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New Linear Block Method for Third Order Ordinary Differential Equations

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Abstract: The diversity of adopting different step by step approach when developing block methods of the form $A^0Y_{n+k} = A^iY_{n-k}+B^iY'_{n-k}+D^iY''_{n-k}+h^3(C^0Y'''_{n+k}+C^iY'''_{n-k})$ is quite rigorous. Hence, this study presents an approach called Linear Block Method (LBM) capable of producing directly any k steplength value of block methods for solving third order ordinary differential equations. The LBM algorithm is validated by recovering certain existing k-step block methods in literature. Likewise, the computational complexity of the LBM algorithm is presented.

Key words: Linear block method, third order, ordinary differential equations, computational complexity

INTRODUCTION

Lambert (1973) stated three approaches for determining the coefficients of linear multistep methods, namely, numerical integration, interpolation and Taylor series approach. These approaches have been modified for the transformation of the discrete linear multistep methods into block methods. These block methods consist of a family of schemes that simultaneously evaluate the solution of the differential equation under consideration at different grid points (Fatunla, 1995).

The transformation of discrete schemes to block methods using numerical integration approach have been explored by Omar (2004), Majid *et al*. (2013) and Phang *et al*. (2011), amongst many others.

Other researcher who have adopted the interpolation approach for developing these block methods include Jator and Li (2009), Badmus and Yahaya (2009) and Adesanya *et al*. (2014). Likewise, the Taylor series expansion has not been left out as it has been adopted in the research by Li (2008) and Chen and Li (2012), however, the methods developed were still restricted to development of the discrete schemes alone and further subject to boundary conditions. The direct solution of third order ordinary differential Eq. 1 of the form:

$$y''' = f(x, y, y', y'') \quad (1)$$

have been considered by several studies such as Adesanya *et al*. (2014), Mohammed and Adeniyi (2014), Adoghe (2014) and Omar and Kuboye (2015) amongst others. Hence, the introduction of the Linear Block Method (LBM) will be of good advantage for researcher when developing block methods of the form:

$$A^0Y_{n+k} = A^iY_{n-K}+B^iY'_{n-k}+D^iY''_{n-k}+h^3(C^0Y'''_{n+k}+C^iY'''_{n-k}) \quad (2)$$

for solving third order ordinary differential equations. Thus, the aim of this study.

MATERIALS AND METHODS

New linear block method for k-step block methods: For $y''' = f(x, y, y', y'')$. For block methods of the form Eq. 2 where, $Y_{n+k} = (y_{n+1}, y_{n+2}, ..., y_{n+k})$ and $Y^{(i)}_{n+k} = (y^{(i)}_{n+1}, y^{(i)}_{n+2}, ..., y^{(i)}_{n+k})$, the following LBM algorithm is given:

$$y_{n+\xi} = \sum_{i=0}^{2} \frac{(\xi h)^i}{i!} y_n^{(i)} + \sum_{i=0}^{k} \phi_i f_{n+i}, \quad \xi = 1, 2, ..., k \quad (3)$$

with derivatives:

$$y_{n+\xi}^{(a)} = \sum_{i=0}^{3-(a+1)} \frac{(\xi h)^i}{i!} y_n^{(i+a)} + \sum_{i=0}^{k} \omega_{ia} f_{n+i}, \quad a = 1_{(\xi = 1, 2, ..., k)}, 2_{(\xi = 1, 2, ..., k)} \quad (4)$$

$\phi_i = A^{-1} B$ and $\omega_{ia} = A^{-1} D$ where:

