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Some Inequalities on Chance Measure for Uncertain Random Variables

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Abstract: In this study, some inequalities for uncertain random variables are first proved based on the concept of chance measure and expected value operator.

Key words: Uncertainty theory, probability theory, uncertain random variable, chance measure, inequality, concept

INTRODUCTION

Uncertainty theory, founded by Liu (2009a, b) and refined by Liu (2010) is a branch of mathematics based on the normality, duality, subadditivity and product measure axioms. In uncertainty theory, uncertain variable (Liu, 2008, 2009a, b) is one of the most important concepts which is defined as a measurable function from an uncertainty space to the set of real numbers. A sufficient and necessary condition of uncertainty distribution was proved by Guo and Iwamura (2011). After introduced the definition of independence by Liu (2010, 2011) presented the operational law of uncertain variable. Up to now, uncertainty theory has already applied to uncertain programming (Liu, 2009a, b, 2010), uncertain process (Yao, 2012).

Liu (2013) proposed chance theory by giving the concepts of uncertain random variable and chance measure in order to describe the situation that uncertainty and randomness appear in a system. In addition, the chance distribution, expected value and variance of an uncertain random variable were also provided. Following that, Liu (2013a, b) gave an operational law of uncertain random variables and proposed uncertain random programming as a branch of mathematical programming involving uncertain random variables. In addition, Yao and Gao (2012) verified a law of large numbers for uncertain random variables, etc.

Preliminary

Definition (2-1); (Liu, 2007): Let Ω be a nonempty set and let F be a σ algebra over Ω. Each element A∈F is called an event. Uncertain measure M was introduced as a set function satisfying the following axioms (Liu, 2011):

- Axiom 1 (Normality) M(Ω) = 1
- Axiom 2 (Monotonicity) M(A)≤M(B) whenever A⊂B
- Axiom 3 (Self-duality) $M(A)+M(A^c) = 1$ for any event
- Axiom 4 (countable subadditivity). For every countable sequence of events {A}, we have:

$$M\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} M(A_i)$$

Axiom 5 (product measure axiom). Let Ω_k be a nonempty sets on which M_k are uncertain measures k = 1, 2, 3, ..., n, respectively. Then the product measure M is an uncertain measure on the product σ algebra $F_1 \times F_2 \times, ..., \times F_n$ satisfying:

$$M\left(\prod_{k=1}^{n} A_k\right) = \min_{1\leq k\leq n} M_k(A_k)$$

where $A_k \in \Omega_k$, k = 1, 2, 3, ..., n.

Definition (2-2); Liu (2007): Let Ω be a nonempty set, let F be a σ algebra over and Ω an uncertain measure. Then the triple (Ω, F, M) is called an uncertainty space.

Definition (2-3); Liu (2007): An uncertain variable is a measurable function X from an uncertainty space (Ω, F, M) to the set of real numbers such that {X∈B} is an event for any set B of real numbers.

Definition (2-4); Liu (2007): The uncertain variables $X_1, X_2, X_3, ..., X_n$ are said to be independent if $M\left(\bigcap_{i=1}^{n}(X_n \in B_i)\right) = \min_{1\leq i\leq n} M_i(X_i \in B_i)$ for any Borel $B_1, B_2, ..., B_n$ of real numbers.

Definition (2-5); Liu (2007): Let X be an uncertain variable. Then the expected value of X is defined by: $E(X) = \int_0^{+\infty} M(X \geq r)dr - \int_{-\infty}^{0} M(X \leq r)dr$ provided that at least one of

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the two integrals is finite. The variance of X is defined by $E(X) = E(X-e)^2$ where X is the finite expected value of X Generally, the expected value $E((X))^k$ is called the k th absolute moment of the uncertain variable X for any positive integer k.

Theorem (2-6); Liu (2007): Let X be an uncertain variable with uncertainty distribution Φ. Then:

$$E(X) = \int_0^{+\infty} (1-\Phi(\alpha)da - \int_{-\infty}^{0} \Phi(\alpha)d\alpha$$

Definition (2-7); Liu (2007): Let (Ω, F, M) be an uncertainty space and (Γ, W, Pr) be a probability space. Then (Ω, F, M)×(Γ, W, Pr) = (Ω×Γ, F×W, M×Pr) is called a chance space.

Definition (2-8); Hou (2014): Let (Ω×Γ, F×W, M×Pr) is called a chance space and A∊F×W be an uncertain random event. Then the chance measure CH of A is defined by:

$$Ch(A) = \int_0^1 Pr(w \in \Gamma | M(y \in \Omega)|(y, w) \geq r)dr$$

Liu (2013a, b) verified that the chance measure Ch satisfies normality, duality and monotonicity properties, that is:

- Ch(Ω×Γ) = 1
- $Ch(A)+Ch(A^c) = 1$ for any event A
- Ch(A)≤Ch(B) for any event A and B with A⊂B

Besides, Hou (2014) proved the subaddivity of chance measure that is $Ch(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} Ch(A_i)$ for a sequence of events A_1, A_2.

