

## Types of Moduli of Smoothness: As Tools for Peoples Working in Approximation Theory

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**Abstract:** Moduli of smoothness are tools for peoples working in approximation theory real analysis, numerical analysis, differential equations, functional analysis and statistical estimation. Differentiating the functions many times to know the number of its derivative is a too ground method. In approximation theory, A sutable tool for measuring smoothness of function is the modulus of smoothness. In Ditzian and Totik the failure of the elassical moduli of smoothness to solve some problems in characterizing the behavior of the degree of best approximation, make Ditzian and Totik introduced a tool for measuring smoothness. Called Ditzian and Totik modulus of smoothness in our paper we introduce some versions of moduli of smoothness and then we show that they are equivalent to Ditzian-Totik (DT) modulus of smoothness of function in  $L_p[-1,1]$  spaces for  $0 < p \leq 1$ .

**Key words:** Smoothness, approximation, functional, statistical, modulus, equivalent

### INTRODUCTION

Let us start with a simple example. Suppose that  $A_p^{\Delta}$  is the space of of all functions in  $L_p[-1,1]$ ,  $0 < p \leq 1$ ,  $\tau_k(f, \delta(t, \cdot))_{q,p}$  is finite whenever  $q \leq p$ . How can we characterize this approximation space? The answer is very well known by now. There are several approaches but the ones that became most that popular in recent decades involve Ivanov moduli of smoothness  $\tau_k(f, \delta(t, \cdot))_{q,p}$  (introduced in 1980-1981) and Ditzian-Totik  $\omega_k^*(f, t)_p$  (introduced around 1984). The Ditzian-Totik (DT) modulus is defined in Eq. 2: by letting  $r = 0$  (Remark 1.1.1) and the Ivanov modulus for example, (Bhaya and Ahmed, 2018; Bhaya and Fadel, 2018; Bhaya, 2010; Bhaya and Al-Sammak, 2018; Bhaya and Anoon, 2018; Bhaya and Madlool, 2018; Kopotun *et al.*, 2015) is given by:

$$t_k(f, d(t, \cdot))_{q,p} = \|\omega_k(f, \dots, d(t, \dots))\|_{q,p}$$

Where:

$$\omega_k(f, x, \delta(t, x))_q^{\Delta} = ((1/2\delta(t, x)) \int_{-d(t, x)}^{d(t, x)} |\Delta_u^k(f, x)|^q dv)^{1/q},$$

$$0 < q < 1$$

It turns out (Ditzian, 2007)) that  $\omega_k^*(f, t)_p \sim \tau_k(f, \Delta(t, \cdot))_{p,p}$  with  $\Delta(t, \cdot) = t\varphi(\cdot)^{t+1}$  but in Ditzian (2007), “The [Ivanov] moduli, ... , are complicated method to describe smoothness than, ... , [DT moduli] with difficult computation”.

### MATERIALS AND METHODS

As alluded to above, we are interested in the constructive characterization of the functions in  $L_p[-1,1]$ ,  $0 < p \leq 1$  and  $c(-1,1)$  when  $p = 1$ . The first section are devoted to introducing the above mentioned DT moduli of smoothness in a new, equivalent form which is more transparent and simpler. We prove the equivalence via K-functionals. For  $p = 1$ , these moduli of smoothness were introduced by Theorem (1.1.4), however, no relations to weighted DT moduli were discussed. In the sequel, we will have constants  $C$  that may depend only on some of the parameters involved ( $p, k, r$ ) but are independent of the function and of as the case may be. The constants  $c$  may be different even if they appear in the same line Define:

$$L_p = \left\{ f : \left( \int_J |f(x)|^p dx \right)^{1/p} < \infty \right\}, 0 < p \leq 1 \text{ and}$$

$$\varphi(x) = \sqrt{1-x^2}$$

For,  $k \in \mathbb{N}_0$ ,  $h \geq 0$  an interval  $J$  and  $f: J \rightarrow \mathbb{R}$  let:

