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Computational Complexity Analysis of Developing Block Methods for Solving Second Order Ordinary Differential Equations using Numerical Integration, Collocation and Linear Block Approach

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Abstract: This study presents three approaches for developing block methods for solving second order ODEs. These approaches include the conventional numerical integration and collocation approaches while in addition, considering a new approach called the linear block approach. A sample two-step block method is developed using the two conventional approaches and from the general form taken by the resulting block method, the linear block approach is adopted to directly obtain the block method. To investigate the rigour involved in adopted these approaches, the computational complexity analysis is investigated and it is observed that the new linear block approach is most suitable for developing block methods.

Key words: Block methods, numerical integration, collocation, linear block, second order, ordinary differential equations

INTRODUCTION

Consider the initial value problem for second order ordinary differential equation:

$$y'' = f(x, y, y'),\ y(a) = \alpha,\ y'(a) = \beta \tag{1}$$

An approximate solution for Eq. 1 is sought within the range $a \le x \le b$ (a, b are finite). In addition it is assumed that f satisfies the conditions of existence and uniqueness of solution (Lambert, 1973). To obtain a numerical approximation to Eq. 1, the initial approach introduced by Lambert (1973) is the general Linear Multistep Method (LMM) which takes the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^m \sum_{j=0}^{k} \beta_j f_{n+j} \tag{2}$$

Where:
α_j and β_j = Constants
k = The stepnumber
h = The steplength

To adopt the LMM for solving (Eq. 1), it is required to provide starting values using methods such as one-step methods or truncated Taylor series. However, these approaches have low level of accuracy in the case of the one-step methods or the required partial derivatives fail to exist for the truncated Taylor series. This led to the introduction of predictor-corrector methods with better accuracy but more computational evaluations and rigour (Fatunla, 1988; Awoyemi, 2003; Butcher, 2008; Olabode, 2009; Kayode and Adeyeye, 2011). Thus, further research birthed block methods. Block methods were first proposed by Milne (1953) as a means to obtain starting values for predictor-corrector methods and this concept was also explored by Sarafyan (1965).

The conventional approaches of numerical integration and collocation for developing LMMs can also be adopted to develop block methods (Omar and Kuboye, 2015). This is discussed in the study as well as introducing a new linear block approach. Details of adopting each approach and the computational rigour encountered are discussed in the following section. The complexity is computed in terms of the number of operations involved in each step of the approaches.

MATERIALS AND METHODS

The two-step block method for second order ODEs is developed as a sample to show which of the approaches require less computations.

Developing two-step block method for second order ODEs using numerical integration approach

Step 1: Evaluate y'_{n+1} by integrating (Eq. 1) once over the interval $[x_n, x_{n+1}]$ as given below:

$$\int_{x_n}^{x_n} y''(x)\,dx = \int_{x_n}^{x_{n+1}} f(x, y\, y')\,dx \tag{3}$$

which gives:

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$$y'(x_{n+1}) - y'(x_n) = \int_{x_n}^{x_{n+1}} f(x, y, y')\, dx \quad (4)$$

Replacing f (x, y, y′) with lagrange polynomial of the interpolation points at (x_n, f_n), (x_{n+1}, f_{n+1}) and (x_{n+2}, f_{n+2}) defined as:

$$P(x) = \frac{(x-x_{n+1})(x-x_{n+2})}{(x_n-x_{n+1})(x_n-x_{n+2})} f_n + \frac{(x-x_n)(x-x_{n+2})}{(x_{n+1}-x_n)(x_{n+1}-x_{n+2})} f_{n+1} + \frac{(x-x_n)(x-x_{n+1})}{(x_{n+2}-x_n)(x_{n+2}-x_{n+1})} f_{n+2} \quad (5)$$

and then taking the integral of Eq. 4 gives:

$$y'_{n+1} = y'_n + \frac{h}{12}(5 f_n + 8 f_{n+1} - f_{n+2}) \quad (6)$$

Step 2: Evaluate y_{n+1} by integrating (Eq. 1) twice over the interval $[x_n, x_{n+1}]$ given as:

$$\int_{x_n}^{x_{n+1}}\int_{x_n}^{x} y''(x)\, dx\, dx = \int_{x_n}^{x_{n+1}}\int_{x_n}^{x} f(x, y, y')\, dx\, dx \quad (7)$$

which gives:

$$y(x_{n+1}) - y''(x)\, dx\, dx = \int_{x_n}^{x_{n+1}}\int_{x_n}^{x} f(x, y, y')\, dx\, dx \quad (8)$$

