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Solving a Large-Scale Nonlinear System of Monotone Equations by using a Projection Technique

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Abstract: In this study, we suggest a new projection algorithm for solving nonlinear systems of monotone equations. The projection methods are an efficient family of derivative free methods for solving nonlinear systems of monotone equation that is in each iteration, the current iterate is strictly separated from the solution set of the problem by an appropriate hyperplane that constructs by the new projection algorithm. Then the current iterate is projected onto this hyperplane to determine the new approximation. Under standard assumptions, the global convergence of the proposed algorithm are proved. The numerical experiments indicate the efficiency of the proposed algorithm.

Key words: Nonlinear system of equations, projection method, monotone strategy, global convergence, experiments, assumptions

INTRODUCTION

The nonlinear systems and their solutions are of great importance in the various sciences which have included different fields and aspects. It is an important part of the sciences of mathematical and physics (since, most physical systems are nonlinear) as well as their importance in engineering, especially, mechanical engineering, electricity, management, economy, population growth, weather and other natural phenomena. The nonlinear systems are studied alongside the linear system because of the possibility of converting nonlinear problems into linear ones from many variables. Consider the following nonlinear system of equations:

$$F(x) = 0 \tag{1}$$

where, F is a continuous and monotone function from R^n to R^n condition of monotony mean:

$$\left(F(x)-F(y)\right)^T (x-y) \geq 0, \forall x, y \in R^n \tag{2}$$

The solution of nonlinear system equations is one of the problem and difficulties in the mathematical and engineering applications that analyzing it by analytical methods is difficult. Therefore, we can only rely on iterative methods that use the iterative procedure to obtain approximate solutions. The Newton method may be one of the best numerical methods that use the iterative method to solve these systems and it is consider a smooth method to find approximate values of equations.

In recent years, many modifications have been made to the Newton method these suggested methods may be equivalent to (or better than) the Newton method to solve the nonlinear system of equations. The line search method and the trust region method are the most important two methods to solve these systems.

The important idea of the line search method is finding the step length in the specified direction but the trust region method always cares to find a neighborhood of the current step x_k, so that, the new iterate falls within the trust region determined by its radius (Amini *et al*., 2016) also this technique is used to solve unconstrained optimization (Shiker and Sahib, 2018).

Some methods proved ineffective for solving large-scale nonlinear system of equations as Newton method and quasi-Newton methods (Hager and Zhang, 2005; Li, 2017; Hassan and Shiker, 2018) because they need to solve the Jacobian matrix or an approximation of it in each iterative.

This research focuses on solving the system of large-scale non-linear equations by a new projection technique. The simple idea of the projection technique is always interested in separating the present approximation from the result set of the problem (Eq. 1) by appropriate hyperplane that is built in each iterate and then projecting this approximation on the same hyperplane to obtain the new approximation (Koorapetse *et al*., 2019). Several researchers use conjugate gradient approaches combining with projection techniques for solving (Eq. 1) as well as optimization issues (Dai, 2002).

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The first projection approach was suggested by Solodov and Svaiter (1998) and it showed the totally convergent of solving nonlinear problems. In this study, the new algorithm is used to solve the nonlinear systems, we proved its global convergence. Then we compare with two famous methods, SBM method by Yan *et al*. (2010) and DFPB1 by Ahookhosh *et al*. (2013), the new algorithm will be more efficient.

The framework: The projection technique is one of the ways that proved to be active in solving nonlinear problems and it is a suitable and applicable way to solve large-scale difficulties these methods use a series of repetitions to arrive to the next iterate:

$$x_{k+1} = x_k + \alpha_k d_k \tag{3}$$

Where:

α_k = A step length
d_k = The step direction

These processes are called an iterative procedures (Ortega and Rheinboldt, 1970), so, the projection techniques are called iterative methods. The projection approaches are family of derivative free. To define these effective methods, we use the projection operator $\Phi_\Omega[.]$. Let $\Phi_\Omega[.]$ be a mapping from R_n to Ω where Ω is non-empty closed convex set (Wang *et al*., 2003):

$$\Phi_\Omega[x] = \arg\min\left\{\|x-z\|, z \in \Omega\right\}, \forall x \in R^n \tag{4}$$