$$\Delta_h^k(f, x, J) = \left\{ \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} f \left( x + \left( i - \frac{k}{2} \right) h \right) \right\}, \text{ if } x \pm \frac{kh}{2} \in J$$

otherwise, be the  $k$ th symmetric difference and let  $\Delta_h^k(f, x) = \Delta_h^k(f, x, [-1,1])$ .  
 Let,  $0 < p \leq 1$  and  $r \in \mathbb{N}$ . Then for  $r \geq 1$ , let

$$\mathfrak{S}_p^r := \left\{ f \mid f^{(r-1)} \in AC_{loc}(-1,1) \text{ and } \|f^{(r)}\|_p < \infty \right\} \quad (1)$$

and set  $\mathfrak{S}_p^0 = L_p[-1,1]$

(Recall that  $AC_{loc}(-1, 1)$  denotes the set of functions which are locally absolutely continuous in  $(-1, 1)$ ):

For,  $f \in \mathfrak{S}_p^r$  define:

$$\omega_{k,r}^f(f^{(r)}, t)_p := \sup_{0 \leq h \leq t} \left\| W_{kh}^r(\cdot) \Delta_{h\varphi(\cdot)}^k(f^{(r)}, \cdot) \right\|_p \quad (2)$$

Where:

$$W_{\delta(x)} = \left( \left( 1-x-\frac{\delta\varphi(x)}{2} \right) \left( 1+x-\frac{\delta\varphi(x)}{2} \right) \right)^{1/2}$$

For  $\delta > 0$ , let:

$$D_\delta := \{x \mid |1-\frac{\delta\varphi(x)}{2}| \geq |x|\} \setminus \{\pm 1\}$$

$$= \left\{ x \mid |x| \leq \frac{4-\delta^2}{4+\delta^2} \right\} = [-1+\mu(\delta), 1-\mu(\delta)]$$

Where:

$$\mu(\delta) = \frac{2\delta^2}{4+\delta^2}$$

Observe that,  $D_\delta = \emptyset$  if  $\delta > 2$  and note that  $\Delta_{h\varphi(x)}^k(f, x)$  is defined to be identically 0 if  $x \in D_{kh}$  and that  $W_\delta$  is well defined on  $D_\delta$  (in fact if  $\delta \leq 2$  then  $\text{Dom}(W_\delta) = D_\delta \cup \{\pm 1\}$  (Kopotun *et al.*, 2015).

Hence:

$$\omega_{k,r}^\varphi(f^{(r)}, t)_p := \sup_{0 \leq h \leq t} \left\| W_{kh}^r(\cdot) \Delta_{h\varphi(\cdot)}^k(f^{(r)}, \cdot) \right\|_{L_p(D_{kh})}$$

And:

$$\omega_{k,r}^\varphi(f^{(r)}, t)_p := \omega_{k,r}^\varphi(f^{(r)}, \frac{2}{k})_p, \text{ for } t \geq \frac{2}{k} \quad (3)$$

**Remark 1.1.1:** When,  $r = 0$  in (1.1.3)  $\omega_k^f(f, t)_p$  reduces to the well known  $k$ th DT modulus of smoothness  $\omega_k^f(f, t)_p$  :

**Definition 1.1.2:** Let  $k \in \mathbb{N}$ ,  $r \in \mathbb{N}_0$  and  $f \in \mathfrak{S}_p^r$ ,  $0 < p \leq 1$ . Then, the averaged modulus of smoothness is defined as:

$$\omega_{k,r}^{*f}(f^{(r)}, t)_p = \left( \frac{1}{t} \int_0^t \int_{D_{kt}} W_{kt}^r(x) \Delta_{t\varphi(x)}^k(f^{(r)}, x)^p dx dt \right)^{1/p}$$

While, the modulus  $\omega_{k,r}^f(f, t)_p$  is obviously a non-decreasing function of  $t$ , the averaged modulus  $\omega_{k,r}^{*f}(f^{(r)}, t)_p$

does not have to be non-decreasing. At the same time, it immediately follows from definition 1.1.2 that:

$$\omega_{k,r}^{*f}(f^{(r)}, t_1)_p \leq \left( \frac{t_2}{t_1} \right)^{1/p} \omega_{k,r}^{*f}(f^{(r)}, t_2)_p, \text{ for } 0 < t_1 \leq t_2 \quad (4)$$

The above moduli of smoothness are equivalent to the following K-functional.