Definition (2-9); Hou (2014): An uncertain random variable is a measurable function X from a chance space (Ω×Γ, F×W, M×Pr) to the set of real numbers such that {X∊B} is an uncertain random event for any Boreal set B.

Definition (2-10); Hou (2014): Let X be an uncertain random variable. Then, the expected value of X is defined by:

$$E(X) = \int_0^{+\infty} Ch(X \geq r)dr - \int_{-\infty}^{0} Ch(X \leq r)dr$$

provided that at least one of the two integrals is finite.

Definition (2-2); Hou (2014): Let X be an uncertain random variable. Then its chance distribution Φ is defined by Φ(x) = Ch(X≤x). For any x∊R.

Theorem (2-6); Hou (2014): Let X be an uncertain random variable with chance distribution Φ. Then:

$$E(X) = \int_0^{+\infty} (1-\Phi)(\alpha)d\alpha - \int_{-\infty}^{0} \Phi(\alpha)d\alpha$$

Theorem (2-4); Liu (2007): Let X be an uncertain random variable with chance distribution Φ. If the expected value exists, then:

$$E(X) = \int_0^1 \Phi^{-1}(\alpha)d\alpha$$

Proof: It follows from the definition of expected value operator and uncertainty distribution that:

$$E(X) = \int_0^{+\infty} Ch(X \geq r)dr - \int_{-\infty}^{0} Ch(X \leq r)dr =$$

$$\int_{\Phi(0)}^{1} \Phi^{-1}(\alpha)d\alpha + \int_0^{\Phi(0)} \Phi^{-1}(\alpha)d\alpha = \int_0^1 \Phi^{-1}(\alpha)d\alpha$$

The theorem is proved.

Some inequalities of uncertain random variables

Lemma(3-1) (rthinequality): Let be X_1, X_2, ..., X_n be uncertain random variables and r>0. Then:

$$E\left[\left|\sum_{i=1}^{n} X_i\right|\right]^r \leq n^r \sum_{i=1}^{n} E|X_i|^r$$

Proof: It is clear that for any number x:

$$\{X_1+X_2+, ..., +X_N \geq x\} \subset \{X_1 \geq \frac{x}{n}\} \cup$$

$$\{X_2 \geq \frac{x}{n}\} \cup, ..., \cup \{X_n \geq \frac{x}{n}\}$$

By the subadditivaty of chance measure, we have:

$$Ch\left[\sum_{i=1}^{n} X_i \geq x\right] \leq \sum_{i=1}^{n} Ch\left[X_i \geq \frac{x}{n}\right]$$

Furthermore, we obtain:

$$Ch\left[\left|\sum_{i=1}^{n} X_i\right| \geq x\right] \leq \sum_{i=1}^{n} Ch\left[|X_i| \geq \frac{x}{n}\right]$$

It follows from the definition of expected value for uncertain random variable:

$$\frac{n^2}{\varepsilon^2}\sum_{i=1}^{n} E|X_i|^2 = \frac{n^2}{\varepsilon^2}\sum_{i=1}^{n} E(X_i)^2$$

The theorem is proved.

Theorem (3-2) (Kolmogrov inequality): Let $X_1, X_2, ..., X_n$ be uncertain random variables and:

$$S_n = \sum_{i=1}^{n} X_i$$

If:

$$E\left[X_i^2\right] < \infty i = 1,\ 2, ...,$$

Then for any given number $\varepsilon>0$, we have:

$$Ch\{\max_{1\le i\le n}|S_i| \ge \varepsilon\} \le \frac{n^2}{\varepsilon^2}\sum_{i=1}^{n} E(X_i)^2$$

Proof: By Markov inequality, we have:

$$Ch\{\max_{1\le i\le n}|S_i| \ge \varepsilon\} \le \frac{E\left[\left|\max_{1\le i\le n}|S_i|\right|^2\right]}{\varepsilon^2}$$

Since:

$$\frac{E\left[\left|\max_{1\le i\le n}|S_i|\right|^2\right]}{\varepsilon^2} \le \frac{1}{\varepsilon^2} E\left|\sum_{i-1}^{n}|X_i|\right|^2$$

then, we have:

$$Ch\{\max_{1\le i\le n}|S_i| \ge \varepsilon\} \le \frac{E\left[\left|\max_{1\le i\le n}|S_i|\right|^2\right]}{\varepsilon^2} \le \frac{1}{\varepsilon^2} E\left|\sum_{i-1}^{n}|X_i|\right|^2$$

By rth inequality, we have:

$$\frac{1}{\varepsilon^2} E\left|\sum_{i=1}^{n}|X_i|\right|^2 \le \frac{n^2}{\varepsilon^2}\sum_{i=1}^{n} E|X_i|^2$$

Finally, since:

$$\frac{n^2}{\varepsilon^2}\sum_{i=1}^{n} E|X_i|^2 = \frac{n^2}{\varepsilon^2}\sum_{i=1}^{n} E(X_i)^2$$