The projection operator has interesting features is non-expansive property:

$$\|\Phi_\Omega(x) - \Phi_\Omega(y)\| \le \|x-y\|, \forall x, y \in R^n \tag{5}$$

As a result produces:

$$\|\Phi_\Omega(x) - y\| \le \|x-y\|, \forall x, y \in \Omega \tag{6}$$

After a series of iterations, in every iteration, the present approximation x_k is isolated from the result set of the problem by the hyperplane H_k that is construction by using a line search technique:

$$H_k = \left\{x \in R^n / F(z_K)^T (x - z_k) = 0\right\} \tag{7}$$

where:

$$z_k = x_k + a_k d_k \tag{8}$$

By Solodov and Svaiter (1998) suggestion, the following iterate x_{k+1} can be resolute by projection z_k onto H_k where:

$$C_K = \left\{x \in R^n / F(z_k)^T (x_k - z_k) \le 0\right\} \tag{9}$$

The approximation that is best among all result of system (Eq. 1) can be determined by projection x_k onto C_K but $x_k \notin C_k$. Then the following approximation, x_{k+1} can be determined by:

$$x_{k+1} = P_C(x_k) = \arg\min\left\{\|x - x_k\| \mid x \in C_K\right\}$$

So:

$$x_{k+1} = x_k - \frac{F(Z_k)^T (X_k - Z_k)}{\|F(Z_k)\|^2} F(z_k) \tag{10}$$

The suggested method, built on the projection free-derivatives method for the system of nonlinear equations, determine a direction d_k, a new direction has foreword as:

$$d_k = \begin{cases} -F(x_k) & \text{if } k = 0 \\ -\mu F(x_k) + \tau_k & \text{otherwise} \end{cases} \tag{11}$$

Where:

$$\mu_k = \frac{s_k^T s_k}{y_k^T s_k}, \; s_k = x_{k+1} - x_k, \; y_k = F(x_{k+1}) - F(x_k)$$

With:

$$\tau_k = \frac{F(x_{k+1}) y_k}{\|F(x_k)\|^2}$$

Generally, used the direction d_k which satisfies:

$$F_k^T d_k \le -C\|F_k\|^2 \tag{12}$$

$$F(z_K)^T (x_k - z_k) > 0 \tag{13}$$

where, C is appositive constant (Ahookhosh *et al*., 2013). By Shiker and Amini (2018), introduced a new line search strategy for separating hyperplane in projection technique, encourage us to take advantages of this line search which needs $\alpha_k = \{\beta\theta^i : i = 0, 1, 2, ...\}$ satisfies the condition:

$$-F(x_k + \alpha_k d_k)^T d_k \ge \theta \lambda_k \alpha_k \|F(z_k)\| \tag{14}$$

where, $\lambda_k = \lambda/1+\|d_k\|^2$ and θ, λ are parameters. Our new algorithm will be state as below.

Algorithm 1 (NBM):

Input: An initial point $x_0 \varepsilon R^n$ and the parameters θ, λ, ε∈ (0.2) and βε (0.1).

Start
  Set k = 0
  $F_0 = F(x_0)$
    $d_0 = -F_0$
  While $\|F_K\|>\varepsilon$

Step 1: Compute $\|F_K\|$. If $\|F_K\|\leq\varepsilon$ stop
  Set $\alpha_k = \beta$;
    Find the minimum index $i_k\in\{1, 2, 3, ...\}$ such that
      $-F(x_k+\alpha_k d_k)^T d_k \geq \theta\lambda_k\alpha_k\|F(z_k)\|$
    where $\lambda_k = \lambda/1+\|d_k\|^2$
      While $\alpha_k = \theta^{ik}\alpha_k$
        Set $z_k \leftarrow x_k+\alpha_k d_k$
      End while

Step 2: If $\|F(z_k)\|\leq\varepsilon$, stop. Otherwise compute x_{k+1} by Eq. 10.

