Journal of Engineering and Applied Sciences 13 (Special Issue 7): 5649-5654, 2018
ISSN: 1816-949X

New Classes of Statistically Convergent Difference Triple Sequence Spaces of Fuzzy Real Numbers

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Abstract: In this study, we have introduced some classes of statistically convergent difference multiple sequences of fuzzy real numbers. We make an effort to study few basic algebraic properties namely complete, normal, monotone, symmetric, convergence free, etc. and inclusion results on these spaces.

Key words: Fuzzy numbers, triple sequence, normal, monotone, convergence free, density, sequence algebra, symmetric, statistical Cauchy, statistical convergenc

INTRODUCTION

After the presentation of study "Fuzzy sets" by Zadeh (1965), the fuzzy sets and theory has made tremendous progress in research works. It has exerted a considerable influence on science and technology such as economics, mathematics, control theory, natural science, signal processing and many more. For practice, it is an easily adaptable theory as compared to other mathematical theories. Fuzzy sets can be used to define an entity in calculus by considering all possible deviations and inexactness in its role and this presentation suits well the uncertainties occurred in practical life which make fuzziness a valuable mathematical tool.

The mathematicians study many of the problems which are concerned with large classes of objects and most of such interesting classes develop into sequence spaces. By introducing a metric in sequence spaces, one can form structure of several new classes of sequence spaces. With the emergence of different techniques and notions of topology and modern analysis, one can studies sequence spaces with larger distant and inspiration. The sequences of numbers have unforeseen and realistic uses in many areas of science and technology.

Fast (1951) introduced the notion of statistical convergence. The research on statistically convergent sequences were further urbanized by Salat (1980), Fridy (1985), Connor (1988), Maddox (1989), Fridy and Orhan (1997), Savas (2001), Tripathy (2003), Savas and Mursaleen (2004), Tripathy and Sen (2001), Tripathy and Dutta (2010), Tripathy and Sarma (2008) and many more. From single real sequences, Moricz (2003) enhanced statistical convergence to multiple real sequences. The notion of statistical convergence and statistically Cauchy for sequences of fuzzy numbers was first introduced by Nuray and Savas (1995). Agnew (1934) investigated the concept of summability theory for multiple sequences and derived some theorems for double sequences. Sahiner *et al.* (2007) first introduced the notions of triple sequences. Sahiner and Tripathy (2008), Savas and Esi (2012), Esi (2013), Tripathy and Dutta (2010), Tripathy and Goswami (2014) and Datta *et al.* (2013), etc. have have investigated different classes of statistically convergent triple sequences. Several classes of sequence spaces of fuzzy numbers have introduced and studied by Nanda (1989), Tripathy and Nanda (2000), Das and Choudhury (1998) and many more. Recent workings on triple sequence spaces of fuzzy numbers found by Nath and Roy (2015, 2016a, b) and Saha and Roy (2017).

MATERIALS AND METHODS

Definition and preliminaries: Throughout N, R, C denote the sets of natural numbers, real numbers and complex numbers respectively and $(w^F)_3$, $(\ell_\infty^F)_3$, $(\overline{\ell}_\infty^F)_3$, $(\overline{c}_0^F)_3^P$, $(\overline{c}_0^F)_3^B$, $(\overline{c}_0^F)_3^{BR}$ $(\overline{c}_0^F)_3^R$, $(\overline{c}^F)_3^P$, $(\overline{c}^F)_3^{BR}$, $(\overline{c}^F)_3^B$, $(\overline{c}^F)_3^R$ denotes the triple sequences spaces of all, bounded, statistically bounded, statistically null, bounded statistically null, bounded statistically regularly null, statistically regularly null, statistically convergent in Pringsheim's sense, bounded statistically regularly convergent, bounded statistically convergent in Pringshiem's sense and statistically regularly convergent of fuzzy numbers.

