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Reduced Differential Transform Method for Solving Fractional-Order Biological Systems

[1]Y.A. Amer, [1,2]A.M.S. Mahdy and [3]H.A.R. Namoos
[1]Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt
[2]Department of Mathematics, Faculty of Science, Taif University, Taif, Saudi Arabia
[3]Department of Mathematics, Faculty of Education Ibn-Al-Haitham,
Baghdad University, Baghdad, Iraq

Abstract: In this study, we want to find the approximate solutions for fractional-order biological systems by using Reduced Differential Transform Method (RDTM). The fractional derivatives are described in the Caputo sense. This method is easy to work out as it gives us very accurate solutions for solving fractional-order biological systems all the results obtained are excellent through our use of a program Maple 14.

Key words: Biological systems, Caputo derivative, reduced differential transform, fractional-order, results obtained, accurate solutions

INTRODUCTION

Mathematical models, using ordinary differential equations with integer order have been proven valuable in understanding the dynamics of biological systems. However, the behavior of most biological systems has memory or after effects. The modelling of these systems by fractional-order differential equations has more advantages than, classical integer-order mathematical modeling in which such effects are neglected. Accordingly, the subject of fractional calculus (i.e., calculus of integral and derivatives of arbitrary order) has gained popularity and importance, mainly due to its demonstrated appli-cations in numerous diverse and widespread-elds of science and engineering. For example, fractional calculus has been successfully applied to system biology (Ahmed *et al*., 2012; Arafa *et al*., 2012; Cole, 1933; El-Sayed *et al*., 2007; Xu, 2009). In some situations, the Fractional-Order Differential Equations (FODEs) models seem more consistent with the real phenomena than the integerorder models. This is due to the fact that fractional derivatives and integrals enable the description of the memory and hereditary properties inherent in various materials and processes. Hence, there is a growing need to study and use the fractional-order differential and integral equations. Ordinary and delay differential equations have long been used in modeling cancer phenomena (Bellomo *et al*., 2010; Gokdogan *et al*., 2011; Kirschner and Panetta, 1998; Rihan *et al*., 2012; Yafia, 2007) but fractionalorder differential equations have short history in modeling such systems with memory. The researchers by Ahmed *et al*. (2012) used a system of fractional-order differential equations in modeling cancerimmune system interaction.Themodel includes two immune effectors; E_1 (t), E_2 (t) (such as cytotoxic T cells and natural killer cells) interacting with the cancer cells, T (t) with a Holling function of type 3. (Holling type 3 describes a situation in which the number of prey consumed perpredator initially rises slowly as the density of prey increases but then, levels off with further increase in prey density. In other words, the response of predators to prey is depressed at low prey density, then, levels off with further increase in prey density). The model takes the form (Rihan, 2013):

$$\begin{cases} D^{\alpha}T = \alpha T - r_1 TE_1 - r_2 TE_2 \\ D^{\alpha}E_1 = -d_1E_1 + \dfrac{T^2E_1}{T^2+k_1} \quad 0<\alpha\leq 1 \\ D^{\alpha}E_2 = -d_2E_2 + \dfrac{T^2E_2}{T^2+k_2} \end{cases} \tag{1}$$

where, D^{α} is Caputo fractional derivative operator (El-Sayed and Salman, 2013; Agarwal *et al*., 2013; Elsadany and Matouk, 2015; El-Shahed *et al*., 2017) with $0<\alpha\leq 1$, T = T (t), $E_1 = E_1$ (t), $E_2 = E_2$ (t) and a, r_1, r_2, d_1, d_2, k_1 and k_2 are positive constants.

MATERIALS AND METHODS

Basic defnitions of fractional calculus: In this study, we present the basic defnitions and properties of the fractional calculus theory which are used further in this study.