**Definition 1.1.3 (K-functional):** For  $k, r \in \mathbb{N}$ ,  $r \in \mathbb{N}_0$ ,  $0 < p \leq 1$  and  $f \in \mathfrak{S}_p^r$  :

$$K_{k,r}^\varphi(f^{(r)}, t^k)_p = \inf_{g \in B_p^{k+r}} \left( \| (f^{(r)} - g^{(r)}) \varphi^r \|_p + t^k \| g^{(k+r)} \varphi^{k+r} \|_p \right)$$

The following weighted DT moduli are defined in (Ditzian and Totik, 1987) (with  $D = (0,1)$ ). For any  $0 < p \leq 1$ :

$$\omega_\psi^k(f, t)_{\omega, p} = \sup_{0 < h \leq t} \left\| \omega \Delta_{h\psi}^k f \right\|_{L_p[t_0^*, 1-t_0^*]} + \sup_{0 < h \leq t_0^*} \left\| \omega \Delta_h^k f \right\|_{L_p[0, 12t_0^*]} + \sup_{0 < h \leq t_1^*} \left\| \omega \Delta_h^k f \right\|_{L_p[1-12t_1^*, 1]}$$

where, if  $\psi = \sqrt{x(1-x)}$ , then  $t_0^* = t_1^* = k^2 t^2$ . In Ditzian and Totik (1987) there is a result for such  $\psi$  and  $\omega \omega_\psi^k(f, t)_{\omega, p}$  is equivalent to the following weighted K-functional:

$$K_{k,\psi}(f, t^k)_{\omega, p} = \inf_{g^{(k-1)} \in AC_{loc}} \left( \| (f-g)\omega \|_{L_p(D)} + t^k \| (\omega \psi^k g^{(k)}) \|_{L_p(D)} \right)$$

With a simple modification we get:

$$\omega_\varphi^k(f, t)_{\varphi, p} = \sup_{0 < h \leq t} \left\| \varphi^r \Delta_{h\varphi}^k f \right\|_{L_p[-1+t^*, 1-t^*]} + \sup_{0 < h \leq t^*} \left\| \varphi^r \Delta_h^k f \right\|_{L_p[-1, -1+At^*]} + \sup_{0 < h \leq t_1^*} \left\| \varphi^r \Delta_h^k f \right\|_{L_p[1-12t_1^*, 1]} \quad (5)$$

where,  $t^* = 2k^2 t^2$  and  $A$  is an absolute constant and note that we can prove that K-functional defined in Definition 1.1.3, satisfies:

$$K_{k,r}^\varphi(f, t^k)_p = K_{k,\varphi}(f, t^k)_{\varphi, p} = \inf_{g^{(k-1)} \in AC_{loc}} \left( \| (f-g)\varphi^r \|_p + t^k \| (\varphi^{k+r} g^{(k)}) \|_p \right)$$

Theorem 6.1.1 in Ditzian and Totik (1987) implies:

$$M^{-1}\omega_f^k(f, t)_{\varphi, p} \leq K_{k,r}^\varphi(f, t^k)_p \leq M\omega_\varphi^k(f, t)_{\varphi, p} \quad (6)$$