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$$A=\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & h & 2h & \cdots & kh \\ 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & \cdots & \frac{(kh)^2}{2!} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & \frac{h^k}{k!} & \frac{(2h)^k}{k!} & \cdots & \frac{(kh)^k}{k!} \end{pmatrix} B=\begin{pmatrix} \frac{(\xi h)^3}{3!} \\ \frac{(\xi h)^4}{4!} \\ \frac{(\xi h)^5}{5!} \\ . \\ . \\ . \\ \frac{(\xi h)^{(3+k)}}{(3+k)!} \end{pmatrix}; D=\begin{pmatrix} \frac{(\xi h)^{(3+a)}}{(3-a)!} \\ \frac{(\xi h)^{(3-a)+1}}{((3-a)+1)!} \\ \frac{(\xi h)^{(3-a)+2}}{((3-a)+2)!} \\ . \\ . \\ . \\ \frac{(\xi h)^{(3-a)+k}}{((3-a)+k)!} \end{pmatrix}$$

However, the following proposition needs to be noted.

Proposition 2.1: There exists only one block form for every k-step block method.

Verification of the LBM algorithm: To verify this algorithm, we develop the 4-step block method using the LBM algorithm and then compare the output to the derived k = 4 block method in literature:

$$A^0 Y_{n+k} = A^i Y_{n-k} + B^i Y'_{n-k} + D^i Y''_{n-k} + h^3 (C^0 Y'''_{n+k} + C^i Y'''_{n-k}),\ k = 4$$

First, the LBM algorithm is expanded together with the expression for the derivatives:

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + \frac{h^2}{2!} y''_n + (\phi_0 f_n + \phi_1 f_{n+1} + \phi_2 f_{n+2} + \phi_3 f_{n+3} + \phi_4 f_{n+4}) \\ y_{n+2} &= y_n + 2hy'_n + \frac{(2h)^2}{2!} y''_n + (\phi_0 f_n + \phi_1 f_{n+1} + \phi_2 f_{n+2} + \phi_3 f_{n+3} + \phi_4 f_{n+4}) \\ y_{n+3} &= y_n + 3hy'_n + \frac{(3h)^2}{2!} y''_n + (\phi_0 f_n + \phi_1 f_{n+1} + \phi_2 f_{n+2} + \phi_3 f_{n+3} + \phi_4 f_{n+4}) \\ y_{n+4} &= y_n + 4hy'_n + \frac{(4h)^2}{2!} y''_n + (\phi_0 f_n + \phi_1 f_{n+1} + \phi_2 f_{n+2} + \phi_3 f_{n+3} + \phi_4 f_{n+4}) \end{aligned} \quad (5)$$

with derivatives:

$$\begin{aligned} y'_{n+1} &= y'_n + hy''_n + (\omega_{01} f_n + \omega_{11} f_{n+1} + \omega_{21} f_{n+2} + \omega_{31} f_{n+3} + \omega_{41} f_{n+4}) \\ y'_{n+2} &= y'_n + 2hy''_n + (\omega_{01} f_n + \omega_{11} f_{n+1} + \omega_{21} f_{n+2} + \omega_{31} f_{n+3} + \omega_{41} f_{n+4}) \\ y'_{n+3} &= y'_n + 3hy''_n + (\omega_{01} f_n + \omega_{11} f_{n+1} + \omega_{21} f_{n+2} + \omega_{31} f_{n+3} + \omega_{41} f_{n+4}) \\ y'_{n+4} &= y'_n + 4hy''_n + (\omega_{01} f_n + \omega_{11} f_{n+1} + \omega_{21} f_{n+2} + \omega_{31} f_{n+3} + \omega_{41} f_{n+4}) \\ y''_{n+1} &= y''_n + (\omega_{02} f_n + \omega_{12} f_{n+1} + \omega_{22} f_{n+2} + \omega_{32} f_{n+3} + \omega_{42} f_{n+4}) \\ y''_{n+2} &= y''_n + (\omega_{02} f_n + \omega_{12} f_{n+1} + \omega_{22} f_{n+2} + \omega_{32} f_{n+3} + \omega_{42} f_{n+4}) \\ y''_{n+3} &= y''_n + (\omega_{02} f_n + \omega_{12} f_{n+1} + \omega_{22} f_{n+2} + \omega_{32} f_{n+3} + \omega_{42} f_{n+4}) \\ y''_{n+4} &= y''_n + (\omega_{02} f_n + \omega_{12} f_{n+1} + \omega_{22} f_{n+2} + \omega_{32} f_{n+3} + \omega_{42} f_{n+4}) \end{aligned} \quad (6)$$