Corresponding Author: Y.A. Amer, Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt

Definition 1: A real function f (z), z>0 is said to be in the space C_α, $\alpha\in\mathbb{R}$ if there exists a real number p>α such that f (z) = t^p f_1 (z) where f_1 (z)∈C [0, ∞) and it is said to be in the space C^m_α if $f^m \in C_\alpha$, m ∈ N

Definition 2: The Riemann-Liouville integral operator of order α>0 with a 0 is defined as (Rihan, 2013; El-Sayed and Salman, 2013; Agarwal *et al*., 2013; Elsadany and Matouk, 2015; Sabatier *et al*., 2007):

$$\left(J_a^\alpha\right)(z)=\frac{1}{\Gamma(\alpha)}\int_a^x (z-t)^{\alpha-1} f(t)dt, x>a \quad (2)$$

Definition 3: The Caputo fractional derivative operator D^α of order α is defined in the following form (Rihan, 2013; El-Sayed and Salman, 2013; Agarwal *et al*.m, 2013; Elsadany and Matouk, 2015):

$$D^\alpha f(z)=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f^{(m)}(\xi)}{(z-\xi)^{\alpha-m+1}}d\xi & 0\le m-1<\alpha<m\\ f^{(m)}(z) & \alpha=m\in N\end{cases} \quad (3)$$

Basic Idea of Reduced Differential Transform Method (RDTM): We consider a general fractional nonlinear differential equation of the form (Garg *et al*., 2011; Keskin and Oturanc, 2010; Keskin and Oturanc, 2009):

$$D_*^\alpha Z(t)+RZ(t)+NZ(t)=g(t) \quad (4)$$

with m-1<α≤m and subject to the initial condition:

$$Z^j(0)=c_j, \quad j=0,1,\dots,m\text{-}1 \quad (5)$$

Where:

$D_*^\alpha Z(t)$	=	The Caputo fractional derivative
g (t)	=	The source term
R	=	The linear operator
N	=	The general nonlinear operator

Let [t_0, T] be the interval over which we want to find the solution of the problem, the kth-order approximate solution of the problem can be expressed by the finite series:

$$Z_i(t)=\sum_{i=0}^{k} Z_i(k)(t-t_0)^{k\alpha i}, t\in[t_0,T] \quad (6)$$

where, Z_i (k) satisfied the recurrence relation:

$$\frac{\Gamma((k+1)\alpha_i+1)}{\Gamma(k\alpha_i+1)}Z_i(k+1)=F_i(k, Z_1, Z_2, \dots, Z_n)$$

By applying the Reduced Differential Transform Method (RDTM) on both sides of Eq. 4 we have:

$$\frac{\Gamma((k+1)\alpha_i+1)}{\Gamma(k\alpha_i+1)}Z_i(k+1)=G_k(Z)\text{-}NU_k(Z)\text{-}U_k(Z) \quad (7)$$

here, U_k (Z), G_k (Z)and Nu_k (Z) are the transformations of the functions RZ (t) g (t) and NZ (t), respectively. From the initial condition, we write:

$$Z_i(0)=c_j \quad (8)$$

Substituting Eq. 8 in to Eq. 7 and by straightforward iterative calculations, we get the following Z(k) values. Then, the inverse transformation of the set of values Z (k) $^n{}_k$ = 0 gives the approximation solution as:

$$Z(t)=\sum_{k=0}^{n} Z(k)t^{\alpha k} \quad (9)$$

where, n is order of approximate solution. Therefore, the exact solution of the problem is given by:

$$Z(t)=\lim_{y\to\infty}\sum_{k=0}^{n} Z(k)t^{\alpha k} \quad (10)$$

RESULTS AND DISCUSSION

Applications: To demonstrate the effectiveness of the proposed algorithm, one special cases of fractional-order biological systems (Table 1). All the results are calculated by using the symbolic calculus Software Mathematica.

Example: Let us consider fractional-order biological systems on homogenous networks is given by Rihan (2013):

$$\begin{cases} D^\alpha T=\alpha T-r_1TE_1\text{-}r_2TE_2 \\ D^\alpha E_1=-d_1E_1+\frac{T^2E_1}{T^2+k_1} & 0<\alpha\le 1 \\ D^\alpha E_2=-d_2E_2+\frac{T^2E_2}{T^2+k_2} \end{cases}$$

where, a = r_1 = r_2 = 1, d_1 = 0:3, d_2 = 0.7, k_1 = 0.3, k_2 = 0.7 and different 0<α1 ≤1 with tinitial condition:

$$T(0)=0.8, E_1(0)=0.1, E_2(0)=0.2 \quad (11)$$

First by applying Reduced Differential Transform Method (RDTM) on both sides of Eq. 1, thus, we get:

Table 1: Basic transformations of RDTM for some functions

Functional form	Transformed form
$u(x, t)$	$U_k(x) = 1/k!\ [\partial^k/\partial t^k u(x, t)]_{t=0}$
$w(x, t) = u(x, t) \pm u(x, t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x, t) = \alpha u(x, t)$	$W_k(x) = \alpha U_k(x)$, (α constant)
$w(x, t) = x^m t^n$	$x^m \delta(k - n)$, $\delta(k) = \{1\ k = 0,\ 0\ k \neq 0$
$w(x, t) = x^m t^n (x, t)$	$W_k(x) = x^m U_{k-n}(x)$
$w(x, t) = u(x, t)\, v(x, t)$	$W_k(x) = \sum_{r=0}^{k} U_r(x) V_{k-r}(x) = \sum_{r=0}^{k} V_r(x) U_{k-r}(x)$
$w(x, t) = \partial^r/\partial t^r u(x, t)$	$W_k(x) = (k + 1)(k + 2), \ldots, (k + r) U_{k-r}(x)$
$w(x, t) = \partial/\partial_x u(x, t)$	$W_k(k) = d/dx\, U_k(x)$
$w(x, t) = \partial^2/\partial t^2 u(x, t)$	$W_k(k) = \partial^2/\partial t^2\, U_k(x)$

$$\begin{cases} \frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} T(k+1) = \left[\alpha T(k) - r_1 \sum_{i=0}^{k} T(i) E_1(k-i) - r_2 \sum_{i=0}^{k} T(i) E_2(k-i)\right] \\ \frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} E_1(k+1) = \left[-d_1 E_1(k) + \frac{\sum_{j=0}^{k}\sum_{i=0}^{j} T(i) T(j-i) E_1(k-j)}{\sum_{i=0}^{k} T(i) T(k-i) k_1}\right] \\ \frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} E_2(k+1) = \left[-d_2 E_2(k) + \frac{\sum_{j=0}^{k}\sum_{i=0}^{j} T(i) T(j-i) E_2(k-j)}{\sum_{i=0}^{k} T(i) T(k-i) k_2}\right] \end{cases} \quad (12)$$

$$\begin{cases} T(k+1) = \frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} \left[\alpha T(k) - r_1 \sum_{i=0}^{k} T(i) E_1(k-i) - r_2 \sum_{i=0}^{k} T(i) E_2(k-i)\right] \\ E_1(k+1) = \frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} \left[-d_1 E_1(k) + \frac{\sum_{j=0}^{k}\sum_{i=0}^{j} T(i) T(j-i) E_1(k-j)}{\sum_{i=0}^{k} T(i) T(k-i) k_1}\right] \\ E_2(k+1) = \frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} \left[-d_2 E_2(k) + \frac{\sum_{j=0}^{k}\sum_{i=0}^{j} T(i) T(j-i) E_2(k-j)}{\sum_{i=0}^{k} T(i) T(k-i) k_2}\right] \end{cases} \quad (13)$$

By using the initial condition Eq. 11 we get the reduced transform form:

$$T(0) = 0.8, E_1(0) = 0.1, E_2(0) = 0.2$$

By substituting Eq. 11-13, we have:

$$\begin{cases} T(t) = \frac{1}{\Gamma(\alpha+1)} \left[\alpha T(0) - r_1 T(0) E_1(0) - r_2 T(0) E_2(0)\right] \\ E_1(1) = \frac{1}{\Gamma(\alpha+1)} \left[-d_1 E_1(0) + \frac{T^2(0) E_1(0)}{T^2(0) + k_1}\right] \\ E_2(k+1) = \frac{1}{\Gamma(\alpha+1)} \left[-d_2 E_2(0) + \frac{T^2(0) E_2(0)}{T^2(0) + k_2}\right] \end{cases} \quad (14)$$

$$T(1) = \frac{0.54}{\Gamma(\alpha+1)},$$

$$E_1(1) = \frac{0.0381}{\Gamma(\alpha+1)},$$

$$E_2(1) = \frac{-0.04448}{\Gamma(\alpha+1)}$$

By similarity we get:

$$T(2) = \frac{0.3971}{\Gamma(2+1)}$$

$$E_1(2) = \frac{-0.01143}{\Gamma(2\alpha+1)} + \frac{0.114 \times \Gamma(\alpha_1+1)}{\Gamma(2\alpha+1)[0.896+0.3\Gamma(\alpha+1)]}$$

$$E_2(2) = \frac{0.031136}{(2\alpha+1)} + \frac{0.1507 \times \Gamma(\alpha_1+1)}{\Gamma(2\alpha+1)[0.896+0.7\Gamma(\alpha+1)]}$$

