Journal of Engineering and Applied Sciences 13 (Special Issue 5): 4771-4773, 2018
ISSN: 1816-949X

L_p, P<1 Approximation Using Radial Basis Function Neural Networks on Ordered Vector Space

Eman Samir Bhaya and Walaa Hussein Ahmed
Department of Mathematics, College of Education for Pure Sciences,
University of Babylon, Hillah, Iraq

Abstract: There are many studies on the approximation by neural networks. In general, we cannot approximate a function f in L_p spaces using radial basis function neural networks. We introduce theorems on the degree of best approximation using neural networks for functions defined overall real line. In addition, we define a version of convolution and use it to estimate the degree of approximation using neural networks. Here, we make the approximation using radial basis function neural networks possible by writing some constrains on the target function f. We use the construction in our methods of proofs. Which we can consider them as a function approximation.

Key words: Articles, approximation, function, radial basis, theorems, addition

INTRODUCTION

Many papers study the approximation by radial basis function neural networks, for example (Liu and Si, 1994; Chen and Chen, 1995; Duy and Cong, 2003; Hartman *et al*., 1990; Scarselli and Tsoi, 1998; Leonard *et al*., 1992; Mhaskar and Micchlli, 1992; Hahm and Hong, 2004; Kurkova, 1995; Park and Sandberg, 1991; Li, 1998). In all these studies, the weight in the neural network mentioned above have many kinds which make the work very difficult to apply in engineering applications. In our research, we fix this weight for the kind of neural network mentioned above and use it to approximate functions in so called L_p^o spaces with a type of discrete quasi norm where $f \in L_p^o$ spaces, then there exists L is apositive integer such that $|f(x)| < \Delta_h^k f(x)$ for $|x| \geq L$. All these to make facilities in applications and improve the works in the articles (Liu and Si, 1994; Chen and Chen, 1995; Duy and Cong, 2003; Hartman *et al*., 1990; Scarselli and Tsoi, 1998; Leonard *et al*., 1992; Mhaskar and Micchelli, 1992; Hahm and Hong, 2004; Kurkova, 1995; Park and Sendberg, 1991; Li, 1998).

APPROXIMATION USING RADIAL BASIS NEURAL NETWORKS

This study consists of our main result discussion and methods of proofs, here, we define the radial basis neural network and use it to introduce a direct estimations for L_p approximation.

Theorem 2.1: Let τ be a radial basis function on the real line R. If $f \in L_p^o$ then there exist $v_i \delta_i \in R$ and positive integer R, L such that:

$$\| f(y) - M(x) \|_p \leq c_{(p)} \omega_k (f, t)_p$$

Where:

$$M(x) := \sum_{i=1}^{L} v_i \tau(Rx + \delta_i)$$

Proof: Since, $f \in L_p^0$ there exist L such that $|f(x)| < \Delta_h^k f(x)$ for $|x| > L$. Divide the interval [-L, L] into $2L^2$ equal segments each with length of 1/L.

Let $-L = x_0 < x_1 <, ..., < x_{2L}{}^2 = L$ and $\boldsymbol{\delta}_i = (x_{i-1} + x_i)/2$, $(1 \leq i \leq 2L^2)$. Since, τ is radial basis function,there exist appositive integer constant K such that $|\tau(x)| < 1/2L$ for $x|x| \geq K$. Choose a positive integer R such that R/2L>K. Now, we construct theneural network as:

$$N(x) = \sum_{i=1}^{2L^2} \Delta_h^k f(x_i) \tau(R(x - \delta_i))$$

If $|x| \geq L$, then $|R(x - \boldsymbol{\delta}_i)| \geq K$ and hence $|\tau(R(x - \boldsymbol{\delta}_i))| < 1/2L$, for i = 1, 2, ..., $2L^2$, therefor, we obtain:

$$\| f(x) - N(x) \|_p \leq \| f(x) \|_p + \| N(x) \|_p \leq$$
$$\Delta_h^k f(x) + \sum_{i=1}^{2L^2} \Delta_h^k f(x) \tau(R(x - \delta_i)) \|_p \leq$$
$$\Delta_h^k f(x) + \| \Delta_h^k f(x) \|_p / (2L) \leq \| \Delta_h^k f(x) \|_p \leq$$
$$c_{(p)} \omega_k (f, t)$$

Corresponding Author: Walaa Hussein Ahmed, Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Hillah, Iraq

If $|x|\leq L$ then there must exist a nonnegative integer a $(0\leq\alpha\leq 2L^2-1)$ such that $x\in[x_\alpha, x_{\alpha+1}]$. So, we can see that $R(x-\delta_i)\leq -K$ for $i = 0, 1, ..., \alpha-1$ and $R(x-\delta_i)\leq -K$. For $i = \alpha+1, \alpha+2, ..., 2L^2-1$ in addition, we can easily see that:

$$\sum_{i=1}^{\alpha-1}\Delta_h^k f(x_i)\tau(R(x-\delta_i)) = \sum_{i=1}^{2L^2}\left(\Delta_h^k f(x_i)\tau(R(x-\delta_i))-1\right)+\Delta_h^k f(x_i)$$