For y_{n+1}:

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & h & 2h & 3h & 4h \\ 0 & \frac{h^2}{2!} & \frac{(2h)^2}{2!} & \frac{(3h)^2}{2!} & \frac{(4h)^2}{2!} \\ 0 & \frac{h^3}{3!} & \frac{(2h)^3}{3!} & \frac{(3h)^3}{3!} & \frac{(4h)^3}{3!} \\ 0 & \frac{h^4}{4!} & \frac{(2h)^4}{4!} & \frac{(3h)^4}{4!} & \frac{(4h)^4}{4!} \end{pmatrix}^{-1} \begin{pmatrix} \frac{h^3}{3!} \\ \frac{h^4}{4!} \\ \frac{h^5}{5!} \\ \frac{h^6}{6!} \\ \frac{h^7}{7!} \end{pmatrix} = \begin{pmatrix} \frac{113h^3}{1120} \\ \frac{107h^3}{1008} \\ -\frac{103h^3}{1680} \\ \frac{43h^3}{1680} \\ -\frac{47h^3}{10080} \end{pmatrix}$$

Likewise for y_{n+2}:

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} \frac{331h^3}{630} \\ \frac{332h^3}{315} \\ -\frac{8h^3}{21} \\ \frac{52h^3}{315} \\ -\frac{19h^3}{630} \end{pmatrix} y_{n+3}, \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} \frac{1431h^3}{1120} \\ \frac{1863h^3}{560} \\ -\frac{243h^3}{560} \\ \frac{45h^3}{112} \\ -\frac{81h^3}{1120} \end{pmatrix}$$

$$y_{n+4}, \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} \frac{248h^3}{105} \\ \frac{2176h^3}{315} \\ \frac{32h^3}{105} \\ \frac{128h^3}{105} \\ -\frac{8h^3}{63} \end{pmatrix} y'_{n+1}, \begin{pmatrix} \omega_{01} \\ \omega_{11} \\ \omega_{21} \\ \omega_{31} \\ \omega_{41} \end{pmatrix} = \begin{pmatrix} \frac{367h^3}{1440} \\ \frac{3h^2}{8} \\ -\frac{47h^2}{240} \\ \frac{29h^2}{360} \\ -\frac{7h^2}{480} \end{pmatrix} y'_{n+2}, \begin{pmatrix} \omega_{01} \\ \omega_{11} \\ \omega_{21} \\ \omega_{31} \\ \omega_{41} \end{pmatrix} = \begin{pmatrix} \frac{53h^2}{90} \\ \frac{8h^2}{5} \\ -\frac{h^2}{3} \\ \frac{8h^2}{45} \\ -\frac{h^2}{30} \end{pmatrix}$$

$$y'_{n+3}, \begin{pmatrix} \omega_{01} \\ \omega_{11} \\ \omega_{21} \\ \omega_{31} \\ \omega_{41} \end{pmatrix} = \begin{pmatrix} \frac{147h^2}{160} \\ \frac{117h^2}{40} \\ \frac{27h^2}{80} \\ \frac{3h^2}{8} \\ -\frac{9h^2}{160} \end{pmatrix}; y'_{n+4}, \begin{pmatrix} \omega_{01} \\ \omega_{11} \\ \omega_{21} \\ \omega_{31} \\ \omega_{41} \end{pmatrix} = \begin{pmatrix} \frac{56h^2}{45} \\ \frac{64h^2}{15} \\ \frac{16h^2}{15} \\ \frac{64h^2}{45} \\ 0 \end{pmatrix}; y''_{n+1}, \begin{pmatrix} \omega_{02} \\ \omega_{12} \\ \omega_{22} \\ \omega_{32} \\ \omega_{42} \end{pmatrix} = \begin{pmatrix} \frac{251h}{720} \\ \frac{323h}{360} \\ -\frac{11h}{30} \\ \frac{53h}{360} \\ -\frac{19h}{720} \end{pmatrix},$$

