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Two Parameter Lindley Distribution:Estimating the Reliability Function with Fuzzy Data

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Abstract: In this study, the maximum likelihood and approximate Bayes estimators to the reliability function of two parameter Lindley distribution have been derived when the data are shown in fuzzy form. Bayes estimators have been derived based on informative gamma priors with squared error and precautionary loss functions according to approximate Lindley's technique. The generated samples that follow the two parameter Lindley distribution are converted to fuzzy data based on a specific fuzzy information system. In addition, obtained estimators to the reliability function have been compared numerically through a Monte-Carlo simulation study in terms of their integrated mean squared error values.

Key words: Lindley, Bayes estimators, gamma, Monte-Carlo, precautionary, simulation study

INTRODUCTION

Lindley distribution: Shanker *et al.* (2013) introduced a Two-Parameter Lindley (TPL) distribution. The probability density and cumulative distribution functions of a TPL distribution are given, respectively by:

$$f_X(x;\alpha,\theta) = \frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x};\ x \geq 0,\ \theta>0,\ \alpha>-\theta \quad (1)$$

$$F_X(x;\alpha,\theta) = 1 - \frac{\theta+\alpha+\alpha\theta x}{\theta+\alpha}e^{-\theta x};\ x \geq 0,\ \theta>0,\ \alpha>-\theta \quad (2)$$

From Eq. 1, it can easily be seen that when $\alpha = 1$, a TPL distribution reduces to the one parameter Lindley distribution. When $\alpha = 0$, a TPL distribution reduces to the exponential distribution with parameter (θ). The reliability function of TPL distribution for a specified period of time, say t ($t\geq 0$) is given, respectively by:

$$R(t;\alpha,\theta) = 1 - F_T(t;\alpha,\theta) = \frac{\theta+\alpha+\alpha\theta t}{\theta+\alpha}e^{-\theta t};\ \theta > 0,\ a>-\theta \quad (3)$$

Maximum likelihood estimation: Suppose $\underline{x} = (x_1, x_2, \ldots, x_n)$ be an i.i.d. random vector of a random sample of size n from a TPL distribution with probability density function given by Eq. 1. If an observation of $\underline{x}$ was observed accurately, then the complete-data likelihood function, $L(\alpha, \theta|\underline{x})$ is given by:

$$L(\alpha, \theta|\underline{x}) = \prod_{i=1}^{n} f_X(x_i; \alpha, \theta) = \frac{\theta^{2n}}{(\theta+\alpha)^n}\prod_{i=1}^{n}(1+\alpha x_i)e^{-\theta\sum_{i=1}^{n} x_i} \quad (4)$$

Now, suppose that a random vector $\underline{x}$ is not observed accurately and only partial information about it is available in the form of a fuzzy subset $\tilde{\underline{x}}$ with the membership function $\mu_{\tilde{x}}(x)$ (Pak *et al.*, 2013). For this state, using the Zadeh's expression of the probability of a fuzzy event $\tilde{x}$ in $\mathbb{R}^n$ which defined as the expectation of $\mu_{\tilde{x}}$ with respect to P where ($\mathbb{R}^n$, X, P) be a probability space, $P(\tilde{x}) = \int \mu_{\tilde{x}}(X)dP$; for all $x\in\mathbb{R}^n$ (Denoeux, 2011; Zadeh, 1968). The observed-data likelihood function can be obtained as:

$$L(\alpha,\theta|\tilde{\underline{x}}) = \prod_{n}^{i=1}\int f_X(x; \alpha,\theta)\mu_{\tilde{x}_i}(x)dx$$
$$L(\alpha,\theta|\tilde{\underline{x}}) = \frac{\theta^{2n}}{(\theta+\alpha)^n}\prod_{i=1}^{n}\int(1+\alpha x)e^{-\theta x}\mu_{\tilde{x}_i}(x)dx \quad (5)$$

Then, the observed-data natural log-likelihood function can be obtained as:

$$\varphi = \varphi(\alpha, \theta|\tilde{\underline{x}}) = \text{in } L(\alpha, \theta\|\tilde{\underline{x}})$$
$$\varphi = 2n \ln\theta - n \ln(\theta+a) + \sum_{i=1}^{n}\ln\int(1+\alpha x)e^{-\theta x}\mu_{\tilde{x}_i}(x)dx \quad (6)$$

The likelihood equations to α and θ are given, respectively by:

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$$\frac{\partial\varphi}{\partial\alpha} = -\frac{n}{\theta+a} + \sum_{i=1}^{n} \frac{\int xe^{-\theta x}\mu_{\tilde{x}_i}(x)dx}{\int(1+ax)e^{-\theta x}\mu_{\tilde{x}_i}(x)dx} \quad (7)$$

$$\frac{\partial\varphi}{\partial\theta} = \frac{2n}{\theta} - \frac{n}{\theta+\alpha} - \sum_{i=1}^{n} \frac{\int(1+\alpha x)e^{-\theta x}\mu_{\tilde{x}_i}(x)dx}{\int(1+\alpha x)e^{-\theta x}\mu_{\tilde{x}_i}(x)dx} \quad (8)$$