Thus:

$$Ch\{\max_{1\le i\le n}|S_i| \ge \varepsilon\} \le \frac{n^2}{\varepsilon^2}\sum_{i=1}^{n} E(X_i)^2$$

Theorem (3-3): Let $X_1, X_2, ..., X_n$ be uncertain random variables and $S_n = \sum_{i=1}^{n} X_i$. If there exists a constant $k>0$ such that $|X|<$ fori = 1, 2, ..., n, then for any given number$\varepsilon>0$ we have:

$$Ch\{\max_{1\le i\le n}|S_i| \ge \varepsilon\} \le \frac{n^2}{\varepsilon^2}\left(\sum_{i=1}^{n} E|X_i - E(X_i)|^2 + k^2\right)$$

Proof: Since:

$$n^2\left(\sum_{i=1}^{n} E|X_i - E(X_i)|^2\right) \ge E\left|\sum_{i=1}^{n}|(X_i - E(X_i))|\right|^2$$

And since:

$$E\left|\sum_{i=1}^{n}|X_i - E(X_i)|\right|^2 = E\left[\sum_{i=1}^{n}|X - E(X_i)|\right]^2 \ge E\left[\sum_{i=1}^{n}(|X_i| - E(|X_i|)\right]^2$$

Then, we have:

$$n^2\left(\sum_{i=1}^{n} E|X_i - E(X_i)|^2\right) \ge E\left[\sum_{i=1}^{n}(|X_i| - E(|X_i|)\right]^2$$

Since, $|X_i|<k$, then $E(|X_i|) \ge -k$. Furthermore:

$$E\left[\sum_{i=1}^{n}(|X_i| - E(|X_i|)\right]^2 \ge E\left[\sum_{i=1}^{n}(|X_i| - k\right]^2$$

Since:

$$E\left[\sum_{i=1}^{n}(|X_i| - k)\right]^2 = E\left((\sum_{i=1}^{n}|x_i|)^2 - 2nk\sum_{i=1}^{n}|x_i| + n^2k^2\right)$$

Furthermore:

$$E\left((\sum_{i=1}^{n}|x_i|)^2 - 2nk\sum_{i=1}^{n}|x_i| + n^2k^2\right) =$$
$$E\left((\sum_{i=1}^{n}|x_i|)^2\right) - 2nkE\left(\sum_{i=1}^{n}|x_i|\right) + n^2k^2$$

So that:

$$E\left[\sum_{i=1}^{n}(|X_i| - k)\right]^2 = E\left((\sum_{i=1}^{n}|x_i|)^2 - 2nk\sum_{i=1}^{n}|x_i|\right) + n^2k^2$$

Thus:

$$n^2\left(\sum_{i=1}^{n} E|X_i - E(X_i)|^2\right) \ge E\left((\sum_{i=1}^{n}|x_i|)^2 - 2nk\sum_{i=1}^{n}|x_i|\right) + n^2k^2$$

Since:

$$|X_i| < k - 2nk\sum_{i=1}^{n}|X_i| \geq -2n^2k^2$$

That is:

$$E\left((\sum_{i=1}^{n}|x_i|)^2 - 2nk\sum_{i=1}^{n}|x_i|\right) + n^2k^2 \geq E\left(\sum_{i=1}^{n}|x_i|\right)^2 - 2n^2k^2 + n^2k^2$$

$$E\left((\sum_{i=1}^{n}|x_i|)^2\right) - 2n^2k^2 + n^2k^2 = E\left(\sum_{i=1}^{n}|X_i|\right)^2 - n^2k^2$$

$$\left(\sum_{i=1}^{n}E|X_i - E(X_i)|^2\right) \geq n^2(E\left(\sum_{i=1}^{n}|X_i|\right)^2 - n^2k^2)$$

By Markov inequality:

$$Ch\{\max_{1\leq i\leq n}|S_i| \geq \varepsilon\} \leq \frac{n^2}{\varepsilon^2}\sum_{i=1}^{n}E(X_i)^2$$

$$n^2\left(\sum_{i=1}^{n}E|X_i - E(X_i)|^2\right) \geq \varepsilon^2\, Ch\{\max_{1\leq i\leq n}|S_i| \geq \varepsilon\} - n^2k^2$$

$$n^2\left(\sum_{i=1}^{n}E|X_i - E(X_i)|^2\right) + n^2k^2 \geq \varepsilon^2 Ch\{\max_{1\leq j\leq n}|S_j| \geq \varepsilon\}$$

$$\varepsilon^2\, Ch\{\max_{1\leq j\leq n}|S_j| \geq \varepsilon\} \leq n^2\left(\sum_{i=1}^{n}E|X_i - E(X_i)|^2\right) + n^2k^2$$

$$Ch\{\max_{1\leq j\leq n}|S_j| \geq \varepsilon\} \leq \frac{n^2}{\varepsilon^2}\left(\sum_{i=1}^{n}E|X_i - E(X_i)|^2 + k^2\right)$$

The theorem is proved.

CONCLUSION

The main aim of this study is to provethe inequalities for uncertain random variables (rth and Kolmgorov inequality) in chance measure.

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