A fuzzy real number on R is a function X: R→L (= [0, 1]) relating every real number t∈R by its membership grade X(t). Each real number r can be articulated as a fuzzy numbe $\bar{r}$ as:

$$\bar{r}(t) = \begin{cases} 1 & \text{if} \quad t = r \\ 0 & \text{otherwise} \end{cases}$$

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The set $[X]^\alpha = \{t\in R: X(t)\geq\alpha\}$ is defined as α-level set of a fuzzy number X, $0<\alpha\leq 1$. A fuzzy real number X is called convex if $X(t)\geq X(s)\wedge X(r) = \min(X(s), X(r))$ where $s<t<r$ and X is called normal if there exists $t_0\in R$ such that $X(t_0) = 1$. Also, X is said to be upper semi-continuous if for each $\varepsilon>0$, X-1[0, a+ε] for all $a\in L$ is open in the usual topology of R. We denote the set of all upper semi continuous, normal, convex fuzzy number by R(L) and its additive and multiplicative identities by $\bar{0}$ and $\bar{1}$, respectively.

If D is the set of all closed bounded intervals $[X = X^L, X^R]$ on R, then $X^L\leq Y^L$ and $X^R\leq Y^R$ holds only when $X\leq Y$. If $d(X, Y) = \max(|X^L-X^R|, |Y^L-Y^R|)$ then (D, d) is a complete metric space. Further, if $\bar{d}: R(L)\times R(L)\rightarrow R$ be defined by $\bar{d}(X, Y) = \sup_{0\leq\alpha\leq 1} d([X]^\alpha, [Y]^\alpha)$, for $X, Y\in R(L)$ then $\bar{d}$ is also a metric on R(L)

Every triple sequence is defined as a function x: $N\times N\times N\rightarrow R(C)$. For triple sequences, the statistical convergence depends on the density of the subsets of $N\times N\times N$. A subset E of $N\times N\times N$ is said to have triple asymptotic density if the limit:

$$\delta_3(E) = \lim_{p, q, r\rightarrow\infty} \sum_{n=1}^{p}\sum_{k=1}^{q}\sum_{l=1}^{r}\chi_E(n, k, l) \text{ exists}$$

where, χ_E is the characteristic function of E. Sahiner *et al*. (2007) first introduced the concept of statistical convergence for triple sequences and they defined: A sequence $x = \langle x_{nkl}\rangle$ is called statistically convergent to the number L if for every $\varepsilon>0$, $\delta_3(\{(n, k, l)\in N\times N\times N: |x_{nkl}-L|\geq\varepsilon\}) = 0$. A sequence $x = \langle x_{nkl}\rangle$ is called statistically Cauchy, if for every $\varepsilon>0$, $\exists$ $p = p(\varepsilon)$, $q = q(\varepsilon)$ and $r = r(\varepsilon)\in N$ such that $\delta_3(\{(n, k, l)\in N\times N\times N: |x_{nkl}-x_{pqr}|\geq\varepsilon\}) = 0$. A triple sequence $x = \langle x_{nkl}\rangle$ of fuzzy numbers is defined as a triple infinite array of fuzzy real numbers x_{nkl} for all n, k, $l\in N$.

A sequence $x = \langle x_{nkl}\rangle$ of fuzzy numbers is called convergent in Pringsheim's sense to the fuzzy real number L if for each $\varepsilon>0$, $\exists$ $n_0 = n_0(\varepsilon)$, $k_0 = k_0(\varepsilon)$ and $l_0 = l_0(\varepsilon)$ such that $\bar{d}(X_{nkl}, L)<\varepsilon$ for all $n>n_0$, $k>k_0$ and $l>l_0$. A sequence $x = \langle x_{nkl}\rangle$ of fuzzy numbers is called bounded if $\sup_{n,k,l}\bar{d}(X_{nkl}, \bar{0})<\infty$.

A sequence $X = \langle X_{nkl}\rangle$ of fuzzy numbers is called a Cauchy sequence if for each $\varepsilon>0$, $\exists$ $p = p(\varepsilon)$, $q = q(\varepsilon)$ and $r = r(\varepsilon)\in N$ such that $\delta_3(\{(n, k, l)\in N\times N\times N: \bar{d}(X_{nkl}, X_{pqr})\geq\varepsilon\}) = 0$. A sequence $X = \langle X_{nkl}\rangle$ of fuzzy real numbers is called regularly convergent if it convergent in Pringsheim's sense and in addition the following statistical limits holds: For every $\varepsilon>0$, $\exists$ $n_1 = n_1(\varepsilon, k, l)$, $k_1 = k_1(\varepsilon, n, l)$ and $l_1 = l_1(\varepsilon, n, k)$ such that $\bar{d}(X_{nkl}, L_{kl})<\varepsilon$, for all $n\geq n_1$ for some $L_{kl}\in R(L)$ for each k $l\in N$; $\bar{d}(X_{nkl}, L_{nl})<\varepsilon$ for all $l\geq l_1$ for some $L_{nl}\in R(L)$ for each n, $l\in N$; And for $\bar{d}(X_{nkl}, L_{nk})<\varepsilon$ all $k\geq k_1$ for some $L_{nk}\in R(L)$ for each n, $k\in N$.