Since, there are no closed forms of the solutions of the above two likelihood equations, a Newton-Raphson (NR) algorithm as an iterative technique can be used to obtain the maximum likelihood estimations of the parameters α and θ. The NR algorithm can be summarized by:

- Given starting values of α and θ, say $\alpha^{(0)}$ and $\theta^{(0)}$ and set iteration I = 0
- At iteration (I+1), estimate the new value of α and θ as

$$\begin{bmatrix}\hat{\alpha}^{(I+1)} \\ \hat{\theta}^{(I+1)}\end{bmatrix} = \begin{bmatrix}\hat{\alpha}^{(I)} \\ \theta^{(I)}\end{bmatrix} - \begin{bmatrix}\frac{\partial^2\varphi}{\partial\alpha^2} & \frac{\partial^2\varphi}{\partial\alpha\partial\theta} \\ \frac{\partial^2\varphi}{\partial\theta\partial\alpha} & \frac{\partial^2\varphi}{\partial\theta^2}\end{bmatrix}^{-1}_{\substack{\alpha=\hat{\alpha}^{(I)} \\ \theta=\hat{\theta}^{(I)}}} \begin{bmatrix}\frac{\partial\varphi}{\partial\alpha} \\ \frac{\partial\varphi}{\partial\theta}\end{bmatrix}_{\substack{\alpha=\hat{\alpha}^{(I)} \\ \theta=\hat{\theta}^{(I)}}} \quad (9)$$

where, $\partial\varphi/\partial\alpha$ and $\partial\varphi/\partial\theta$ as in Eq. 7 and 8, respectively:

$$\frac{\partial^2\varphi}{\partial\alpha^2} = \frac{n}{(\theta+\alpha)^2} - \sum_{i=1}^{n}\left(\frac{\int xe^{-\theta x}\mu_{\tilde{x}}(x)dx}{\int(1+\alpha x)e^{-\theta x}\mu_{\tilde{x}}(x)dx}\right)^2 \quad (10)$$

$$\frac{\partial^2\varphi}{\partial\theta^2} = -\frac{2n}{\theta^2} + \frac{n}{(\theta+\alpha)^2} + \sum_{i=1}^{n}\left[\frac{\int x^2(1+\alpha x)e^{-\theta x}\mu_{\tilde{x}}(x)dx}{\int(1+\alpha x)e^{-\theta x}\mu_{\tilde{x}}(x)dx} - \left(\frac{\int x(1+\alpha x)e^{-\theta x}\mu_{\tilde{x}}(x)dx}{\int(1+\alpha x)e^{-\theta x}\mu_{\tilde{x}}(x)dx}\right)^2\right] \quad (11)$$

$$\frac{\partial^2}{\partial\alpha\partial\theta} = \frac{\partial^2\varphi}{\partial\theta\partial\alpha} = \frac{n}{(\theta+\alpha)^2}\sum_{i=1}^{n}\frac{\int x^2e^{-\theta x}\mu_{\tilde{x}_i}(x)dx}{\int(1+\alpha x)e^{-\theta x}\mu_{\tilde{x}_i}(x)dx} + \sum_{i=1n}^{n}\frac{\int xe^{-\theta x}\mu_{\tilde{x}_i}(x)dx\int x(1+\alpha x)e^{-\theta x}\mu_{\tilde{x}_i}(x)dx}{\left(\int(1+\alpha x)e^{-\theta x}\mu_{\tilde{x}_i}(x)dx\right)^2} \quad (12)$$

Repeat the second step until convergence occurs, that is the absolute difference between two successive iterations is less than pre-specified error tolerance, $\varepsilon>0$. When the convergence occurs then the current estimation of α and θ at iteration (I+1) be the Maximum Likelihood Estimates (MLE's) of that parameters which we referred to as $(\hat{\alpha}_{ML}, \hat{\theta}_{ML})$.

Then, according to an invariant property of the ML estimation, the estimate of reliability function at time (t) of TPL can be obtained by replacing α and θ in Eq. 3 by their MLE's as:

$$\hat{R}_{ML}(t) = \frac{\hat{\theta}_{ML}+\hat{\alpha}_{ML}+\hat{\alpha}_{ML}\hat{\theta}_{ML}t}{\hat{\theta}_{ML}+\hat{\alpha}_{ML}}e^{-\hat{\theta}_{ML}t}; t \geq 0 \quad (13)$$

MATERIALS AND METHODS

Bayes estimation: In this study, we derive Bayes estimators for the reliability function of TPL distribution when the available data are in the form of fuzzy numbers. Consider the prior distributions of TPL distribution are taken to be independent gamma (a, b) and gamma (c, d), respectively with probability density functions:

$$P(\alpha) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}e^{-b\alpha};\ \alpha>0,\ a,\ b>0 \quad (14)$$

$$P(\theta) = \frac{d^c}{\Gamma(c)}\theta^{c-1}e^{-d\theta};\ \theta>0,\ c,\ d>0 \quad (15)$$