$M > 1$  and  $0 < t \leq t_0$

A similar proof of 6.1.9 in Ditzian and Totik (1987), we have for  $t^* = 2k^2t^2$ :

$$\begin{aligned} \omega_\varphi^{*k}(f, t)_{\varphi, p} &= \left( \frac{1}{t} \int_0^t \int_{-1+t^*}^{1-t^*} |\omega(x) \Delta_{\varphi(x)}^k(f, x)|^p dx dt \right)^{1/p} + \\ &\left( \frac{1}{t^*} \int_0^{t^*} \int_{-1-A1^*}^{1-A1^*} |\omega(x) \Delta_u^k(f, x)|^p dx du \right)^{1/p} + \\ &\left( \frac{1}{t^*} \int_0^{t^*} \int_{1-A1^*}^1 |\omega(x) \Delta_u^k(f, x)|^p dx du \right)^{1/p} \end{aligned} \quad (7)$$

where,  $0 < p \leq 1$

As a proof of a result in Ditzian and Totik (1987), we have for  $t > 0$  that:

$$K_{k,r}^\varphi(f, t^k)_p \leq M_1 \omega_\varphi^{*k}(f, t)_{\varphi, p} \quad (8)$$

**Theorem 1.1.4:**

$$\begin{aligned} cK_{k,r}^\varphi(f^{(r)}, t^k)_p &\leq \omega_{k,r}^{*\varphi}(f^{(r)}, t)_p \leq \omega_{k,r}^\varphi(f^{(r)}, t)_p \leq \\ cK_{k,r}^\varphi(f^{(r)}, t^k)_p \end{aligned} \quad (9)$$

C is constant depends in k, r and p.

**Remark 1.1.5:** Note that with an additional restriction that  $t \leq t_0$  in the case  $r = 0$ , Theorem 1.1.4 becomes [3, Theorem 2.1.1 ] (with  $\varphi(x) = \sqrt{1-x^2}$ ) and that  $t_0$  can be taken to be  $(2k)^{-1}$  as was shown in [1, Theorem 6.6.2].

**RESULTS AND DISCUSSION**

**Auxiliary results:** In this section we shall introduce.

**Proposition:** (properties of  $W_\delta(x)$  and  $D_\delta$ ) (DeVore and Lorentz, 1993):

- $W_\delta(x) \leq \varphi(u)$ , for  $x \in D_\delta$  and  $u \in [-|x| - \delta\varphi(x)/2, |x| + \delta\varphi(x)/2]$
- $W_\delta(x) \leq \varphi(x)$ , for  $x \in D_\delta$
- $\varphi(x) \leq 2W_\delta(x)$ , for  $x \in D_{2\delta}$
- $|\delta\varphi'(x)| \leq 1$ , for  $x \in D_\delta$
- If  $y(u) = x + \delta_1\varphi(x)/2$  and  $|\delta_1| \leq \delta$ , then  $1/2 \leq y'(x) \leq 3/2$  for all  $x \in D_\delta$
- If  $\delta_1 > \delta_2$ , then  $D_{1\delta} \subset D_{2\delta}$

**Lemma 1.2.2:** For any  $0 < p \leq 1, r \in \mathbb{N}_0, \lambda \geq 0$  such that  $\lambda > r-1$ . If  $g \in \mathfrak{S}_p^{r+\lambda}$ , then:

$$\|\varphi^\lambda g^{(r)}\|_p < \infty$$

**Proof:** Suppose that when  $g \in \mathfrak{S}_p^{r+\lambda}$ . Since,  $\varphi(u) \geq \varphi(x)$  for  $|u| \leq |x|$ , we have:

$$\begin{aligned} \|\varphi^\lambda g^{(r)}\|_p^p &= \int_{-1}^1 |\varphi^{\lambda p}(x) \left| \int_{-1}^1 g^{(r+1)}(u) du \right|^p dx \leq \\ \int_{-1}^1 |\varphi^{\lambda p-r-1}(x)| \left( \int_0^x |\varphi^{r+1}(u)|^p du \right)^{p/p} dx &\leq \\ \|\varphi^{r+1} g^{(r+1)}\|_p^p \int_{-1}^1 |\varphi^{\lambda p-r-1}(x)| dx &\leq c_p \|\varphi^{r+1} g^{(r+1)}\|_p^p \leq \\ c_p \|\varphi^{r+1} g^{(r+1)}\|_p^p &< \infty \end{aligned}$$

As a direct consequence of above lemma we get the following corollary.