A sequence $X = \langle X_{nkl}\rangle$ of fuzzy numbers is called statistically convergent in Pringsheim's sense to the fuzzy real number X_0 if for all $\varepsilon>0$:

$$\delta_3(\{(n, k, l)\in N\times N\times N: \bar{d}(X_{nkl}, X_0)\geq\varepsilon\}) = 0$$

A sequence $X = \langle X_{nkl}\rangle$ of fuzzy numbers is called statistically null if it is statistically convergent to zero. A sequence $X = \langle X_{nkl}\rangle$ of fuzzy numbers is called statistically bounded if there exists a real number μ such that:

$$d_3(\{(n, k, l)\in N\times N\times N: \bar{d}(X_{nkl}, \bar{0})>\mu\}) = 0$$

A sequence $X = \langle X_{nkl}\rangle$ of fuzzy numbers is called statistically Cauchy if for each $\varepsilon>0$, $\exists$ $p = p(\varepsilon)$, $q = q(\varepsilon)$ and $r = r(\varepsilon)\in N$ such that:

$$\delta_3(\{(n, k, l)\in N\times N\times N: \bar{d}(X_{nkl}, X_{pqr})\geq\varepsilon\}) = 0$$

A sequence $X = \langle X_{nkl}\rangle$ of fuzzy numbers is called statistically regularly convergent if it convergent in Pringsheim's sense and also the following statistical limits holds:

$$\text{stat}_3\text{-}\lim_{n\rightarrow\infty} X_{nkl} = L_{kl}(k, l\in N)$$

$$\text{stat}_3\text{-}\lim_{k\rightarrow\infty} X_{nkl} = L_{nl}(n, l\in N)$$

And:

$$\text{stat}_3\text{-}\lim_{l\rightarrow\infty} X_{nkl} = L_{nl}(n, k\in N)$$

Let E^F denote a sequence space of fuzzy numbers. E^F is called normal or solid if $\langle Y_{nkl}\rangle\in E^F$ whenever $\langle X_{nkl}\rangle\in E^F$ and $d(Y_{nkl}, \bar{0})\leq d(X_{nkl}, \bar{0})$ for all n, k, $l\in N$.

E^F is called monotone if it contains the canonical pre-image of all its step spaces. E^F is called symmetric if $\langle X_{\pi(nkl)}\rangle\in E^F$ whenever $\langle X_{nkl}\rangle\in E^F$ where π is a permutation on $N\times N\times N$. E^F is called sequence algebra if $\langle X_{nkl}\otimes Y_{nkl}\rangle\in E^F$ whenever $\langle X_{nkl}\rangle$, $\langle Y_{nkl}\rangle\in E^F$. E^F is called convergence free if $\langle Y_{nkl}\rangle\in E^F$ whenever $\langle X_{nkl}\rangle\in E^F$ an $X_{nkl} = \bar{0}$ d implies $Y_{nkl} = \bar{0}$.

Let $\langle X_{nkl}\rangle$ and $\langle Y_{nkl}\rangle$ be two triple sequences of fuzzy real numbers, then $X_{nkl} = Y_{nkl}$ for almost all n, k and l (in short a.a. n, k and l) if $\delta_3(\{(n, k, l)\in N\times N\times N: X_{nkl}\neq Y_{nkl}|\geq\varepsilon\}) = 0$. Kizmaz (1981) first introduced the concept of difference sequence spaces as follows: $Z(\Delta) = \{x = (x_k)\in w: (\Delta x_k)\in Z\}$, for $Z = \ell_\infty$, c, c_0 where $\Delta x_k = x_k - x_{k+1}$ for all $k\in N$. Tripathy and Esi (2006) introduced the difference sequence by $\Delta mx = (\Delta_m x_k) = x_k - x_{k+m}$ for all $k\in N$ and $m\in N$ be fixed.