Replacing f (x, y, y′) with Lagrange polynomial (Eq. 5) and then taking the integral of Eq. 8 gives:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{24}(7 f_n + 6 f_{n+1} - f_{n+2}) \quad (9)$$

Step 3: Evaluate y'_{n+2} by integrating (Eq. 1) once over the interval $[x_n, x_{n+2}]$ given as:

$$\int_{x_n}^{x_{n+2}} y''(x)\, dx = \int_{x_n}^{x_n} f(x, y, y')\, dx \quad (10)$$

which gives:

$$y'(x_{n+2}) - y'(x_n) = \int_{x_n}^{x_{n+2}} f(x, y, y')\, dx \quad (11)$$

Replacing f (x, y, y′) with Lagrange polynomial (Eq. 5) and then taking the integral of Eq. 11 gives:

$$y'_{n+2} = y'_n + \frac{h}{3}(f_n + 4 f_{n+1} + f_{n+2}) \quad (12)$$

Step 4: Evaluate y_{n+2} by integrating (Eq. 1) twice over the interval $[x_n, x_{n+2}]$ given as:

$$\int_{x_n}^{x_{n+2}}\int_{x_n}^{x} y''(x)\, dxdx = \int_{x_n}^{x_{n+2}}\int_{x_n}^{x} f(x, y, y')\, dxdx \quad (13)$$

which gives:

$$y(x_{n+2}) - y(x_n) - 2hy'(x_n) = \int_{x_n}^{x_{n+2}}\int_{x_n}^{x} f(x, y, y')\, dxdx \quad (14)$$

Replacing f (x, y, y′) with Lagrange polynomial (Eq. 5) and then taking the integral of Eq. 14 gives:

$$y_{n+2} = y_n + 2hy'_n + \frac{h^2}{3}(2 f_n + 4 f_{n+1}) \quad (15)$$

Combining (Eq. 6, 9, 12 and 15) gives the block method:

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + \frac{h^2}{24}(7 f_n + 6 f_{n+1} - f_{n+2}) \\ y_{n+2} &= y_n + 2hy'_n + \frac{h^2}{3}(2f_n + 4 f_{n+1}) \\ y'_{n+1} &= y'_n + \frac{h}{12}(5 f_n + 8 f_{n+1} - f_{n+2}) \\ y'_{n+2} &= y'_n + \frac{h}{3}(f_n + 4 f_{n+1} + f_{n+2}) \end{aligned} \quad (16)$$

The correctors of the block method (Eq. 16) takes the form:

$$A^0 Y_{n+k} = A^1 Y_{n-k} + B^1 Y'_{n-k} + h^2 (C^0 Y''_{n+k} + C^1 Y''_{n-k}) \quad (17)$$

Where:

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, Y_{n+k} = \begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix}, A^1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, Y_{n-k} = \begin{pmatrix} y_{n-1} \\ y_n \end{pmatrix},$$

$$B^1 = \begin{pmatrix} 0 & h \\ 0 & 2h \end{pmatrix}, Y'_{n-k} = \begin{pmatrix} y'_{n-1} \\ y'_n \end{pmatrix}, C^0 = \begin{pmatrix} \frac{6h^2}{24} & -\frac{h^2}{24} \\ 4h^2 & 0 \end{pmatrix},$$

$$Y''_{n+k} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix}, C^1 = \begin{pmatrix} 0 & \frac{7h^2}{24} \\ 0 & \frac{2h^2}{3} \end{pmatrix} \text{ and } Y''_{n-k} = \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix}$$

Developing two-step block method for second order ODEs using collocation approach

Step 1: Consider an approximate solution to the second order ODE in Eq. 1 in form of the power series:

$$y(x) = \sum_{j=0}^{k+2} a_j x^j \quad (18)$$

which can be expanded to take the form:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \quad (19)$$

Since, the ODE under consideration is a second order ODE, the power series approximate solution (Eq. 19) is differentiated twice to give:

$$y''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 \quad (20)$$

Step 2: Equation 19 is interpolated at x_{n+i}, i = 0, 1 while Eq. 20 is collocated at points x_{n+i}, i = 0, 1, 2 to give a system of equations:

$$\begin{aligned} y_n &= a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 \\ y_{n+1} &= a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + a_4 x_{n+1}^4 \\ f_n &= 2a_2 + 6a_3 x_n + 12a_4 x_n^2 \\ f_{n+1} &= 2a_2 + 6a_3 x_{n+1} + 12a_4 x_{n+1}^2 \\ f_{n+2} &= 2a_2 + 6a_3 x_{n+2} + 12a_4 x_{n+2}^2 \end{aligned} \quad (21)$$

which can be written in matrix form:

