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On Polish Groups and their Applications

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Abstract: A top-space X is Polish if it is separable and completely metrizable (com-metrizable) and with Polish topological space G, it is a Polish topological group. In this study, we study new applications of Polish group and new relations with several certain classes of topological spaces such as metrizable space, seprable space, Nagata space and M3-space. If G is a top-gp and (X, dθ) is a separable com-θ-metric space then G is Polish group. If G is a top-gp and X a top-space and each point in X has a neighbourhood which is homeomorphic to an open subset of Rn then G is a Polish group.

Key words: Polish group, topological group, metrizable space, separable space, completely metrizable space, M3-space

INTRODUCTION

The subject of Polish group have been studied by several researchers. "Let us recall that a metrizable topological space (X, τ) issaid to be completely metrizable (com-metrizable) if it admits a com-metric" (Engelking, 1977). By Cech (1937), "a Hausdorff space is com-metrizable if it has a compatible com-metric". By Michael (1986), "if ametrizable space has a com-sequence of exhaustive covers then it is com-metrizable". The study of Khojasteh *et al*. (2013) "explained if (X, d_θ) is a θ-metric space then the collectionof open sets forms a topology, denoted by τd_θ". Also, in the study of Khojasteh *et al.* (2013), we founded some results about the notion θ-metric. By Ceder (1961), "a T_1-space X is metrizable if and only if X is a Nagata space with a Nagata structure $\langle\{U_n(x)\}$ n = 1, ..., ∞ $\{S_n(x)\}$; n = 1, ..., ∞$\rangle$ with the property that $x\in S_n(y)$ implies $S_n(x)\subset U_n(y)$ for all x, y∈X". Also, by Stubblefield (1972), "a regular space X is metrizable if and only if X has a σ-closure preserving base $B = \cup B_n$, n = 1, ..., ∞ where each B_n is point finite and A topological space (X, τ) is a Lindel of space if every open cover of X has a countable sub-cover". The aim of this research is to study a Polish group topology by several top-spaces.

MATERIALS AND METHODS

Preliminaries

Defnition 2.1: Let (X, d) be a metric space. Then if $\bar{A} = X$ this means A is densein X. So, a top-space (X, τ) is called separable if there exists a dense D⊆X.

Examples 2.2:

- If the metric space is compct, it is necessary to be separable. Alsometrizable spaceis separable
- If $(X, \tau) = \cup A_i$ such that A_i is countable number of separable subspaces then it is also separable
- All the continuous functions K→ℝ is separable space where K is compact and subset χℝ

Examples 2.3: Non-separable spaces:

- The Ω_1 is not separable (Ω_1 is the first uncountable ordinal)
- The Banach space l^∞ of $B\{X_n\}$ is not separable such that $\{X_n\}$ is all bounded real sequences

Definition 2.4: Let d be a metric on X and $\tau_d = \tau$ then (X, τ) is called metrizable.

Examples 2.5:

- (X, τ_u) is a metrizable space
- If (X, τ) is a discrete topological space then it is a metrizable space

Example 2.6: The (0, 1) is not complete in its usual metric but it is homeomorphic to R, so, completely metrizable.

Lemma 2.7; Michael (1986): "A metrizable space X is com-metrizable iffit has a complete exhaustive sieve (com-exhaustive sieve)".

Lemman 2.8; Ceder (1961): " Let X be an M_1-space. Then X is separable iff it is Lindelof."

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Lemma 2.9; Ceder (1961): Any com-metrizable is Nagata space and com-Nagata structure.

Remarks 2.10; Carderi and Maitre (2016):

- "The space (X, T) is called first-countable if the topology has a countable local basis at each point $x \in X$"
- "The space (X, T) is called second-countable if the topology T has a countable basis".
- If G is a Polish group and let $H \leq G$ then H is also, Polish iff H is closed in G
- If G is a Polish group, so, a quotient topology G/H is called a Polish group if $H \leq G$
- A Polish subgroup H of a Polish group G is closed in G

For several facts about Polish groups (Becker and Kechris, 1996; Carderi and Maitre, 2016; Malicki, 2007).

RESULTS AND DISCUSSION

Definition 3.1: A Polish space is a com-metrizable, separable topological space and a Polish group is a top-group whose topology is Polish.

Examples 3.2:

- Lie group is Polish group. So, top-group which is also, a manifold is Polish group such that a topological manifold is a Hausdorff, second countable, locally Euclidean space
- Countable lie group is Polish group
- Separable banach space group is Polish group
- Countable discrete group is Polish group
- All locally compact second countable groups, like (R, +)

Lemma 3.3: Let (X, τ) be a regular space. If X have countable basis then X is metrizable.