For crisp set, Tripathy and Sarma (2008) introduced the concept of difference double sequences of fuzzy numbers as: $\Delta X_{nk} = X_{n,k} - X_{n,k+1} - X_{n+1,k} + X_{n+1,k+1}$ In this study, for triple sequence spaces of fuzzy numbers we introduce it as follows:

$$Z(\Delta) = \{X = (X_{nkl}) \in {}_3 w^F : (\Delta X_{nkl}) \in Z\}$$

For:

$$Z = (\bar{\ell}_\infty^F)_3, (\bar{c}_0^F)_3^P, (\bar{c}_0^F)_3^B, (\bar{c}_0^F)_3^{BR}, (\bar{c}_0^F)_3^R, (\bar{c}^F)_3^P, (\bar{c}^F)_3^{BR}, (\bar{c}^F)_3^B, (\bar{c}^F)_3^R$$

Where:

$$\Delta X_{nkl} = X_{nkl} - X_{n,k+1,l} - X_{n,k,l+1} + X_{n,k+1,l+1} - X_{n+1,k,l} + X_{n+1,k+1,l} + X_{n+1,k,l+1} - X_{n+1,k+1,l+1}$$

for all n, k, l∈N.

Lemma 2.1: Every normal sequence spaceis monotone.

RESULTS AND DISCUSSION

Theorem 3.1: The spaces $Z(\Delta)$ for $Z = (\bar{\ell}_\infty^F)_3, (\bar{c}_0^F)_3^B$ are $(\bar{c}_0^F)_3^{BR}, (\bar{c}^F)_3^{BR}, (\bar{c}^F)_3^B$ complete metric spaces with respect to the metric ρ defined by:

$$\rho(X,Y) = \sup_{n,l} \bar{d}(X_{nl1}, Y_{nl1}) + \sup_{n,k} \bar{d}(X_{n1k}, Y_{n1k}) + \sup_{l,k} \bar{d}(X_{1lk}, Y_{1lk}) + \sup_{n,k,l} \bar{d}(\Delta X_{nkl}, \Delta Y_{nkl})$$

Proof: Consider the space $(\bar{\ell}_\infty^F)_3$. Let $\langle X^{(i)} \rangle$ be a Cauchy sequence in $(\bar{\ell}_\infty^F)_3$. Then for each $\varepsilon>0$, $\exists$ n_0 such that:

$$\rho\left(X^{(i)}, X^{(j)}\right) < \varepsilon, \text{ for all } i, j \geq n_0$$
$$\Rightarrow \sup_{n,l} \bar{d}(X_{nl1}^{(i)}, X_{nl1}^{(j)}) + \sup_{n,k} \bar{d}(X_{n1k}^{(i)}, X_{n1k}^{(j)}) + \sup_{l,k} \bar{d}(X_{1lk}^{(i)}, X_{1lk}^{(j)}) + \sup_{n,k,l} \bar{d}(\Delta X_{nkl}^{(i)}, \Delta\Delta X_{nkl}^{(j)}) < \varepsilon,$$
$$\text{for all } i, j \geq n_0$$

This implies for each fixed n, k, l∈N:

$$\bar{d}\,(X_{nl1}^{(i)}, X_{nl1}^{(j)}) < \frac{\varepsilon}{4}, \ \bar{d}(X_{n1k}^{(i)}, X_{n1k}^{(j)}) < \frac{\varepsilon}{4}$$

$$\bar{d}(X_{1lk}^{(i)}, X_{1lk}^{(j)}) < \frac{\varepsilon}{4}, \quad \bar{d}(\Delta X_{nkl}^{(i)}, \Delta X_{nkl}^{(j)}) < \frac{\varepsilon}{4}$$

for each $i, j \geq n_0$. Hence, $X_{nl1}^{(i)}$ is a Cauchy sequence in R(L) for all n, l∈N, $X_{nl1}^{(i)}$ is a Cauchy sequence in R(L) for all n, k∈N $X_{1k1}^{(I)}$ is a Cauchy sequence in R(L) for all l, k ∈N and $\langle \Delta X_{nkl}^{(I)} \rangle$ is a Cauchy sequence in R(L) for all n, k, l∈N.