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ f_n \\ f_{n+1} \\ f_{n+2} \end{pmatrix} \quad (22)$$

Step 3: Using Gaussian elimination method to obtain the values of a_i, i = 0, 1, ..., 4 in Eq. 22 gives:

$$\begin{aligned} a_0 &= \frac{f_n - 2f_n + 1 + f_n + 2}{24h^2} \\ a_1 &= \frac{f_n + (h + x_n)}{3h^2} - \frac{f_n(3h + 2x_n)}{12h^2} - \frac{f_{n+2}(h + 2x_n)}{12h^2} \\ a_2 &= \frac{f_{n+2}(x_n^2 + hx_n)}{4h^2} - \frac{f_{n+1}(x_n^2 + 2hx_n)}{2h_2} + \frac{f_n(2h^2 + 3hx_n + x_n^2)}{4h^2} \\ a_3 &= \frac{y_{n+1}}{h} - \frac{y_n}{h} - \frac{f_n(7h^3 + 24h^2 x_n + 18hx_n^2 + 4x_n^3)}{24h^2} - \\ &\frac{f_{n+2}(-h^3 + 6hx_n^2 + 4x_n^3)}{24h^2} + \frac{f_{n+1}(-3h^3 + 12hx_n^2 + 4x_n^3)}{12h_2} \\ &- \frac{x_n y_n + 1}{h} + \frac{y_n(h + x_n)}{h} \end{aligned} \quad (23)$$

Substituting a_i, i = 0, 1, ..., 4 in Eq. 21 gives a continuous linear multistep method of the form:

$$y(x) = \sum_{j=0}^{1} \alpha_j(x)\, y_{n+j} + h^2 \sum_{j=0}^{2} \beta_j(x)\, f_{n+j} \quad (24)$$

Step 4: Using the transformation $z = x - x_{n+k-1}/h$ implies:

$$x = zh + x_n + h \quad (25)$$

Substituting Eq. 25 in Eq. 24 and simplifying gives the following:

$$\begin{aligned} \alpha_0(z) &= -z \\ \alpha_0(z) &= 1 + z \\ \beta_0(z) &= \frac{1}{24}(z^4 - 2z^3 + 3z) \\ \beta_1(z) &= \frac{1}{24}(-z^4 + 6z^2 + 5z) \\ \beta_2(z) &= \frac{1}{24}(z^4 + 2z^3 - z) \end{aligned} \quad (26)$$

Evaluating Eq. 26 at the non-interpolating point which implies z = 1 results in the following scheme:

$$y_{n+2} = -y_n + 2y_{n+1} + \frac{h^2}{12}(f_n + 10f_{n+1} + f_{n+2}) \quad (27)$$

Step 5: Since, the block method to be derived is to solve second order ODEs, the first derivative of Eq. 26 is computed to evaluate the first derivative schemes at all grid points. The derivatives are obtained given as:

$$\begin{aligned} \alpha_0'(z) &= -1 \\ \alpha_0'(z) &= 1 \\ \beta_0'(z) &= \frac{1}{24}(4z^3 - 6z^2 + 3) \\ \beta_1'(z) &= \frac{1}{12}(-4z^3 + 12z + 5) \\ \beta_2'(z) &= \frac{1}{24}(4z^3 + 6z^2 - 1) \end{aligned} \quad (28)$$

Evaluating Eq. 28 at all grid points which implies z = 0, 1, -1 results in the following schemes:

$$y_n' = \frac{1}{h}(-y_n + y_{n+1}) + \frac{h}{24}(-7f_n - 6f_{n+1} + f_{n+2}) \quad (29)$$

$$y_{n+1}' = \frac{1}{h}(-y_n + y_{n+1}) + \frac{h}{24}(3f_n + 10f_{n+1} - f_{n+2}) \quad (30)$$

$$y_{n+2}' = \frac{1}{h}(-y_n + y_{n+1}) + \frac{h}{24}(f_n + 26f_{n+1} + 9f_{n+2}) \quad (31)$$

