r_j)\, T(r_j, A_0) \quad (3.9)$$

From Eq. 3.1, we have:

$$A_0(z) = -\frac{f^{(n)}}{f}\text{-}A_{n-1}(z)\frac{f^{(n-1)}}{f}, ..., A_1(z)\frac{f'}{f}$$

From $\rho(A_j)<\mu(A_0)$ and Eq. 3.4, we can obtain:

$$\int_{F_j}\log^+\left|A_0\left(r_j e^{i\theta}\right)\right| d\theta \leq \int_{F_j}\left(\sum_{i=1}^{n}\log^+\left|\frac{f^{(i)}\left(r_j e^{i\theta}\right)}{f\left(r_j e^{i\theta}\right)}\right| + \log^+\left|A_q\left(r_j\, e^{i\theta}\right)\right|\right)$$
$$d\theta+0(1) \leq \operatorname{mes}(F_j)\,(T(r_j, A_q)+O(\log r_j))$$
$$\leq c_0\operatorname{mes}(F_j)\,\Lambda^2(r_j)T(r_j, A_0), \; q = 1, 2, ..., n-1 \quad (3.10)$$

where, C_0 is a positive constant. Regarding Eq. 3.9 and 3.10 yields a contradiction. Therefore, there must have:

$$\text{mes } I(f) \geq \sigma := \min\left\{2\pi, \frac{\pi}{\mu(A_0)}\right\}$$

CONCLUSION

The Nevanlinna theory of value distribution of meromorphic functions in the finite plan is very powerful tool to study the higher order complex differential equations.

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