Since, R(L) is complete, so, $X_{nl1}^{(I)}, X_{n1k}^{(I)}, X_{1lk}^{(I)}$ and $x_{nkl}^{(i)}$ are convergent for each n, k, l ∈N. Let $\lim_{i\to\infty} X_{nll}^{(i)} = X_{nll}$ for each n, k, l∈N, $\lim_{i\to\infty} X_{nlk}^{(i)} = X_{nlk}$ for each n, k∈N and $\lim_{i\to\infty} X_{1lk}^{(i)} = X_{1lk}$ for each l, k, n∈N. Now consider $x_{111}^{(i)}$. Then:

$$\Delta X_{111} = X_{111} - X_{121} - X_{112} + X_{122} - X_{211} + X_{221} + X_{212} - X_{222}$$

Therefore, $x_{111}^{(i)}, x_{121}^{(i)}, x_{112}^{(i)}, x_{122}^{(i)}, x_{211}^{(i)}, x_{221}^{(i)}, x_{212}^{(i)}$ are convergent. Hence, $x_{222}^{(i)}$ converges. Let $\lim_{i\to\infty} X_{222}^{(i)} = X_{222}$. Proceeding in this way inductively, we have $x_{nkl}^{(i)}$ converges for each n, k, l ∈N. Let $\lim_{i\to\infty} X_{nkl}^{(i)} = X_{nkl}$ for all n, k, l ∈N. Then we get a set:

$$A = \left\{(n, k, l) : \bar{d}\,(X_{nkl}^{(i)}, X_{nkl}) < \frac{\varepsilon}{2}\right\} \subseteq N \times N \times N$$

Such that $\delta_3(A) = 1$ as $I\to\infty$. Let i_0 be such that $\delta_3(A) = 1-\gamma$ for $i>i_0$ where γ is a small positive real number. Taking limit as $j\to\infty$ in Eq. 1:

$$\rho\left(X^{(i)}, X\right) < \varepsilon, \text{ for all } i \geq n_0$$

So, the Cauchy sequence $\langle X^{(i)} \rangle$ converges to X.
Now for all $i > n_0$:

$$\rho\left(X, \bar{0}\right) \leq \rho\left(X, X^{(i)}\right) + \rho\left(X^{(i)}, \bar{0}\right) \quad \leq \varepsilon + K < \infty$$

Then, we get a set $B = \left\{(n, k, l) : \bar{d}\,(X_{nkl}^{(i)}, \bar{0}) < \frac{\varepsilon}{2}\right\}$ such that: $\delta_3(B) = 1$. For $i>i_0$ let $C = A \cap B$. Then $\delta_3(C) = \delta_3(A) \cap \delta_3(B) = 1-\gamma$. $\therefore \lim_{i\to\infty} \delta_3(C) = 1$. This gives $X \in (\bar{\ell}_\infty^F)_3$ and hence, $(\bar{\ell}_\infty^F)_3$ is complete. Similarly, we can prove the result for other spaces.

Proposition 3.2: The spaces $Z(\Delta)$ for $Z = (\bar{\ell}_\infty^F)_3, (\bar{c}_0^F)_3^P, (\bar{c}_0^F)_3^B, (\bar{c}_0^F)_3^{BR}, (\bar{c}_0^F)_3^R, (\bar{c}^F)_3^P, (\bar{c}^F)_3^{BR}, (\bar{c}^F)_3^B, (\bar{c}^F)_3^R$ are neither normal nor monotone.

Proof: The spaces $(\bar{c}_0^F)_3^P, (\bar{c}_0^F)_3^B, (\bar{c}_0^F)_3^{BR}$ are not normal. To justify this, we cite an example.