Step 6: To obtain the correctors of the block method, combine Eq. 27 and 29 in the form:

$$\begin{pmatrix}-24 & 12\\ -24 & 0\end{pmatrix}\begin{pmatrix}y_{n+1}\\ y_{n+2}\end{pmatrix} = \begin{pmatrix}0 & 12\\ 0 & -24\end{pmatrix}\begin{pmatrix}y_{n-1}\\ y_n\end{pmatrix} + h\begin{pmatrix}0 & 0\\ 0 & -24\end{pmatrix}\begin{pmatrix}y'_{n-1}\\ y'_n\end{pmatrix} + h^2\begin{pmatrix}10 & 1\\ -6 & 1\end{pmatrix}\begin{pmatrix}f_{n+1}\\ f_{n+2}\end{pmatrix} + h\begin{pmatrix}0 & -1\\ 0 & -7\end{pmatrix}\begin{pmatrix}f_{n-1}\\ f_n\end{pmatrix} \quad (32)$$

Multiplying individual terms in Eq. 32 by the inverse of $\begin{pmatrix}-24 & 12\\ -24 & 0\end{pmatrix}$ gives:

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}y_{n+1}\\ y_{n+2}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}y_{n-1}\\ y_n\end{pmatrix} + h\begin{pmatrix}0 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}y'_{n-1}\\ y'_n\end{pmatrix} + h^2\begin{pmatrix}\frac{1}{4} & -\frac{1}{24}\\ \frac{4}{3} & 0\end{pmatrix}\begin{pmatrix}f_{n+1}\\ f_{n+2}\end{pmatrix} + h^2\begin{pmatrix}0 & \frac{7}{24}\\ 0 & \frac{2}{3}\end{pmatrix}\begin{pmatrix}f_{n-1}\\ f_n\end{pmatrix} \quad (33)$$

which can be written as:

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + \frac{h^2}{24}(7\,f_n + 6\,f_{n+1} - f_{n+2}) \\ y_{n+2} &= y_n + 2hy'_n + \frac{h^2}{3}(2\,f_n + 4\,f_{n+1}) \end{aligned} \quad (34)$$

Substituting Eq. 34 in Eq. 30 and 31 gives the first derivative schemes:

$$\begin{aligned} y'_{n+1} &= y'_n + \frac{h^2}{12}(5\,f_n + 8\,f_{n+1} - f_{n+2}) \\ y'_{n+2} &= y'_n + \frac{h}{3}(f_n + 4\,f_n + f_{n+2}) \end{aligned} \quad (35)$$

where the Eq. 34 also, takes the form Eq. 17.

Developing two-step block method for second order ODEs using linear block approach: The linear block form is as stated below:

$$y_{n+\zeta} = \sum_{i=0}^{1} \frac{(\xi\, h)^i}{i\,!} y_n^{(i)} + \sum_{i=0}^{2} \phi_{i\xi}\, f_{n+i},\ \xi = 1, 2 \quad (36)$$

with first derivative:

$$y'_{n+\zeta} = y'_n + \sum_{i=0}^{2} \omega_{i\,\xi}\, f_{n+i},\ \xi = 1, 2 \quad (37)$$

where, $\phi_{i\xi} = A^{-1}B$ and $\omega_{i\xi} = A^{-1}D$ with:

$$A = \begin{pmatrix}1 & 1 & 1\\ 0 & h & 2h\\ 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!}\end{pmatrix},\ B = \begin{pmatrix}\frac{(\xi h)^2}{2!}\\ \frac{(\xi h)^3}{3!}\\ \frac{(\xi h)^4}{4!}\end{pmatrix},\ D = \begin{pmatrix}\frac{(\xi h)^1}{1!}\\ \frac{(\xi h)^2}{2!}\\ \frac{(\xi h)^3}{3!}\end{pmatrix}$$

This implies that Eq. 36 takes the following form:

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + (\phi_{01}\, f_n + \phi_{11}\, f_{n+1} + \phi_{21}\, f_{n+2}) \\ y_{n+2} &= y_n + 2hy'_n + (\phi_{02}\, f_n + \phi_{12}\, f_{n+1} + \phi_{22}\, f_{n+2}) \end{aligned} \quad (38)$$

with first derivative gotten from Eq. 37 as:

$$\begin{aligned} y'_{n+1} &= y'_n + (\omega_{01}\, f_n + \omega_{11}\, f_{n+1} + \omega_{21}\, f_{n+2}) \\ y'_{n+2} &= y'_n + (\omega_{02}\, f_n + \omega_{12}\, f_{n+1} + \omega_{22}\, f_{n+2}) \end{aligned} \quad (39)$$