In view of the differential inverse transform, the differential transform series solution for the system Eq. 13 can be obtained as:

$$T(t) = \sum_{n=0}^{N} T(n) t^{\alpha n},\ E_1(t) = \sum_{n=0}^{N} E_1(n) t^{\alpha n},\ E_2(t) = \sum_{n=0}^{N} E_2(n) t^{\alpha n}$$

We get the solution as series:

$$T(t) = 0.8 + \frac{0.54}{\Gamma(\alpha+1)} t^{\alpha} + \frac{0.3971}{\Gamma(2+1)} t^{2\alpha} + \cdots$$

$$E_1(t) = 0.1 + \frac{0.0381}{\Gamma(\alpha+1)} t^{\alpha} - \frac{0.01143}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{0.114 \times \Gamma(\alpha_1+1)}{\Gamma(2\alpha+1)[0.896+0.3\Gamma(\alpha+1)]} t^{2\alpha} +$$

$$E_2(t) = 0.2 - \frac{0.04448}{\Gamma(\alpha+1)} t^{\alpha} + \frac{0.031136}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{0.1507 \times \Gamma(\alpha 1+1)}{\Gamma(2\alpha+1)[0.896+0.7\Gamma(\alpha+1)]} t^{2\alpha} +$$

It is evident that the effciency of this approach can dramatically enhanced by com-puting further terms of $T(t)$, $E_1(t)$ and $E_2(t)$ when the Reduced differential transform method is used. The results in Fig. 1-3 show the the behavior of $T(t)$, $E_1(t)$ and $E_2(t)$ at $\alpha = 0{:}8$ and the results are in full agreement with the results obtained by Garg *et al*. (2011) using the reduced differential transform.

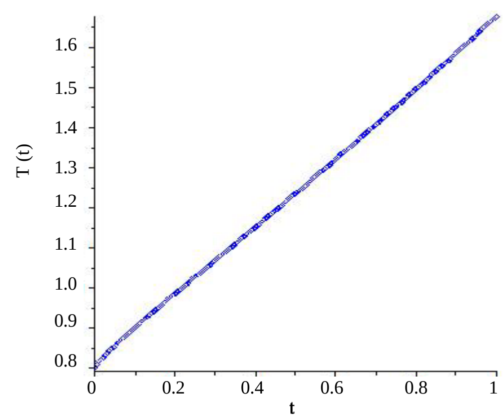

Fig.1: The behavior of approximate solution of $T(t)$ at $\alpha = 0{:}8$

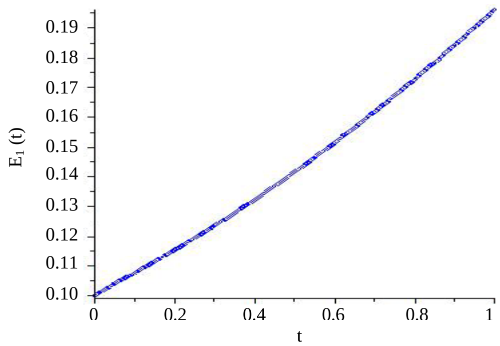

Fig. 2: The behavior of approximate solution of $E_1(t)$ at $\alpha = 0{:}8$

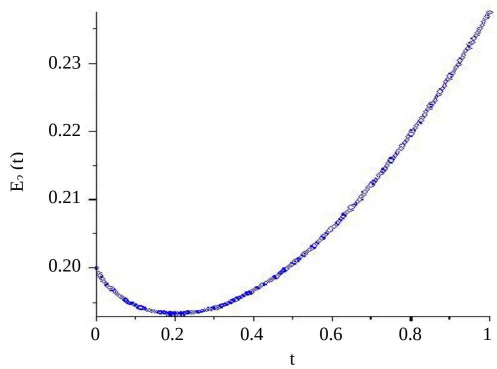

Fig. 3: The behavior of approximate solution of $E_2(t)$ at $\alpha = 0{:}8$

CONCLUSION

This present analysis exhibits the applicability of the Reduced Differential Transform Method (RDTM) to solve systems of fractional-order biological systems The research emphasized our belief that the method is a reliable technique to handle linear and non-linear fractional di¤erential equations. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, restrictive assumptions. The results of this method are in good agreement with those obtained by using the variational iteration method and the Adomian decomposition method. Generally speaking, the proposed method is promising and applicable to a broad class of linear and nonlinear problems in the theory of fractional calculus.

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