Example 3.1: Consider the sequence $\langle X_{nkl} \rangle$ defined as: $X_{nkl} = \bar{1}$ for all n, k, l ∈N. Then $\Delta X_{nkl} = \bar{0}$ for all n, k, l ∈N. Thus:

$$\langle X_{nkl} \rangle \in Z(\Delta)$$

Where:

$$Z = (\bar{c}_0^F)_3^P, (\bar{c}_0^F)_3^B, (\bar{c}_0^F)_3^{BR}, (\bar{c}_0^F)_3^R, (\bar{c}^F)_3^P, (\bar{c}^F)_3^{BR}, (\bar{c}^F)_3^B, (\bar{c}^F)_3^R$$

Next consider the sequence $\langle Y_{nkl}\rangle \in R(L)$ defined as: $Y_{nkl}(t) = \overline{(-1)^{n+k+l}}$ for all n, k, l $\in$N. Then $\Delta Y_{nkl} = \bar{-8}$ for all n, k, l odd $\Delta Y_{nkl} = \bar{8}$ and for all n, k, l even.

Also, $d(Y_{nkl}, \bar{0}) \le d(X_{nkl}, \bar{0})$ for all n, k, l$\in$N and $\langle \Delta y_{nkl}\rangle \notin Z(\Delta)$. Hence, the paces $Z(\Delta)$ are not normal. Now from Lemma 2.1; It follows that $Z(\Delta)$ are not monotone. For the space $(\bar{\ell}_\infty^F)_3$ we cite an example.

Example 3.2: Let the sequence $\langle X_{nkl}\rangle$ be defined by:
For $n \ne i^2$, i $\in$N and for all k, l $\in$N
$X_{nkl} = \begin{cases} \bar{i}, & \text{for } l = i^2, \ i \in N \text{ and } n = k = 1 \\ \bar{1}, & \text{Otherwise} \end{cases}$. Then $\Delta X_{nkl} = \bar{0}$ for all n, k, l $\in$N and hence, $\langle X_{nkl}\rangle \in (\bar{\ell}_\infty^F)_3(\Delta)$. Now the sequence $\langle Y_{nkl}\rangle$ is defined as:

$$Y_{nkl} = \begin{cases} \bar{i}, & \text{for } l = i^2, i \in N \text{ and } n = k\ 1 \\ \bar{1}, & \text{otherwise} \end{cases}$$

Then $\Delta Y_{nkl} = \bar{0}$ for $l \ne i^2$ and $\Delta Y_{nkl} = \overline{i-1}$ for $I = i^2$ for n = k = 1. Thus $d(Y_{nkl}, \bar{0}) \le d(X_{nkl}, \bar{0})$ for all n, k, l $\in$N and $\langle Y_{nkl}\rangle \notin (\bar{\ell}_\infty^F)_3(\Delta)$. Therefore $(\bar{\ell}_\infty^F)_3(\Delta)$ is not normal and hence not monotone.

Proposition 3.3: The spaces $Z(\Delta)$ for:

$$Z = (\bar{\ell}_\infty^F)_3, (\bar{c}_0^F)_3^P, (\bar{c}_0^F)_3^B, (\bar{c}_0^F)_3^{BR}, (\bar{c}_0^F)_3^R, (\bar{c}^F)_3^P, (\bar{c}^F)_3^{BR}, (\bar{c}^F)_3^B, (\bar{c}^F)_3^R$$

are not symmetric.

Proof: The spaces $Z(\Delta)$ are not symmetric which follows from the Example 3.3.

Example 3.3: Let the sequence $\langle X_{nkl}\rangle$ be defined by:

$$X_{nkl} = \begin{cases} \bar{-i}, & \text{for all } k, l \in N, \ n = 1 \\ \bar{1}, & \text{otherwise} \end{cases}$$

Then $\langle X_{nkl}\rangle \in Z(\Delta)$. Let $\langle Y_{nkl}\rangle$ be a readjustment of $\langle X_{nkl}\rangle$ defined by:

$$Y_{nkl} = \begin{cases} \bar{i}, & \text{for } (n+k+1) \text{ even, for all } n, k, l \in N \\ -\bar{1}, & \text{otherwise} \end{cases}$$

Then $\Delta Y_{nkl} = \bar{8}$ for (n+k+l) even and $\Delta Y_{nkl} = \bar{-8}$ for (n+ k + l) odd. Thus, $\langle X_{nkl}\rangle \notin Z(\Delta)$. So, $Z(\Delta)$ are not symmetric. For the space $(\bar{\ell}_\infty^F)_3$ we consider Example.