Step 3: Compute d_k by Eq. 11:
    $F_{K+1}\leftarrow F(x_{k+1})$;
    If $F_k^T d_k>-\varepsilon\|F_k\|^2$
      $d_k = -F_k$
    End if
      $k\leftarrow k+1$
  End while
End

Remark (R1): Shiker and Amini (2018) from stage 3 of algorithm 1, it is easy to note that the introduced direction satisfy the sufficient descent condition and for any k, $F_k^T d_k \leq -\varepsilon\|F_k\|^2$.

Convergence possessions: In this part, we need some interesting lemmas and assumptions in showing the global convergence of algorithm 1.

Assumption (B_1): The result set of (Eq. 1) is nonempty.

Assumption (B_2): The mapping F (x) is Lipschitz continuous on R^n such that there exists a positive constant M, i.e:

$$\|F(x)-F(y)\|\leq M\|x-y\|, \forall x, y\in R^n$$

Assumption (B_3): The mapping F (x) is monotone on R^n such that:

$$(F(x)-F(y))^T (x-y)\geq 0, \forall x, y\in R^n$$

Lemma (L1): Zarantonello (1971) let the set $\Omega\subseteq R^n$ be nonempty closed convex set and the projection operator $\Phi_\Omega(x)$ be the projection of x onto closed convex set Ω. For any x, $y\in R^n$, the next statements hold:

- $\forall i\in\Omega$, $\langle\Phi_\Omega(x)-x, z-\Phi_\Omega(x)\rangle\geq 0$
- $\langle\Phi_\Omega(x)-\Phi_\Omega(y), x-y\rangle\geq 0$ and the inquality is strict when $\Phi_\Omega(x)\neq\Phi_\Omega(y)$
- $\|\Phi_\Omega(x)-\Phi_\Omega(y)\|\leq\|x-y\|$

Lemma (L2): Solodov and Svaiter (1998) assume the assumption B_1, B_2 and B_3 hold and the sequence $\{x_k\}$ is generated via. algorithm1. For any x^* such that $F(x^*) = 0$ then:

$$\|x_{k+1}-x^*\|^2 \leq \|x_k-x^*\|^2-\|x_{k+1}-x_k\|^2 \quad (15)$$

And the sequence $\{x_k\}$ is bounded. Moreover, either the sequence $\{x_k\}$ is finite although, the last iterate is a solution of (Eq. 1) or the sequence $\{x_k\}$ is infinite and:

$$\lim_{k\to\infty}\|x_{k+1}-x_k\| = 0 \quad (16)$$

Proof: Let $x^*\in R^n$ be any point such that $F(x^*) = 0$ by monotonicity of $F\langle F(y), x^*-y\rangle\leq 0$. The hyperplane $H = \{s\in R^n/\langle F(y), s-y\rangle = 0\}$, separates x_k from x^*, it is easy to satisfy that x_{k+1} is the projection of x_k onto the hyperplane H. Sine x^* belongs to this hyperplane from properties of the projection operator (Zarantonello, 1971) we get:

$$\|x_k-x^*\|^2 = \|x_k-x_{k+1}\|^2+\|x_{k+1}-x^*\|^2+2\left\langle x_k-x_{k+1}, x_{k+1}-x^*\right\rangle \geq \left(\frac{\langle F(y), x_k-y\rangle}{\|F(y)\|}\right)^2+\|x_{k+1}-x^*\|^2$$

Lemma (L3): Solodov and Svaiter (1998) assume that the assumption B_1, B_2 and B_3 holds and the sequences $\{x_k\}$ and $\{z_k\}$ are generated by algorithm 1 then:

$$\alpha_k \geq \min\left\{\beta, \frac{\theta^c\|F_k\|^2}{\left(M\|d_k\|^2+\lambda\|F(z_k)\|\right)}\right\} \quad (17)$$

Proof: By the line search rule (Eq. 14), if $\alpha_k\neq\beta$ then $\hat{\alpha}_k = \theta^{-1}\alpha_k$ does not satisfy (Eq. 14) this mean that:

$$-F\left(x_k+\theta^{-1}\alpha_k d_k\right)^T d_k < \theta\lambda\theta^{-1}\alpha_k\gamma_k\|F(z_k)\| \leq \lambda\alpha_k\|F(z_k)\|$$

where, $\gamma_k = 1/1+\|d_k\|^2$. By the Lipchitz continuity of F and (Eq. 12) we get:

$$C\|F_k\|^2 \leq -F_k^T d_k = \left(F(z_k)-F(x_k)\right)^T d_k - F(z_k)^T d_k \leq \|F(z_k)-F(x_k)\|\,\|d_k\|+\lambda\alpha_k\|F(z_k)\| = \alpha_k\left(M\|d_k\|^2+\lambda\|F(z_k)\|\right)$$