**Corollary 1.2.3:** For any  $0 < p \leq 1, r \in \mathbb{N}_0$  if  $g \in \mathfrak{S}_p^r$ .

Using a method as in (Kopotun *et al.*, 2015)), we can prove the following:

**Lemma 1.2.4:** For any  $k \in \mathbb{N}, 0 < p \leq 1, r \in \mathbb{N}_0$  and  $f \in \mathfrak{S}_p^r$ . Then

$$\begin{aligned} \omega_\varphi^{*k}(f^{(r)}, t)_{\varphi, p} &\leq c_{(\varphi)}(k, r) \omega_{k,r}^{*k}(f^{(r)}, c(k)t)_p, \\ 0 < t &\leq c(k) \end{aligned}$$

**Proof:** We estimate each of the three terms in Definition (1.1.7) separately. First, recall that  $t^* = 2k^2t^2$  and note that  $[-1+t^*, 1-t^*]_{D_{2kt}}$  and so using (iii) of Proposition 1.2.1 to get

$$\begin{aligned} &\frac{1}{t} \int_0^t \int_{-1+t^*}^{1-t^*} |\varphi^r(x) \Delta_{\varphi(x)}^k(f^{(r)}, x)|^p dx dt \leq \\ &\frac{2^{2p}}{t} \int_0^t \int_{-1+t^*}^{1-t^*} |W_{kt}^r(x) \Delta_{\varphi(x)}^k(f^{(r)}, x)|^p dx dt \leq \\ &\frac{2^{2p}}{t} \int_0^t \int_{D_{2kt}} |W_{kt}^r(x) \Delta_{\varphi(x)}^k(f^{(r)}, x)|^p dx dt \leq \\ &2^{2p} \omega_{k,r}^{*\varphi}(f^{(r)}, t)_p^p \end{aligned}$$

Let us now take the second term and the third term is the same If  $t \leq (2k\sqrt{A+K}/2)^{-1}$ , then using the fact:

$$\begin{aligned} \varphi(y-kh\varphi(y)/2) &\leq \sqrt{2}W_{kh}(y) \text{ if } 0 \leq h \leq \\ 2(y+1)/(k\varphi(y)) &\text{ and } y \leq -1/2 \end{aligned}$$

$$\begin{aligned} & \frac{1}{t^*} \int_0^{t^*} \int_{-1}^{-1+At^*} \left| \varphi^r(x) \Delta_u^k(f^{(t)}, x) \right|^p dx du = \\ & \frac{1}{t^*} \int_0^{t^*} \int_{-1}^{-1+At^*} \left| \varphi^r(x) \Delta_u^k(f^{(t)}, x+ku/2) \right|^p dx du \leq \\ & \frac{1}{t^*} \int_0^{t^*} \int_{-1+ku/2}^{-1+(A+k/2)t^*} \left| \varphi^r(y-ku/2) \Delta_u^k(f^{(t)}, y) \right|^p dy du \leq \\ & \frac{1}{t^*} \int_{-1}^{-1+(A+k/2)t^*} \int_0^{2(y+1)/k} \left| \varphi^r(y-ku/2) \Delta_u^k(f^{(t)}, y) \right|^p dy du = \\ & \frac{1}{t^*} \int_{-1}^{-1+(A+k/2)t^*} \int_0^{2(y+1)/(k\varphi(y))} \varphi(y) \left| \varphi^r(y-kh\varphi(y)/2) \Delta_{h\varphi(y)}^k(f^{(t)}, y) \right|^p dh dy \leq \\ & c \frac{1}{t^*} \int_{-1}^{-1+(A+k/2)t^*} \int_0^{2(y+1)/(k\varphi(y))} \varphi(y) \left| W_{kh}^r(y) \Delta_{h\varphi(y)}^k(f^{(t)}, y) \right|^p dh dy \leq \\ & c \frac{1}{\sqrt{t^*}} \int_{-1}^{-1+(A+k/2)t^*} \int_0^{2(y+1)/(k\varphi(y))} \left| W_{kh}^r(y) \Delta_{h\varphi(y)}^k(f^{(t)}, y) \right|^p dh dy \leq \\ & c \frac{1}{\sqrt{t^*}} \int_0^{c\sqrt{t^*}} \int_{D_{kh,r}[-1,-1+(A+k/2)t^*]} \left| W_{kh}^r(y) \Delta_{h\varphi(y)}^k(f^{(t)}, y) \right|^p dy dh \leq \\ & \omega_{k,r}^* \varphi(f^{(t)}, t)_p^c \end{aligned}$$