Example 3.4: Consider the sequence $\langle X_{nkl}\rangle$ defined as:

$$X_{nkl} = \begin{cases} \bar{1}, & \text{for } l = i^2, \ i \in N, \text{ for all } n, k \in N \\ \bar{0}, & \text{otherwise} \end{cases}$$

Then $\Delta X_{nkl} = \bar{0}$ for all n, k, l $\in$N and hence, $\langle X_{nkl}\rangle \in (\bar{\ell}_\infty^F)_3(\Delta)$
Let $\langle Y_{nkl}\rangle$ be a readjustment of $\langle X_{nkl}\rangle$ defined by:

$$Y_{nkl} = \begin{cases} \overline{k1}, & \text{for n odd, for all } k, l \in N \\ \bar{0}, & \text{Otherwise} \end{cases}$$

Then $\Delta Y_{nkl} = \bar{-1}$ for n odd and for $\Delta Y_{nkl} = \bar{1}$ n even. Thus $\langle Y_{nkl}\rangle \notin (\bar{\ell}_\infty^F)_3(\Delta)$. Hence, $\left(\bar{\ell}_\infty^{-F}\right)_3(\Delta)$ is not symmetric.

Proposition 3.4: The spaces $Z(\Delta)$ for:

$$Z = (\bar{\ell}_\infty^F)_3, (\bar{c}_0^F)_3^P, (\bar{c}_0^F)_3^B, (\bar{c}_0^F)_3^{BR}, (\bar{c}_0^F)_3^R, (\bar{c}^F)_3^P, (\bar{c}^F)_3^{BR}, (\bar{c}^F)_3^B, (\bar{c}^F)_3^R$$

are not convergence free. Proof. To prove this result, we cite an example.

Example 3.5: Let the sequence $\langle X_{nkl}\rangle$ be defined by:
For $n \ne i^3$, i$\in$N and for all K, L$\in$N:

$$X_{nkl} = \begin{cases} \bar{0}, & \text{for } n = i^2, i \in N, \text{ for all } k, l \in N, \\ \bar{3}, & \text{Otherwise} \end{cases}$$

Then:

$$\langle X_{nkl}\rangle \in Z(\Delta)$$

Where:

$$Z = (\bar{\ell}_\infty^F)_3, (\bar{c}_0^F)_3^P, (\bar{c}_0^F)_3^B, (\bar{c}_0^F)_3^{BR}, (\bar{c}_0^F)_3^R, (\bar{c}^F)_3^P, (\bar{c}^F)_3^{BR}, (\bar{c}^F)_3^B, (\bar{c}^F)_3^R$$

Next we define the sequence $\langle Y_{nkl}\rangle$ by:

$$Y_{nkl} = \begin{cases} \bar{0}, & \text{for } n = i^2 \ \ i \in N, \text{ for all } k, l \in N, \\ \overline{(k+1)(1+1)}, & \text{for all n even, } n \ne i^2, i \in N, \\ \text{for all } k, l \in N & \\ \overline{(k+1)(1+1)}, & \text{for all n odd, } n \ne i^2, i \in N, \\ \text{for all } k, \ l \in N, & \end{cases}$$

Then $\Delta Y_{nkl} = \bar{1}$ for $n = i^2$ for all n, k, l $\in$N, $\Delta Y_{nkl} = \bar{0}$ for n even, $n \ne i^2$ for all k, l $\in$N and $\Delta Y_{nkl} = \bar{-1}$ for n odd, for all $n \ne i^2$

for all k, l∈N. Then $\langle Y_{nkl}\rangle \notin Z(\Delta)$ where:

$$Z = (\overline{\ell}_\infty^F)_3, (\overline{c}_0^F)_3^P, (\overline{c}_0^F)_3^B, (\overline{c}_0^F)_3^{BR}, (\overline{c}_0^F)_3^R, (\overline{c}^F)_3^P, (\overline{c}^F)_3^{BR}, (\overline{c}^F)_3^B, (\overline{c}^F)_3^R$$

Hence, the spaces are not convergence free.