Proof: Suppose $\{f_n\}$ is a functions define by f_n: $X \rightarrow [0, 1]$ $x_0 \in X$ and $Y = R^N$. We take a map F: $X \rightarrow Y$ follows:

$$F(x)=f_1(x),f_2(x)$$

Here, we need to show F(x) is continuous injective mapping. Also, we need to prove F (U) is open inside F (X) where U is open set in X. Let a_0 be a point of F(U). Let $b_0 \in U \ni F(b_0) = a_0$ and let I be an index $\ni f_I(b_0)$ and $F_I(X \backslash U) = 0$. Assume that $K = (\beta_I)^{-1}((0, \infty)) \cap f(X)$; β_I is the projection $Y \rightarrow R$ onto the Ith multiple. Thus, K is an open subset of F (X) $\ni b_0 \in K \subset F(U)$.

Theorem 3.4: Let (X, τ) be a top-space and G is a top-gp. If G satisfy the following conditions:

- G is T_0-space
- G is separable space
- (X, τ) is a com-countble top-space.Then G is a Polish group

Proof: From condition (3), every point has a countable local basis. So, G is a first countable. But G is T_0-space. Therefore, G is metrizable space but we have (X, τ) is a com-countable top-space. So (X, τ) is a metrizable space with condition (2), we get G is Polish top-space.Thus, G is Polish top-gp (Polish group).

Note that if $P = (P_1, P)$ of subsets of X and $P_1 \subset P_2 \forall P \in P$ then, we denote cushioned to P if for every $P' \subset P$:

$$\left(\{p_1 : p \in P'\}\right) \{p_2 : p\ p \in\} P'$$

and if P is the union of countably many cushioned then we denote σ-cushioned to P.

Definition 3.5: "An M3-space is a T_1-space with a σ-cushioned pairbase" (Ceder, 1961).

Definition 3.6: "A Nagata space X is a T_1-space such that for each $x \in X$ there exist sequences of neighbor hoods of x, $\{U_n(x)\}$, n = 1, …, ∞ and $\{S_n(x)\}$, n = 1, …, ∞ such that:

- For each $x \in X$, $\{U_n(x)\}$, n = 1, …, ∞ is a local base of neighborhoods of x
- For all x, $y \in X$, $S_n(x) \cap S_n(y) \neq \phi$ yield $x \in U_n(y)$" (Ceder, 1961)

Lemma 3.7: "A topological space is a Nagata space if and onlyif it is first countable and M3-space" (Ceder, 1961).

Theorem 3.8: Let G be a group. If X is first countable and M3 with com-exhaustive sieve such that $[U_a: a \in A_n]$ is disjoint for all n then G is polish group.

Proof: Let $(\{U_a : a \in A_n\}, \pi_n)$ be a sieve on X and d and disjoint for all n. let $\square_n = \{U_a : a \in A_n\}$. Then each $\square_n$ is an exhaustive cover of X by Lemma 2.7. To show that $(\square_n)$ is a complete sequence of covers, let $\mathfrak{C}$ be a filter base on X such that each $\square_n$ has an element $\square_{an}$ $(a_n \in A_n)$ containing some $F \in \mathfrak{C}$. Then $U\, a_{n+1} n\, U_n \neq \varnothing$ for all n and hence, (since, $\square_n$ is disjoint) $\pi_n(a_{n+1}) = a_n$. Since, $(\{U_n : a \in A_n\}, \pi_n)$ is complete, it follows that $n \cap \{\bar{F} : F \in \mathfrak{C}\} \neq \varnothing$. Thus, X has a com-sequence of exhaustive covers. But X is first countable and M3. So, X is Nagata space.

Hence, is T_1-space (metirizable property). But X has com-exhaustive sieve. Hence, X is com-metrizable with separable property yield G is a polish group.

Lemma 3.9: A topologically com-M_1-space is com-metrizable.

Corollary 3.10: A locally compact space is com-metrizable.

Proof: Since, any locally compact is open in any Hausdorff space then X is openin β(X) and so, X is a com-metrizable.

Definition 3.11: Let X be a nonempty set and dθ: X×X→[0,⁺∞). Sowe said d θ-metric on X with respect to B-action θ∈Y if dθ satisfies the following:

- (A1) (a, b) = 0 if, a = b
- (A2) (a, b) = d (b, a), for all a, b∈X
- (A3) (a, b) ≤ (d_θ (a, c),d_θ (c, b)), a, b, c∈X

Theorem 3.12: Let G be a top-gp. If (X, d_θ) be a separable com-θ-metric space then G is Polish group.

Proof: We have that {U_n}, ∈N∋U is uniformity on X with τd_θ where U_n = {(x, y)∈X×: d(x, y)<1/n}, n∈N. Then d is a metric space on X and coincides with U. Know, we must prove that d is complete. Let {x_n} is a Cauchy in X, d for ∈>0. If k∈N∋1/k<∈ then ∃n_0∈N with (x_m, x_n) ∈∪k, n, m≥n_0. For all n, m≥n_0, (x_n, x_m) = 1/k<∈. So, {x_n} is a Cauchy sequence in the com-θ-metric space (X, d) and convergent to X, d. Thus, (X, d) is a com-metric space. So, (X, d_θ) is com-metrizable space. But (X, d_θ) is separable space. Thus (X, d_θ) is Polish top-space and hence, Polish top-group (G is Polish group).