A joint prior distribution of α and θ of the form P (α, θ) = P (α) P (θ) will be:

$$P(\alpha,\theta) = \frac{b^a d^c}{\Gamma(a)\Gamma(c)}\alpha^{a-1}\theta^{c-1}e^{-(b\alpha+d\theta)} \quad (16)$$

The joint posterior density function of α and θ given fuzzy data can be obtained by combining likelihood Eq. 5 with Eq. 16 as:

$$(\alpha,\theta|\underline{\tilde{x}}) = \frac{L(\alpha,\theta|\underline{\tilde{x}})P(\alpha,\theta)}{\int_\theta\int_\alpha L(\alpha,\theta|\underline{\tilde{x}})P(\alpha,\theta)d\alpha d\theta} = \frac{P(\alpha,\theta|\underline{\tilde{x}})}{\int_\theta\int_\alpha P(\alpha,\theta|\underline{\tilde{x}})d\alpha d\theta} \quad (17)$$

Where:

$$P(\alpha,\theta|\underline{\tilde{x}}) = \frac{b^a d^c\alpha^{a-1}\theta^{2n+c-1}}{\Gamma(a)\Gamma(c)(\theta+\alpha)^n}e^{-(b\alpha+d\theta)}\prod_{i=1}^{n}\int(1+\alpha x)e^{-\theta x}\mu_{\tilde{x}}(x)dx$$

In our situation we have two parameters to be estimated, so, Bayes estimation of any function of the parameters, say u (α, θ), relative to squared error and precautionary loss functions, u_{BS} (α, θ) and u_{BP} (α, θ) can be obtained, respectively as:

$$\hat{u}_{BS}(\alpha,\theta) = E\left[u(\alpha,\theta)|\underline{\tilde{x}}\right] = \frac{\int_0^\infty\int_0^\infty u(\alpha,\theta)P(\alpha,\theta|\underline{\tilde{x}})d\alpha d\theta}{\int_0^\infty\int_0^\infty P(\alpha,\theta|\underline{\tilde{x}})d\alpha d\theta} \quad (18)$$

$$\hat{u}_{BP}(\alpha,\theta) = \left[E\left(u(\alpha,\theta)\,|\,\underline{\tilde{x}}\right)\right]^{\frac{1}{2}} = \left[\frac{\int_0^\infty\int_0^\infty u(\alpha,\theta)P(\alpha,\theta\,|\,\underline{\tilde{x}})d\alpha d\theta}{\int_0^\infty\int_0^\infty P(\alpha,\theta\,|\,\underline{\tilde{x}})d\alpha d\theta}\right]^{\frac{1}{2}} \tag{19}$$

Equation 18 and 19 are of the form of ratio of two integrals which cannot be simplified into a closed form. However, we use Lindley's approximation form to approximate these Bayes estimators. Lindley (1980) developed an approximate procedure for assessment of the ratio of two integrals. Consider $I(\tilde{x})$ defined as:

$$I(\underline{\tilde{x}}) = \frac{\int_0^\infty\int_0^\infty u(\alpha,\theta)e^{\varphi+\rho(\alpha,\theta)}d\alpha d\theta}{\int_0^\infty\int_0^\infty e^{\varphi+\rho(\alpha,\theta)}d\alpha d\theta} \tag{20}$$

where, u (α, θ) is a function of α and θ only, φ is the natural log-likelihood function defined by Singh *et al*. (2013), ρ(α, θ) is the natural log-joint prior density function. Then for sufficiently large sample size, the ratio of two integrals can be approximated as Singh *et al*. (2013):

$$\begin{aligned} I(\underline{\tilde{x}}) = {} & u(\hat{\alpha},\hat{\theta}) + \frac{1}{2}[(\hat{u}_{\alpha\alpha}+2\hat{u}_\alpha\hat{\rho}_\alpha)\hat{\sigma}_{\alpha\alpha} + (\hat{u}_{\alpha\theta}+2\hat{u}_\alpha\hat{\rho}_\theta)\hat{\sigma}_{\alpha\theta} + \\ & (\hat{u}_{\theta\alpha}+2\hat{u}_\theta\hat{\rho}_\alpha)\hat{\sigma}_{\theta\alpha} + (\hat{u}_{\theta\theta}+2\hat{u}_\theta\hat{\rho}_\theta)\hat{\sigma}_{\theta\theta}] + \frac{1}{2}[(\hat{u}_\theta\hat{\sigma}_{\alpha\theta} + \\ & \hat{u}_\alpha\hat{s}_{\alpha\alpha})(\hat{\varphi}_{\alpha\theta\theta}\hat{\sigma}_{\theta\theta} + \hat{\varphi}_{\theta\alpha\alpha}\hat{\sigma}_{\theta\alpha} + \hat{\varphi}_{\alpha\theta\alpha}\hat{\sigma}_{\alpha\theta} + \hat{\varphi}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}) + \\ & (\hat{u}_\theta\hat{\sigma}_{\theta\theta} + \hat{u}_\alpha\hat{\sigma}_{\theta\alpha})(\hat{\varphi}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta} + \hat{\varphi}_{\theta\alpha\theta}\hat{\sigma}_{\theta\alpha} + \hat{\varphi}_{\alpha\theta\theta}\hat{\sigma}_{\alpha\theta} + \hat{\varphi}_{\alpha\alpha\theta}\hat{\sigma}_{\alpha\alpha})] \end{aligned} \tag{21}$$