Proposition 3.5: $Z \subset Z(\Delta)$ for:

$$Z = (\overline{\ell}_\infty^F)_3, (\overline{c}_0^F)_3^P, (\overline{c}_0^F)_3^B, (\overline{c}_0^F)_3^{BR}, (\overline{c}_0^F)_3^R, (\overline{c}^F)_3^P, (\overline{c}^F)_3^{BR}, (\overline{c}^F)_3^B, (\overline{c}^F)_3^R$$

$$(\overline{c}^F)_3 \subset (\overline{c}_0^F)_3(\Delta)$$

Proof: Let us consider the space $(\overline{c}_0^F)_3$. Let $\langle X_{nkl}\rangle \in (\overline{c}_0^F)_3$. Then for each ε>0, ∃K⊂N×N×N such that $\bar{d}(X_{nkl}, \bar{0}) < \frac{\varepsilon}{8}$ for all n, k, l∈N with $\delta_3(K) = 1$. Now, for all (n, k, l)∈N:

$$\begin{aligned}&\bar{d}(\Delta X_{nkl}, \bar{0}) = \bar{d}(X_{nkl} - X_{n,k+1,l} - X_{n,k,l+1} + X_{n,k+1,l+1}\\ &- X_{n+1,k,l} + X_{n+1,k+1,l} + X_{n+1,k,l+1} - X_{n+1,k+1,l+1}, \bar{0}) =\\ &\bar{d}(X_{nkl}, \bar{0}) + \bar{d}(X_{n,k+1,l}, \bar{0}) + \bar{d}(X_{n,k,l+1}, \bar{0}) +\\ &\bar{d}(X_{n,k+1,l+1}, \bar{0}) + \bar{d}(X_{n+1,k,l}, \bar{0}) + \bar{d}(X_{n+1,k+1,l}, \bar{0}) +\\ &\bar{d}(X_{n+1,k,l+1}, \bar{0}) + \bar{d}(X_{n+1,k+1,l+1}, \bar{0})\\ &< \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \varepsilon\end{aligned}$$

Then there exist the sets:

$$A_1 = \left\{(n,k,l): \bar{d}(X_{nkl}, \bar{0}) < \frac{\varepsilon}{8}\right\},$$

$$A_2 = \left\{(n,k,l): \bar{d}(X_{n,k+1,l}, \bar{0}) < \frac{\varepsilon}{8}\right\},$$

$$A_3 = \left\{(n,k,l): \bar{d}(X_{n,k,l+1}, \bar{0}) < \frac{\varepsilon}{8}\right\},$$

$$A_4 = \left\{(n,k,l): \bar{d}(X_{n,k+1,l+1}, \bar{0}) < \frac{\varepsilon}{8}\right\},$$

$$A_5 = \left\{(n,k,l): \bar{d}(X_{n+1,k,l}, \bar{0}) < \frac{\varepsilon}{8}\right\},$$

$$A_6 = \left\{(n,k,l): \bar{d}(X_{n+1,k+1,l}, \bar{0}) < \frac{\varepsilon}{8}\right\},$$

$$A_7 = \left\{(n,k,l): \bar{d}(X_{n+1,k,l+1}, \bar{0}) < \frac{\varepsilon}{8}\right\},$$

$$A_8 = \left\{(n,k,l): \bar{d}(X_{n+1,k+1,l+1}, \bar{0}) < \frac{\varepsilon}{8}\right\}$$

such that:

$$\delta_3(A_1) = 1, \delta_3(A_2) = 1, \delta_3(A_3) = 1, \delta_3(A_4) = 1$$

$$\delta_3(A_5) = 1, \delta_3(A_6) = 1, \delta_3(A_7) = 1, \delta_3(A_8) = 1$$

Let:

$$A = A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6 \cap A_7 \cap A_8$$

Then clearly $\delta_3(A) = 1$. This completes the proof. The proof of the result for other spaces can be derived similarly. In a similar way the result can be proved. So, it is omitted.

CONCLUSION

Convergence theory is used as a simple device in measure spaces, sequences of random variables, information theory, etc. The results obtained in this study can be helpful as theoretical tools to study generalized convergence of sequences of fuzzy measurable functions and sequences of random variables which are broadly used in fuzzy information theory and fuzzy signal systems.

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