$$y''_{n+2},\begin{pmatrix}\omega_{02}\\ \omega_{12}\\ \omega_{22}\\ \omega_{32}\\ \omega_{42}\end{pmatrix}=\begin{pmatrix}\frac{29h}{90}\\ \frac{62h}{45}\\ \frac{4h}{15}\\ \frac{2h}{45}\\ -\frac{h}{90}\end{pmatrix} y''_{n+3},\begin{pmatrix}\omega_{02}\\ \omega_{12}\\ \omega_{22}\\ \omega_{32}\\ \omega_{42}\end{pmatrix}=\begin{pmatrix}\frac{27h}{80}\\ \frac{51h}{40}\\ \frac{9h}{10}\\ \frac{21h}{40}\\ -\frac{3h}{80}\end{pmatrix} y''_{n+4},\begin{pmatrix}\omega_{02}\\ \omega_{12}\\ \omega_{22}\\ \omega_{32}\\ \omega_{42}\end{pmatrix}=\begin{pmatrix}\frac{14h}{45}\\ \frac{64h}{45}\\ \frac{8h}{15}\\ \frac{64h}{45}\\ \frac{14h}{45}\end{pmatrix}$$

Combining these results give the block form:

$$\begin{aligned}
y_{n+1} &= y_n+hy'_n+\frac{h^2}{2}y''_n+\frac{h^3}{30240}(3051f_n+3210f_{n+1}-1854f_{n+2}+774f_{n+3}-141f_{n+4})\\
y_{n+2} &= y_n+2hy'_n+2h^2y''_n+\frac{h^3}{1890}(993f_n+1992f_{n+1}-720f_{n+2}+312f_{n+3}-57f_{n+4})\\
y_{n+3} &= y_n+3hy'_n+\frac{9h^2}{2}y''_n+\frac{h^3}{3360}(4293f_n+11178f_{n+1}-1458f_{n+2}+1350f_{n+3}-243f_{n+4})\\
y_{n+4} &= y_n+4hy'_n+8h^2y''_n+\frac{h^3}{2205}(5208f_n+15232f_{n+1}-672f_{n+2}+2688f_{n+3}-280f_{n+4})\\
y'_{n+1} &= y'_n+hy''_n+\frac{h^2}{1440}(367f_n+540f_{n+1}-282f_{n+2}+116f_{n+3}-21f_{n+4})\\
y'_{n+2} &= y'_n+2hy''_n+\frac{h^2}{90}(53f_n+144f_{n+1}-30f_{n+2}+16f_{n+3}-3f_{n+4})\\
y'_{n+3} &= y'_n+3hy''_n+\frac{h^2}{160}(147f_n+468f_{n+1}+54f_{n+2}+60f_{n+3}-9f_{n+4})\\
y'_{n+4} &= y'_n+4hy''_n+\frac{h^2}{45}(56f_n+192f_{n+1}+48f_{n+2}+64f_{n+3})\\
y''_{n+1} &= y''_n+\frac{h}{720}(251f_n+646f_{n+1}-264f_{n+2}+106f_{n+3}-19f_{n+4})\\
y''_{n+2} &= y''_n+\frac{h}{90}(29f_n+124f_{n+1}-24f_{n+2}+4f_{n+3}-f_{n+4})\\
y''_{n+3} &= y''_n+\frac{h}{80}(27f_n+102f_{n+1}-72f_{n+2}+42f_{n+3}-3f_{n+4})\\
y''_{n+4} &= y''_n+\frac{h}{45}(14f_n+64f_{n+1}-24f_{n+2}+64f_{n+3}-14f_{n+4})
\end{aligned} \qquad (7)$$

This block expression corresponds with the coefficients presented by Adesanya *et al*. (2012) and Osa and Olaoluwa (2015) where the block method was used as a block predictor. Hence, this verifies the proposed LBM algorithm for developing block methods for solving third order ordinary differential equations.