where, $\hat{\alpha}$ and $\hat{\theta}$ are the ML's of α and θ, respectively. σ_{ij} is the (I, j)[th] elements of matrix $[-\partial^2\varphi/\partial\alpha\partial\theta]^{-1}$ where sub-scripts (I, j) refer to α, θ, respectively. $\hat{\mu}_\alpha$ and $\hat{\mu}_\theta$ are the first derivative of the function u (α, θ) with respect to α and θ, respectively evaluated at $\hat{\alpha}$ and $\hat{\theta}$.$\hat{\mu}_{\alpha\alpha}$ is the second derivative of the function u (α, θ) with respect to α evaluated at $\hat{\alpha}$ and $\hat{\theta}$. Other expressions can be inferred exactly in similar style:

$$\hat{\rho}_\alpha = \left.\frac{\partial \ln P(\alpha,\theta)}{\partial\alpha}\right|_{\substack{\alpha=\hat{\alpha}\\ \theta=\hat{\theta}}} = \frac{a-1}{\hat{\alpha}} - b \tag{22}$$

$$\hat{\rho}_\theta = \left.\frac{\partial \ln P(\alpha,\theta)}{\partial\theta}\right|_{\substack{\alpha=\hat{\alpha}\\ \theta=\hat{\theta}}} = \frac{c-1}{\hat{\theta}} - d \tag{23}$$

And from Eq. 10-12, we can get:

$$\hat{\varphi}_{\alpha\alpha\alpha} = \left.\frac{\partial^3\varphi}{\partial\alpha^3}\right|_{\substack{\alpha=\hat{\alpha}\\ \theta=\hat{\theta}}} = -\frac{2n}{(\hat{\theta}+\hat{\alpha})^3} + 2\sum_{i=1}^{n}\left(\frac{\int x e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx}{\int(1+\hat{\alpha}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx}\right)^3 \tag{24}$$

$$\begin{aligned} \hat{\varphi}_{\alpha\alpha\theta} = {} & \left.\frac{\partial^3\varphi}{\partial\alpha\partial\alpha\partial\theta}\right|_{\substack{\alpha=\hat{\alpha}\\ \theta=\hat{\theta}}} = \hat{\varphi}_{\alpha\theta\alpha} = \left.\frac{\partial^3\varphi}{\partial\alpha\partial\theta\partial\alpha}\right|_{\substack{\alpha=\hat{\alpha}\\ \theta=\hat{\theta}}} = \hat{\varphi}_{\theta\alpha\alpha} = \left.\frac{\partial^3\varphi}{\partial\theta\partial\alpha\partial\alpha}\right|_{\substack{\alpha=\hat{\alpha}\\ \theta=\hat{\theta}}} = \\ & -\frac{2n}{(\hat{\theta}+\hat{\alpha})^3} + 2\sum_{i=1}^{n}\left[\frac{\int x e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx\int x^2 e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx}{\left(\int(1+\hat{\alpha}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx\right)^2} - \frac{\left(\int x e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx\right)^2\int x(1+\hat{\alpha}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx}{\left(\int(1+\hat{\alpha}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx\right)^3}\right] \end{aligned} \tag{25}$$

$$\begin{aligned} \hat{\varphi}_{\theta\theta\theta} = {} & \left.\frac{\partial^3\varphi}{\partial\theta^3}\right|_{\substack{\alpha=\hat{\alpha}\\ \theta=\hat{\theta}}} = \frac{4n}{\hat{\theta}^3} - \frac{2n}{(\hat{\theta}+\hat{\alpha})^3} - \sum_{i=1}^{n}\frac{\int x^3(1+\hat{\alpha}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx}{\int(1+\hat{\alpha}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx} + \\ & 3\sum_{i=1}^{n}\frac{\int x(1+\hat{\alpha}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx\int x^2(1+\hat{\alpha}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx}{\left(\int(1+\hat{\alpha}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx\right)^2} - \\ & 2\sum_{i=1}^{n}\left(\frac{\int x(1+\hat{\alpha}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx}{\int(1+\hat{\alpha}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx}\right)^3 \end{aligned} \tag{26}$$