A top-space X is called Baire space if ∩C_i is dense of X such that Ci is countably dense. Or equivalently, if ∪F_i has empty interior points in X such that F_i is countable family of closed subsets.

Theorem 3.13; (The Baire category theorem): Every locally compact Hausdorff space and every locally com-metrizable space is a Baire space.

Theorem 3.14: Let G be a group. If (X, J) is Nagata space with a com-Nagata structure and second countable then G is a Polish group.

Proof: First we must prove that (X, J) is separable space. Let (X, J) is a second countable top-space. So, ∃D = {B_1, B_2, ...,} is countable base for (X, J). When we write D = {B_1, B_2, ...,}. Thus, means B is not countably infinite set. So, either D = {B_1, B_2, ..., B_n} or D = ∅ or D is a countably infinite set; n in N. If X≠∅, D≠∅. If Bk = ∅ then D' = {B_1, B_2, ..., B_{k-1}, B_{k+1}; ...,} is a basis for (X, J), k in N. Assume B_n∅. Since, B_n∅, for each n∈N, we can say, let x_n∈B_n (by axiom of choice ∃ f: N→B_n, n = 1, ..., ∞∋x_n = f (n)∈B_n) and A = {x_1, x_2, x_3, ...,}. Also, A is a finite set. To show $\bar{A}$ = X, we need x in X and U⊃x. Since, D is a basis for (X, J), implies B_n∈D∋x∈B_n, B_n⊆U and x_n∈B_n. Hence, x_n∈ U∩A. So, U∩A≠∅. Hence, x ∈ $\bar{A}$ and so, x in X. Therefore, x ∈ $\bar{A}$ and $\bar{A}$ = X. Also, (X, J) has a countable dense subset. Hence, (X, J) is a separable. But (X, J) is Nagata space with a com-Nagata structure. Thus, (X, J) is a com-metrizable space. So, G is Polish top-space and so is a Polish group.

Proposition 3.15: Let G be top-gp and (X, J) be com-metrizable topological space. If (X, J) is a countable space over group G, so, G is a Polish group.

Proof: If, we have an element of a countable base, the set is at most countable and dense. So, (X, J) is a separable space with com-metrizable property we get G is a Polish Top-space. Hence, G is a Polish top-group (G is a Polish group). The Zariski topology on R is the family Z of all complements to finite sets.

Theorem 3.16: Let G be a top-gp. If the Zariski com-metrizable topology space (Z, J) given on the real line R with the structure of a top-space (R, Z) then G is a Polish group.

Proof: (Z, J) is separable, since, it is weaker than the standard topology in R. We have (Z, J) is com-metrizable space. So, (Z, J) is a Polish top-space. Hence, is a Polish top-group (G is a Polish group).

Definition 3.17: Let (X, τ) be a top-space. If ∃{G_n:n∈Z^+} of X,∋G_n is a sequence of open covers and ∀×∈X, {st (x, G_n)|n∈Z} forms a neighborhood base at x then (X, τ) is called a developable space and {G_n|n∈Z} a development for X.

Theorem 3.18: Let (X, τ) be a space and G is a top-gp. If:

- (X, τ) is a compact
- (X, τ) is a semi-metric
- (X, τ) is a com-metrizable then G is a Polish group

Proof: A compact semi-metric space is a com-regular p-space and is a developable space. A compact developable space is second countable. Thus, a compact semi-metric space is regular and second countable. Hence, a compact semi-metric space is a separable metric space. But (X, τ) is com-metrizable space. Hence, (X, τ) is a Polish top-space and so is a Polish top-group (G is a Polish group).

Remarks 3.19:

- Compact space is a Lindelof
- A second axiom is a Lindelof
- Let (X, τ) is a discrete top-space with X as a infinite countable set. This space is a Lindelof space
- It is easily seen that a regular Lindelof, developable space is a separable metric space

Corollary 3.20: Every paracompact β-space with a quasi development is metrizable.

Corollary 3.21: Let G be a top-gp and X be a top-space and each point in X has a neighbourhood which is homeomorphic to an open subset of R^n. Then gis a Polish group.

Proof: For x in X. $\exists U_x$, $x \in U_x$ and the mappingh_x: $U_x \rightarrow V$. If V is open, so, $\exists r_x$ such that $N_r(h_x(x)) \subseteq V$. Hence, we get $(h_x)^{-1}$ $(N_r(z))$ gives a homeomorphism $W_x \rightarrow N_r(z)$. Hence, X = $U_{x \in X} W_x$ and each W_x is homeomorphic to $N_{rx}(Z)$ for some Z homeomorphic to R^n. So, X is a manifold with G top-group yield G is a Polish group.

Corollary 3.22: Let the unitary group U (n) is defined as U(n) = {T$\in$GL (n, C); T•T^* = 1} where T^* denotes the conjugate transpose (adjoint) of T be a top-group. Then U (n) is a polish group.

CONCLUSION

Also, if we have Polish group is paracompact β-space with a quasi development is metrizable Mathematical Review subject classi_cation. MSC (2010):Primary: 20E15; Secondary: 20F19.

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