**Proof of Theorem 1.1.4:** We have:

$$\omega_{k,r}^\varphi(f^{(t)}, t)_p \leq c(k, r, p) K_{k,r}^\varphi(f^{(t)}, t^k)_p, \text{ for all } t > 0$$

Using (1.1.3) and since  $K$ -functional is monotone with respect to  $t$ , we may assume that  $t \leq 2/k$ . Take any  $g \in \mathfrak{S}_p^{st}$ . Corollary 1.2.3 implies that  $g \in \mathfrak{S}_p^{st}$ , so:

$$\omega_{k,r}^\varphi(f^{(t)}, t)_p \leq \omega_{k,r}^\varphi(f^{(t)}-g^{(t)}, t)_p + \omega_{k,r}^\varphi(g^{(t)}, t)_p$$

Let  $0 < h \leq t$ :

Let  $0 \leq i \leq k$ ,  $y_i(x) = x + (i-k/2)h\varphi(x)$ . Using (v) proposition 1.2.1 to get  $Y_i(x) \geq 1/2$  for  $x \in D_{kh}$  and so, we have:

$$\begin{aligned} & \left\| \varphi^r(y_i) (f^{(t)}(y_i) - g^{(t)}(y_i)) \right\|_{L_p(D_{kh})} = \\ & \left( \int_{D_{kh}} \varphi^p(y_i(x)) \left| (f^{(t)}(y_i(x)) - g^{(t)}(y_i(x))) \right|^p dx \right)^{1/p} \leq \\ & 2^{1/p} \left( \int_{-1}^1 \varphi^p(y) \left| (f^{(t)}(y) - g^{(t)}(y)) \right|^p dy \right)^{1/p} = \\ & 2^{1/p} \left\| \varphi^r(f^{(t)} - g^{(t)}) \right\|_p \end{aligned}$$

For the term  $\omega_{k,r}^\varphi(g^{(t)}, t)_p$  using the relation:

$$\Delta_h^k(f, x) = \int_{-h/2}^{h/2} \dots \int_{-h/2}^{h/2} f^{(k)}(x+u_1+\dots+u_k) du_1, \dots, du_k$$

We obtain:

$$\begin{aligned} \omega_{k,r}^\varphi(g^{(t)}, t)_p &= \sup_{0 < h \leq t} \left\| W_{kh}^r \Delta_{h\varphi}^k(g^{(t)}, \cdot) \right\|_{L_p(D_{kh})} = \\ & \sup_{0 < h \leq t} \left\| W_{kh}^r \int_{-h\varphi/2}^{h\varphi/2} \int_{-h\varphi/2}^{h\varphi/2} g^{(k+r)}(\cdot+u_1+\dots+u_k) du_1, \dots, du_k \right\|_{L_p(D_{kh})} \end{aligned}$$