$$\begin{aligned} \hat{\varphi}_{\theta\alpha\theta} = {} & \left.\frac{\partial^3\varphi}{\partial\theta\partial\alpha\partial\theta}\right|_{\substack{\alpha=\hat{\alpha}\\ \theta=\hat{\theta}}} = \hat{\varphi}_{\alpha\theta\theta} = \left.\frac{\partial^3 f}{\partial\alpha\partial\theta\partial\theta}\right|_{\substack{\alpha=\hat{\alpha}\\ \theta=\hat{\theta}}} = \\ & -\frac{2n}{(\hat{\theta}+\hat{\alpha})^3} + \sum_{i=1}^{n}\frac{\int x^3 e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx}{\int(1+\hat{\alpha}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx} - \\ & \sum_{i=1}^{n}\frac{\int x^2(1+\hat{a}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx\int x e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx}{\left(\int(1+\hat{\alpha}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx\right)^2} + \\ & 2\sum_{i=1}^{n}\left[\frac{\left(\int x(1+\hat{a}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx\right)^2\int x e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx}{\left(\int(1+\hat{a}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx\right)^3} - \frac{\int x(1+\hat{a}x)e^{\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx\int x^2 e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx}{\left(\int(1+\hat{a}x)e^{-\hat{\theta}x}\mu_{\tilde{x}_i}(x)dx\right)^2}\right] \end{aligned} \tag{27}$$

Now, according to the above defined expressions, we can obtain the approximate Bayes estimators of the reliability function of TPL distribution relative to squared errorloss function by assuming that (α, θ) = $R^{(t)}$ = θ+α+α θt/θ+α$e^{-\theta t}$ and then:

$$u_a = \frac{\theta^2 t}{(\theta+\alpha)^2} e^{-\theta t}, u_? = -\frac{\theta t\left(\theta+2\alpha+\alpha\theta t+\alpha^2 t\right)}{(\theta+\alpha)^2} e^{-\theta t},$$

$$u_{\alpha\alpha} = -\frac{2\theta^2 t}{(\theta+a)^3} e^{-\theta t}, u_{\theta\theta} =$$

$$\frac{\left(2\alpha^2\theta^2+\alpha?^3+a^3\theta\right)t^3+\left(\theta^3+3\alpha\theta^2+\alpha^2\theta-\alpha^3\right)t^2-2\alpha^2 t}{(\theta+\alpha)^3} e^{-\theta t}, u_{\alpha\theta} =$$

$$u_{\theta\alpha} = -\frac{\theta t\left(\theta^2 t+\alpha\theta t-2\varepsilon\right)}{(\theta+\alpha)^3} e^{-\theta t}$$

$$\hat{R}_{BS}(t) = E\left(R(t)\,|\,\underline{x}\right) = \frac{\hat{\theta}+\hat{\alpha}+\hat{\alpha}\hat{\theta}t}{\hat{\theta}+\hat{\alpha}} e^{-\hat{\theta}t} + \frac{1}{2}[\left(\hat{u}_{\theta\theta}+2\hat{u}_\theta\hat{\rho}_\theta\right)\hat{\sigma}_{\theta\theta} + \left(\hat{u}_{\alpha\theta}+2\hat{u}_\alpha\hat{\rho}_\theta\right)\hat{\sigma}_{\alpha\theta} + \left(\hat{u}_{\theta\alpha}+2\hat{u}_\theta\hat{\rho}_\alpha\right)\hat{\sigma}_{\theta\alpha} + \left(\hat{u}_{\alpha\alpha}+2\hat{u}_\alpha\hat{\rho}_\alpha\right)\hat{\sigma}_{\alpha\alpha} + \left(\hat{u}_\theta\hat{\sigma}_{\theta\theta}+\hat{u}_\alpha\hat{\sigma}_{\theta\alpha}\right)\left(\hat{\varphi}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta}+\hat{\varphi}_{\theta\alpha\theta}\hat{\sigma}_{\theta\alpha}+\hat{\varphi}_{\alpha\theta\theta}\hat{\sigma}_{\alpha\theta}+\hat{\varphi}_{\alpha\alpha\theta}\hat{\sigma}_{\alpha\alpha}\right) + \left(\hat{u}_\theta\hat{\sigma}_{\alpha\theta}+\hat{u}_\alpha\hat{\sigma}_{\alpha\alpha}\right)\left(\hat{\varphi}_{\alpha\theta\theta}\hat{\sigma}_{\theta\theta}+\hat{\varphi}_{\theta\alpha\alpha}\hat{\sigma}_{\theta\alpha}+\hat{\varphi}_{\alpha\theta\alpha}\hat{\sigma}_{\alpha\theta}+\hat{\varphi}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\right)] \quad (28)$$

and the approximate Bayes estimations for reliability function relative to precautionary loss function, would be as follows: Assume that u $(\alpha, \theta) = R^2(t) = \theta+\alpha+\alpha\ \theta t/\alpha+\theta\ e^{-\theta t})^2$ and then:

$$u_\alpha = \frac{2\theta^2 t(\theta+\alpha+\alpha\theta t)}{(\theta+\alpha)^3} e^{-2\theta t}, u_\theta =$$

$$-\frac{2\theta t(\theta+\alpha+\alpha\theta t)\left(\theta+2\alpha+\alpha\theta t+\alpha^2 t\right)}{(\theta+\alpha)^3} e^{-2\theta t}, u_{\alpha\alpha} =$$

$$-\frac{2\theta^2 t\left(2\theta+2\alpha+2\alpha\theta t-\theta^2 t\right)}{(\theta+\alpha)^4} e^{-2\theta t}, u_{\alpha\theta} = u_{\theta\alpha} \quad (29)$$

$$= \frac{2\theta t\left(2\alpha^2+2\alpha\theta+\alpha^2\theta t-2\theta^3 t-2\alpha\theta^3 t^2-4\alpha\theta^2 t-2\alpha^2\theta^2 t^2\right)}{(\theta+\alpha)^4}$$

$$e^{-2\theta t}\hat{R}_{BP}(t) = \left[E\left(R^2(t)\,|\,\tilde{x}\right)\right]^{\frac{1}{2}}$$