From the fact that τ is bounded function, we have:

$$\|f(x)-N(x)\|_p = \|f(x)-\sum_{i=1}^{\alpha-1}\Delta_h^k f(x_i)\tau(R(x-\delta_i))-\Delta_h^k f(x_\alpha)\tau(R(x-\delta_i))+\sum_{i=\alpha+1}^{2L^2-1}\Delta_h^k f(x_i)\tau(R(x-\delta_i))\|_p \leq c_{(p)}\ \|f(x)+\Delta_h^k f(x_i)\|_p + \sum_{i=1}^{\alpha-1}\Delta_h^k f(x_i)\tau(R(x-\delta_i))\|_p+\|\Delta_h^k f(x_\alpha)\tau(R(x-\delta_\alpha))\|_p + \sum_{i=\alpha+1}^{2L^2-1}\Delta_h^k f(x_i)\tau(R(x-\delta_i))\|_p \leq c_{(p)}\left(\tau(R(x-\delta_i))\right)w_k(f, t)$$

To prove our second result we need the following definition.

Definition: For two functions f, $h\in L_p$, $p<1$ on R, their convolution defined by:

$$(f\odot h)(x_i) = \sum_{j=0}^{n}\sum_{i=0}^{n} f(y_j)h(x_i-y_j)$$

$-\infty = x_0<x_1< ... < x_n = \infty$, $y_j\in\{x_i\}$. For any $x_i\in R$, we define a function:

$$F(x) = \begin{cases} de^{\frac{-1}{1-x^2}}, & \text{if } |x| < 1 \\ 0, & \text{if } |x|\Delta > 1 \end{cases}$$

where, d is chosen, so that:

$$\lim_{n\to\infty}\sum_{i=0}^{n} F(x_i) = 1$$

Then, $F\in L_p$ for any positive integer K, we define $F_k(x) = KF(kx)$, then we obtain:

$$\lim_{n\to\infty}\sum_{i=0}^{n} F_k(x_i) = 1$$

In the next theorem, we use the convolution to prove a Jackson type approximation theorem.

Theorem 2.3: If $f\in L_p^0$ on R, then:

$$\|(F_k\odot f)-f\|_p\leq c_{(p)}w_k(f, t)_p$$

Proof: Since, $f\in L_p^0$, then there exists an L is apositive integer such that:

$$|f(x)|<\Delta_h^K f(x) \text{for } |x|\geq L$$

Then:

$$\|(F_k\odot f)-f\|_p = \sup_{n\in N}\left(\sum_{i=0}^{n}\frac{c}{n}\left|\sum_{j=0}^{n}F_k(y_j)f(x_i-y_{ij})-\sum_{j=0}^{n}F_k(y_j)f(x_i)\right|^p\right)^{1/p} = \sup_{n\in N}\left(\sum_{i=0}^{n}\frac{c}{n}\left|\sum_{j=0}^{n}KF(K_{yj})(f(x_i-y_{ij})-f(x_i))\right|^p\right)^{1/p} \leq c_{(p)}\sup_{n\in N}\left(\sum_{i=0}^{n}\sum_{j=0}^{n}|F(y_j)|^p\,|f(x_i)|^p\right)^{1/p} \leq c_{(p)}\sup_{n\in N}\left(\sum_{i=0}^{n}\sum_{j=0}^{n}|F(y_j)|^p|\Delta_h^k f(x_i)|^p\right)^{1/p} \leq c_{(p)}\omega_k(f, t)_p$$

Our third result deals with the radial basis neural networks approximation on [a, b] subset of R.

Theorem 2.4: Let, $f\in L_p^0$ defined on a bounded closed interval [a, b] subset of R if τ is a radial basis function on R, then there exist constants v_i, $\delta_i\in R$ and positive integers R and L such that:

$$\|f(y)-N(x)\|_p\leq c_{(p)}\omega_k(f, t)_p$$

Where:

$$N(x) = \sum_{i=1}^{2L^2}\Delta_h^k f(x_i)\tau(R(x-\delta_i))$$

Proof: We construct a function f^0 on R as follows:

$$f^0 = \begin{cases} f(\alpha)x+(\alpha-1)f(\alpha), \text{ if } x\in[\alpha-1, \alpha] \\ f(x), \text{ if } x\in[\alpha,b] \\ -f(b)x+(b+1)f(b), \text{ if } x\in[b, b+1] \\ 0, \text{ if } x\in(-\infty, \alpha-1]\cup[b+1, \infty) \end{cases}$$