For any  $u$  satisfying  $-1 < x+u-h\varphi(x)/2 < x+u+h\varphi(x)/2 < 1$  and using (ii) of Proposition 1.2.1, we get:

$$\begin{aligned} \omega_{k,r}^\varphi(g^{(t)}, t)_p &\leq \\ & c \frac{h\varphi}{2} \left| \int_{D_{kh}} \int_{-h\varphi(y)/2}^{h\varphi(y)/2} \dots \int_{-h\varphi(y)/2}^{h\varphi(y)/2} \varphi^r(x) g^{(k+r)}(x+u_2+\dots+u_k) du_1, \dots, du_k \right| dx \leq \\ & c \left( \frac{h\varphi}{2} \right)^2 \left| \int_{D_{kh}} \int_{-h\varphi(y)/2}^{h\varphi(y)/2} \dots \int_{-h\varphi(y)/2}^{h\varphi(y)/2} \varphi^r(x) g^{(k+r)}(x+u_3+\dots+u_k) du_1, \dots, du_k \right| dx \leq \\ & \int_{D_{kh}} c \left( \frac{h\varphi}{2} \right)^3 \left| \int_{-h\varphi(y)/2}^{h\varphi(y)/2} \dots \int_{-h\varphi(y)/2}^{h\varphi(y)/2} \varphi^r(x) g^{(k+r)}(x+u_4+\dots+u_k) du_1, \dots, du_k \right| dx \leq \\ & c \left( \frac{h\varphi}{2} \right)^k \left| \int_{D_{kh}} \int_{-h\varphi(y)/2}^{h\varphi(y)/2} \varphi^r(x) g^{(k+r)}(x+u_k) du_k \right| dx \leq c \left( \frac{h\varphi}{2} \right)^k \left\| \varphi^r g^{(k+r)} \right\|_p \end{aligned}$$

If  $0 < p \leq 1$ :

$$\begin{aligned} \int_{-hf/2}^{hf/2} &= \int_{-hf(y)/2}^{hf(y)/2} + \int_{-hf(y)/2}^{hf(y)/2} + \dots + \int_{-hf(y)/2}^{hf(y)/2} \\ I(p) &= I_1(p) \cup I_2(p) \cup I_3(p) \cup \dots, \end{aligned}$$

Hence:

$$\omega_{k,r}^\varphi(f^{(t)}, t)_p \leq c(k, r, p) K_{k,r}^\varphi(f^{(t)}, t^k)_p$$

Now show:

$$c K_{k,r}^\varphi(f^{(t)}, t^k)_p \leq \omega_{k,r}^* \varphi(f^{(t)}, t)_p,$$

Estimates (1.1.6) and (1.1.8) with Lemma 1.2.4 imply that, for  $f \in \mathfrak{S}_p^r$ ,  $0 < p \leq 1$ :

$$K_{k,r}^\varphi(f^{(t)}, t^k)_p \leq c \omega_{k,r}^* \varphi(f^{(t)}, c_1 t)_p, 0 < t \leq c_2,$$

where,  $c_1 = c_1(k)$  and  $c_2 = c_2(k)$  are positive constants

Suppose,  $0 < t \leq 2/k$  and let  $\mu = \max\{1, c_1, 2/(kc_2)\}$ . Then, since,  $t/\mu \leq c_2$  using (1.1.4), we get:

$$\begin{aligned} K_{k,r}^\varphi(f^{(t)}, t^k)_p &\leq \mu^k K_{k,r}^\varphi \left( f^{(t)}, \left( \frac{t}{\mu} \right)^k \right)_p \leq c \omega_{k,r}^* \varphi \left( f^{(t)}, c_1 \frac{t}{\mu} \right)_p \leq \\ & c \left( \frac{\mu}{c_1} \right)^{1/2} \omega_{k,r}^* \varphi(f^{(t)}, t)_p \end{aligned}$$

Thus:

$$c K_{k,r}^\varphi(f^{(t)}, t^k)_p \leq \omega_{k,r}^* \varphi(f^{(t)}, t)_p$$

### CONCLUSION

We can define new modulus of smoothness for functions in  $L_p[-1,1]$  quasinormed spaces for  $0 < p \leq 1$ . This modulus of smoothness equivalent to Ditzian-Totik modulus of smoothness.

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