Where:

$$E\left(R^2(t)\,|\,\underline{x}\right) = \left(\frac{\hat{\theta}+\hat{\alpha}+\hat{\alpha}+\hat{\theta}t}{\hat{\theta}+\hat{\alpha}} e^{-\hat{\theta}t}\right)^2 + \frac{1}{2}[\left(\hat{u}_{\theta\theta}+2\hat{u}_\theta\hat{\rho}_\theta\right)\hat{\sigma}_{\theta\theta} + \left(\hat{u}_{\alpha\theta}+2\hat{u}_\alpha\hat{\rho}_\theta\right)\hat{\sigma}_{\alpha\theta} + \left(\hat{u}_{\theta\alpha}+2\hat{u}_\theta\hat{\rho}_\alpha\right)\hat{\sigma}_{\theta\alpha} + \left(\hat{u}_{\alpha\alpha}+2\hat{u}_\alpha\hat{\rho}_\alpha\right)\hat{\sigma}_{\alpha\alpha} + \left(\hat{u}_\theta\hat{\sigma}_{\theta\theta}+\hat{u}_\alpha\hat{\sigma}_{\theta\alpha}\right)\left(\hat{\varphi}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta}+\hat{\varphi}_{\theta\alpha\theta}\hat{\sigma}_{\theta\alpha}+\hat{\varphi}_{\alpha\theta\theta}\hat{\sigma}_{\alpha\theta}+\hat{\varphi}_{\alpha\alpha\theta}\hat{\sigma}_{\alpha\alpha}\right) + \left(\hat{u}_\theta\hat{\sigma}_{\alpha\theta}+\hat{u}_\alpha\hat{\sigma}_{\alpha\alpha}\right)\left(\hat{\varphi}_{\alpha\theta\theta}\hat{\sigma}_{\theta\theta}+\hat{\varphi}_{\theta\alpha\alpha}\hat{\sigma}_{\theta\alpha}+\hat{\varphi}_{\alpha\theta\alpha}\hat{\sigma}_{\alpha\theta}+\hat{\varphi}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\right)] \quad (30)$$

RESULTS AND DISCUSSION

Simulation study: In simulation study, the sample sizes are selected to be n = 20, 30 and 50 in order to represent small, moderate and large sample sizes, respectively. Set the true (default) values for the TPL distribution, $\alpha = 0.5$ and 5, $\theta = 1.5$ and 3. Select different values of hyper-parameters associated with gamma priors to deal with non-informative and informative priors, respectively as: prior 1: a = b = c = d = 0.0001 (use very small non-negative values of the hyper-parameters, it will make the informative priors as proper non-informative priors: (Press and Judith, 2001) and prior 2: = 3, b = 2, c = 2, d = 2. Select four times (t = 1, 2, 3, 4) to evaluate the estimating reliability function. Since, the explicit form of the inverse function of Lindley distribution cannot be obtained, the random samples of size n followed TPL distribution can be generated depending on the fact that Lindley distribution is a special mixture of exponential distribution with parameter (θ) and gamma distribution with parameters (2, θ) and mixing proportion $\theta/\theta+\alpha$. Generate u_i~ Uniform (0, 1), I = 1, 2, ..., n. Generate ν_i~ Exponemtial (θ), I = 1, 2, ..., n. Generate w_i~ Gamma (0, θ), I = 1, 2, ..., n. If $u_i \leq p = \theta/\theta+\alpha$, then set $x_i = \nu_i$ otherwise, set $x_i = w_i$. Then based on the fuzzy information system which appears in Fig. 1, encode the generated data where each observation in sample will be fuzzy based on a suitable selected membership function of the following eight membership functions:

$$\mu\tilde{x}(x) = \begin{cases} 1 & ; \quad x \leq 0.05, \\ \frac{0.25-x}{0.2} & ; 0.05 \leq x \leq 0.25, \\ 0 & ; \text{otherwise}, \end{cases}$$

$$\mu_{\tilde{x}_2}(x) = \begin{cases} \frac{x-0.05}{0.2} & ; 0.05 \leq x \leq 0.25, \\ \frac{0.5-x}{0.25} & ; 0.25 \leq x \leq 0.5, \\ 0 & ; \text{otherwise}, \end{cases}$$