From Theorem 2.3, we can see that:

$$\|(F_k\odot f^0)-f^0\|_p\leq\omega_k(f^0, t)_p$$

where, $f^0\in L_p^0$ on R then:

$$\|(F_k\odot f^0)-f\|_p\leq\omega_k(f^0,\ t)_p$$

Where:

$$f^0 \in L_p^0 \text{ on } [a, b]$$

Since:

$$\lim_{n\to\infty} \sum_{i=0}^{n} \sum_{i=0}^{n} Fk(x_i\text{-}y_j) f^0(y_j) < \infty$$

For each positive integer k, there exist M_k and constants y_i, v_i for i = 1, 2, ..., M_k such that:

$$\| (F_k \odot f^0) - \sum_{i=1}^{M_k} v_i F_k (x_i\text{-}y_j) f^0(y_i) \| \leq \frac{1}{3} c_{(p)} \omega_k (f^0, t)_p \quad (1)$$

From Definition 2.2, we can see that, $F_k \in L_p^0$. Let:

$$\sum_{j=1}^{M_k} v_i f^0(y_i) = B$$

By using Theorem 2.1, we know that there exist constants $\theta_{j,k}$, $\varepsilon_{j,k}$ and a positive integer k such that:

$$\left\| F_K(x_i - y_j) - \sum_{j=1}^{n} \sum_{i=1}^{M_K} \varepsilon_{j,k} \tau(k(x_i - y_j) + \theta_{j,k}) \right\|_p \leq \frac{1}{3B} C_{(p)} \omega_k (f^0, t)_p \quad (2)$$

As $\|(F_K \odot f^0) - f\|_p \leq c_{(p)} \omega_k (f^0, t)_p$ where, $f \in L_p^0$ on R, so, we can choose a positive integer k such that:

$$\| (F_k \odot f) - f \|_p \leq \frac{1}{3} c_{(p)} \omega_k (f, t)_p \quad (3)$$

From Eq. 1-3, we have:

$$\left\| f(y) - \sum_{j=1}^{n} v_i f^0(y_i) \sum_{i=1}^{M_k} \varepsilon_{j,k} \, \tau(k(x_i - y_j) + \theta_{j,k}) \right\|_p \leq$$

$$\| f - (F_K \odot f) \|_p +$$

$$\left\| (F_K \odot f^0) - \sum_{i=1}^{M_k} v_i F_k (x_i - y_j) f^0(y_i) \right\|_p +$$

$$\left\| F_K(x_i - y_j) - \sum_{j=1}^{n} v_i f^0(y_i) \sum_{i=1}^{M_k} \varepsilon_{j,k} \, \tau(k(x_i - y_j) + \theta_{j,k}) \right\|_p \leq$$

$$C(p) \omega_k (f, t)_p$$

CONCLUSION

The approximation using radial basis function netural network is possible wich make the work very easy to apply in engineering applications. We can approximate a function in L_p spaces by a radial basis function netural network.

REFERENCES

Chen, T.P. and H. Chen, 1995. Approximation capability to function of several variables, nonlinear functionals and operators by radial basis function neural networks. IEEE. Trans. Neural Netw., 6: 904-910.

Duy, N.M. and T.T. Cong, 2003. Approximation of function and its derivations using radial basis function networks. Applied Math. Modell., 27: 197-220.

Hahm, N. and B.I. Hong, 2004. An approximation by neural networks with a fixed weight. Comput. Math. Appl., 47: 1897-1903.

Hartman, E.J., J.D. Keeler and J.M. Kowalski, 1990. Layered neural networks with gaussian hidden units as universal approximators. Neural Comput., 2: 210-215.

Kurkova, V., 1995. Approximation of functions by perceptron networks with bounded number of hidden units. Neural Netw., 8: 745-750.

Leonard, J.A., M.A. Kramer and L.H. Ungar, 1992. Using radial basis functions to approximate a function and its error bounds. IEEE. Trans. Neural Netw., 3: 624-627.

Li, X., 1998. On simultaneous approximations by radial basis function neural networks. Appl. Math. Comput., 95: 75-89.

Liu, B. and J. Si, 1994. The best approximation to C/sup 2/functions and its error bounds using regular-center Gaussian networks. IEEE. Trans. Neural Netw., 5: 845-847.

Mhaskar, H.N. and C.A. Micchelli, 1992. Approximation by superposition of sigmoidal and radial basis functions. Adv. Math., 13: 350-373.

Park, J. and I.W. Sandberg, 1991. Universal approximation using radial-basis-function networks. Neural Comput., 3: 246-257.

Scarselli, F. and A.C. Tsoi, 1998. Universal approximation using feedforward neural networks: A survey of some existing methods and some new results. Neural Netw., 11: 15-37.