RESULTS AND DISCUSSION

Computational complexity of the new linear block method for developing four-step block method: The development of the four-step block method using the new LBM algorithm involved obtaining the coefficients for the block corrector schemes and its corresponding derivatives at grid points x_{n+1}, x_{n+2}, x_{n+3} and x_{n+4}.

LBM algorithm:

Step 1: Evaluate:

$$y_{n+\xi}=\sum_{i=0}^{2}\frac{(\xi h)^i}{i!}y_n^{(i)}+\sum_{i=0}^{k}\phi_i f_{n+i},\ \xi=1,2,...,k$$

with derivatives:

$$y_{n+\xi}^{(a)}=\sum_{i=0}^{3-(a+1)}\frac{(\xi h)^i}{i!}y_n^{(i+a)}+\sum_{i=0}^{k}\omega_{ia}f_{n+i},\ a=1_{(\xi=1,2,...,k)},2_{(\xi=1,2,...,k)}$$

$\phi_i = A^{-1}B$ and $\omega_{ia} = A^{-1}D$ where

$$A=\begin{pmatrix}1&1&1&\cdots&1\\0&h&2h&\cdots&kh\\0&\frac{(h)^2}{2!}&\frac{(2h)^2}{2!}&\cdots&\frac{(kh)^2}{2!}\\ .&.&.&\cdots&.\\ .&.&.&\cdots&.\\ .&.&.&\cdots&.\\0&\frac{h^k}{k!}&\frac{(2h)^k}{k!}&\cdots&\frac{(kh)^k}{k!}\end{pmatrix},B=\begin{pmatrix}\frac{(\xi h)^3}{3!}\\ \frac{(\xi h)^4}{4!}\\ \frac{(\xi h)^5}{5!}\\ .\\ .\\ .\\ \frac{(\xi h)^{(3+k)}}{(3+k)!}\end{pmatrix},D=\begin{pmatrix}\frac{(\xi h)^{(3-a)}}{(3+a)!}\\ \frac{(\xi h)^{(3-a)+1}}{((3-a)+1)!}\\ \frac{(\xi h)^{(3-a)+2}}{((3-a)+2)!}\\ .\\ .\\ .\\ \frac{(\xi h)^{(3-a)+k}}{((3-a+k))!}\end{pmatrix}$$

and $k = 4$

Step 2: Stop

The two major mathematical operations required are matrix inverse and matrix multiplication. Recall that the computational complexity of taking the inverse of an $n\times n$ matrix is $O(n^3)$ while the computational complexity of the matrix multiplication of one $n\times m$ matrix with one $n\times p$ matrix is $O(nmp)$. Hence, the computational complexity of developing the four step block method using the LBM algorithm is obtained from:

$$3k\left\{\left[O\left((k+1)^3\right)+O\left((k+1)^2\right)\right]\right\} \qquad (8)$$

as $O((k+1)^3)$.

CONCLUSION

This study has presented a new linear block method for developing any k-step block method for solving third order ordinary differential equations. The algorithm is seen to be valid as verified with previously developed k = 4 block methods in literature. Hence, this algorithm is suitablefor adoption when developing block methods of this kind as it bypasses the rigour attached to the step by step approach used by previous studies. The computational complexity of the method is also presented, so that, new algorithms developed in the future can be compared to this algorithm in terms of computational complexity. Likewise, this present research can be extended to using this algorithm to develop suitable k-step block methods for solving both initial and boundary value problems of third order ordinary differential equations.

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