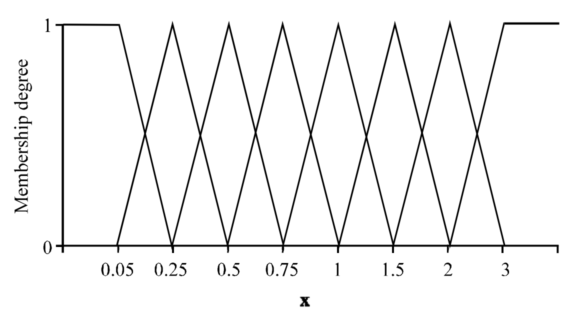

Fig. 1: Fuzzy information system used to encode the simulated data (Pak *et al*., 2013)

$$\mu_{\tilde{x}_3}(x) = \begin{cases} \dfrac{x-0.25}{0.25} & ; 0.25 \le x \le 0.5, \\ \dfrac{0.75-x}{0.25} & ; 0.5 \le x \le 0.75, \\ 0 & ; \text{otherwise}, \end{cases}$$

$$\mu_{\tilde{x}_4}(x) = \begin{cases} \dfrac{x-0.5}{0.25} & ; 0.5 \le x \le 0.75, \\ \dfrac{1-x}{0.25} & ; 0.75 \le x \le 1, \\ 0 & ; \text{otherwise}, \end{cases}$$

$$\mu_{\tilde{}}(x) = \begin{cases} \dfrac{\quad}{0.25} & ; 0.75 \quad x \quad 1, \\ \dfrac{1.5-x}{0.5} & ; 1 \le x \le 1.5, \\ 0 & ; \text{otherwise}, \end{cases} \quad \mu_{\tilde{}}(x) = \begin{cases} \dfrac{\quad}{0.5} & ; 1 \quad x \quad 1.5, \\ \dfrac{2-x}{0.5} & ; 1.5 \le x \le 2, \\ 0 & ; \text{otherwise}, \end{cases}$$

$$\mu_{\tilde{}}(x) = \begin{cases} \dfrac{\quad}{0.5} & \le \quad \le \\ 3-x & ; 2 \le x \le 3, \\ 0 & ; \text{otherwise}, \end{cases} \quad \mu_{\tilde{}}(x) = \begin{cases} x-2 & ; 2 \le x \le 3, \\ 1 & ; x \ge 3, \\ 0 & ; \text{otherwise} \end{cases}$$

Calculate the ML and Bayes estimates of the reliability function of TPL distribution based on the formulas that obtained in the previous studies. The iterative process of NR algorithm stops when the absolute difference between two successive iterations become $<\varepsilon = 0.0001$. Repeat the steps 1000 times and then compare the obtained estimates of reliability function with different times according to the average integrated mean square error (IMSE) as:

$$\text{IMSE}\left(\hat{R}(t)\right) = \frac{1}{L}\sum_{j=1}^{L}\left(\frac{1}{n_t}\sum_{i=1}^{nt}\left(\hat{R}_j(t_i) - R(t_i)\right)^2\right) \quad (31)$$

where, $R_j(t_i)$ is the estimates of R (t) at the j^{th} replicate and i^{th} time. L is the number of sample replicated chosen to be (1000). n_t is the number of times chosen to be Eq. 4. The results of simulation study have been summarized in Table 1-3 and Fig. 2-4.

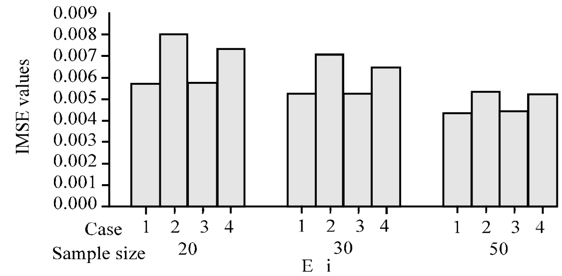

Fig. 2: IMSE values for ML estimators of R(t) with different cases

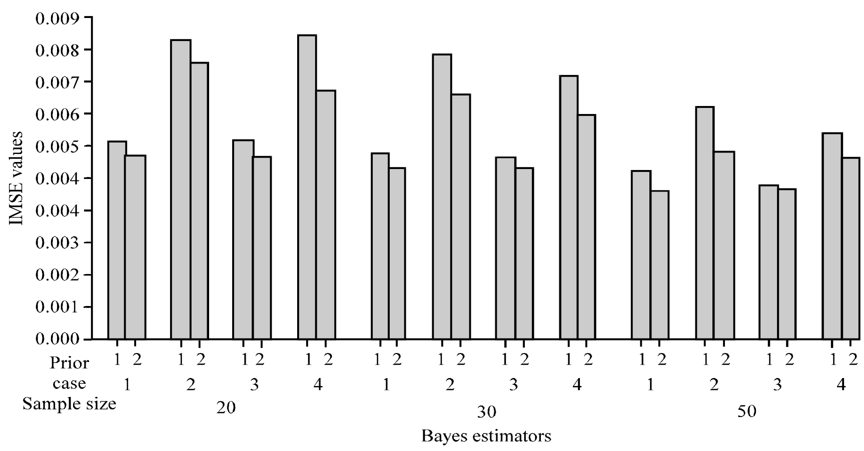

Fig. 3: IMSE values for Bayes estimators of R(t) under squared error loss function with different cases and prior distributions