Step 1:

$$\begin{pmatrix}\phi_{01}\\ \phi_{11}\\ \phi_{21}\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ 0 & h & 2h\\ 0 & \frac{h^2}{2!} & \frac{(2h)^2}{2!}\end{pmatrix}\begin{pmatrix}\frac{h^2}{2!}\\ \frac{h^3}{3!}\\ \frac{h^4}{4!}\end{pmatrix} = \begin{pmatrix}\frac{7h^2}{24}\\ \frac{h^2}{4}\\ -\frac{h^2}{24}\end{pmatrix}$$

$$\begin{pmatrix}\phi_{02}\\ \phi_{12}\\ \phi_{22}\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ 0 & h & 2h\\ 0 & \frac{h^2}{2!} & \frac{(2h)^2}{2!}\end{pmatrix}\begin{pmatrix}\frac{(2h)^2}{2!}\\ \frac{(2h)^3}{3!}\\ \frac{(2h)^4}{4!}\end{pmatrix} = \begin{pmatrix}\frac{2h^2}{3}\\ \frac{4h^2}{3}\\ 0\end{pmatrix}$$

Step 2:

$$\begin{pmatrix}\omega_{01}\\ \omega_{11}\\ \omega_{21}\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ 0 & h & 2h\\ 0 & \frac{h^2}{2!} & \frac{(2h)^2}{2!}\end{pmatrix}^{-1}\begin{pmatrix}\frac{(h)^1}{2!}\\ \frac{(h)^2}{3!}\\ \frac{(h)^3}{4!}\end{pmatrix} = \begin{pmatrix}\frac{5h^2}{12}\\ \frac{2h^2}{3}\\ -\frac{h}{12}\end{pmatrix}$$

$$\begin{pmatrix}\omega_{02}\\ \omega_{12}\\ \omega_{22}\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ 0 & h & 2h\\ 0 & \frac{h^2}{2!} & \frac{(2h)^2}{2!}\end{pmatrix}^{-1}\begin{pmatrix}\frac{(2h)^2}{1!}\\ \frac{(2h)^2}{2!}\\ \frac{(2h)^3}{3!}\end{pmatrix} = \begin{pmatrix}\frac{h}{3}\\ \frac{4h}{3}\\ \frac{h}{3}\end{pmatrix}$$

Combining these $\phi_i\xi$ and $\omega_i\xi$ results gives the two-step block method:

Table 1: Computational complexity comparison for developing two-step block method

Steps	Collocation approach	Numerical integration approach	Linear block approach
1	2+ and 6×	4+ and 2×	12+ and 27×
2	-	99+ and 24×	8+ and 12×
3	84+ and 130×	3+ and 2×	-
4	-	98+ and 23×	-
5	13+ and 31×	-	-
5	-	-	-
Σ =	99+ and 167×	204+ and 51×	20+ and 39×

$$
\begin{aligned}
y_{n+1} &= y_n + hy'_n + \frac{h^2}{24}(7\,f_n + 6\,f_{n+1} - fn+2),\\
y_{n+2} &= y_n + 2hy'_n + \frac{h^2}{3}(2\,f_n + 4\,f_{n+1}),\\
y'_{n+1} &= y'_n + \frac{h^2}{12}(5\,f_n + 8\,f_{n+1} - f_{n+2}),\\
y'_{n+1} &= y'_n + hy'_n + \frac{h}{3}(f_n + 4\,f_{n+1} - f_{n+2})
\end{aligned}
\quad (40)
$$

The block method obtained in Eq. 40 is same as the block method obtained with the numerical integration and collocation approaches.

RESULTS AND DISCUSSION

Computational complexity comparison for developing two-step block method: This section will highlight the number operations involved in each step of adopting the collocation, numerical integration and new linear block approach (Table 1). Where + denotes the number of addition operations while X denotes the number of multiplication operations. It is observed that the linear block approach requires less computations than the collocation and numerical integration approach.

CONCLUSION

This study has presented three approaches of developing block methods for solving second order ODEs. The first observation is that all three approaches resulted in the same block method which leads to investigating which of the approaches is less cumbersome. Using the computational complexity analysis concept, it is observed that the linear block approach is simpler and less computationally burdensome. Therefore, the new linear block approach is more suitable when developing block methods.

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