Table 1: IMSE values for ML estimators of R (t)

	Cases			
N	$\alpha = 0.5, \theta = 1.5$	$\alpha = 0.5, \theta = 3$	$\alpha = 5, \theta = 1.5$	$\alpha = 5, \theta = 3$
20	0.0058073	0.0081462	0.0058708	0.0074983
30	0.0053200	0.0071459	0.0053213	0.0066047
50	0.0044161	0.0054454	0.0044941	0.0052971

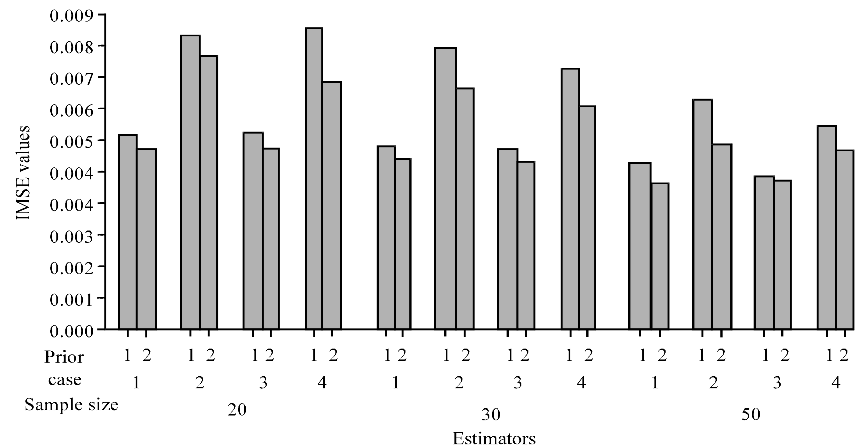

Fig. 4: IMSE values for Bayes estimators of R(t) under precautionary loss function with different cases and prior distributions

Table 2: IMSE values for Bayes estimators of R (t) under squared error loss function with different prior distributions

	Cases							
	$\alpha = 0.5, \theta = 1.5$		$\alpha = 0.5, \theta = 3$		$\alpha = 5, \theta = 1.5$		$\alpha = 5, \theta = 3$	
N	Prior 1	Prior 2	Prior 1	Prior 2	Prior 1	Prior 2	Prior 1	Prior 2
20	0.0057046	0.0052733	0.0086770	0.0079854	0.0057615	0.0053293	0.0087801	0.0072215
30	0.0053491	0.0049422	0.0082312	0.0070467	0.0052495	0.0049313	0.0076164	0.0064927
50	0.0048902	0.0042774	0.0066847	0.0053930	0.0044689	0.0043518	0.0059570	0.0052611
Best Prior	Prior 2		Prior 2		Prior 2		Prior 2	

Table 3: IMSE values for Bayes estimators of r (T) under precautionary loss function with different prior distributions

	Cases							
	$\alpha = 0.5, \theta = 1.5$		$\alpha = 0.5, \theta = 3$		$\alpha = 5, \theta = 1.5$		$\alpha = 5, \theta = 3$	
N	Prior 1	Prior 2	Prior 1	Prior 2	Prior 1	Prior 2	Prior 1	Prior 2
20	0.0056137	0.0051031	0.0082211	0.0078562	0.0056684	0.0052577	0.0081607	0.0071770
30	0.0051820	0.0048897	0.0081800	0.0070342	0.0051882	0.0049068	0.0073411	0.0064731
50	0.0043667	0.0040967	0.0058550	0.0053725	0.0044470	0.0042337	0.0057352	0.0052587
Best Prior	Prior 2		Prior 2		Prior 2		Prior 2	

CONCLUSION

The estimation methods adopted to estimate the reliability function of TPL distribution based on eight fuzzy linear membership functions are the maximum likelihood estimation and Bayesian estimation methodswhen prior distributions are specified as independent gamma distributions with squared error and precautionary loss functions as symmetric and asymmetric loss functions, respectively. The important conclusions from the empirical part with a Monte Carlo simulation study that performed to evaluate the behavior of obtained estimators for the reliability function of TPL distribution based on fuzzy data with different cases can be summarized by.

With different cases and all sample sizes under study, the performance of obtained Bayes estimators according to Lindley's approximation is better than that of maximum likelihood estimators.

With different cases and all sample sizes under study, the performance of Bayes estimates with informative priors assumption (prior 2) is better than that with non-informative priors assumption (prior 1).

The performance of Bayes estimates according to Lindley's approximation under precautionary loss function is better than that according to squared error loss function for different cases and all sample sizes under study.

Increase the value of the parameter θ increasing the values of integrated mean square error associated with all considered estimators for different cases and all sample sizes under study.

The integrated mean square error values associated with maximum likelihood and Bayes estimates are decreasing as the sample size increases. The performance of different estimators is almost identical with large sample size.

Hence, based on fuzzy data with such different cases and sample sizes under study, we recommended to use Bayes estimators according to Lindley's approximation under precautionary loss function with informative gamma prior to estimate the reliability function of TPL distribution comparing with squared error loss function.

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