So:

$$\alpha_k \geq \frac{\theta c\|F_k\|^2}{\left(M\|d_k\|^2+\lambda\|F(z_k)\|\right)}$$

The proof is complete and (Eq. 17) is correct. The results of Lemma L3 found that the line search of algorithm 1 is well defined.

Theorem (T1): Assume that B_2 and B_3 hold and the sequence $\{x_k\}$ is generated by algorithm 1 then:

$$\lim_{k\to\infty}\|F_k\| = 0 \tag{18}$$

Proof: From Eq. 10 and Eq. 14 we get:

$$\|x_{k+1}-x_k\|^2 \geq \frac{|F(z_k)^T(x_k-z_k)|}{\|F(z_k)\|} = \frac{-\alpha_k F(z_k)^T d_k}{\|F(z_k)\|} \geq \frac{\lambda\alpha_k^2\|F(z_k)\|}{(1+\|d_k\|^2)\|F(z_k)\|} = \frac{\lambda\alpha_k^2}{(1+\|d_k\|^2)} \tag{19}$$

By lemma 3 from Ahookhosh *et al*. (2013), the sequence of direction $\{d_k\}$ that generated by algorithm 1 are bounded there is a constant N>0 such that:

$$\|F(x_k)\| \leq N \tag{20}$$

And result that for all k there exists a constant L>0 such that:

$$\|d_k\| \leq L \tag{21}$$

By the Lipschitz continuity of F, it can be concluded that:

$$\|F(z_k)\| \leq \|F(z_k)-F(x_k)\|+\|F(x_k)\| \leq M(z_k-x_k)+N = M\alpha_k\|d_k\|+N \tag{22}$$

From (Eq. 19) together with (Eq. 21) gives:

$$\|x_{k+1}-x_k\|^2 \geq \frac{\lambda\alpha_k^2}{1+L^2}$$

So:

$$\lim_{k\to\infty}\|x_{k+1}-x_k\|^2 \geq \lim_{k\to\infty}\left(\frac{\lambda\alpha_k^2}{1+L^2}\right) \Rightarrow \lim_{k\to\infty}\alpha_k\|d_k\|0 \tag{23}$$

Now by using Cauchy Schwartz inequality along with (Eq. 12), we get:

$$C\|F_k\|^2 \leq -F_k^T d_k \leq \|F_k\|\,\|d_k\|$$

So:

$$\|d_k\| \geq C\|F_k\| \tag{24}$$

For all k. Giving to this condition and (Eq. 23), it follows that:

$$\lim_{k\to\infty}\alpha_k = 0 \tag{25}$$

On the other hand, multiplying (Eq. 17) by $\|d_k\|^2$ result that:

$$\alpha_k\|d_k\|^2 \geq \min\left\{\beta\|d_k\|^2, \frac{\theta CL^2\|F_k\|^2}{(ML^2+\lambda(M\alpha_k\|d_k\|+N))}\right\} \tag{26}$$

From (24) and (26) we have:

$$\lambda_0\|F_k\|^2 \leq \alpha_k\|d_k\|^2 \tag{27}$$

Where:

$$\lambda_0 = \min\left\{\beta c^2, \frac{\theta CL^2}{(ML^2+\lambda(M\alpha_k\|d_k\|+N))}\right\}$$

The relation (23) and (27) conclude that $\lim_{k\to\infty}\|F_k\| = 0$.

Numerical experiment: In this study, we compare the performance of the new algorithm (NBM) with tow famous algorithms:

SBM: This technique is taken from Yan *et al*. (2010) and it uses two modified HS approaches with the projection technique by Solodov and Svaiter (1998).

DFPB1: This technique is taken from Ahookhosh *et al*. (2013), it uses a three-term PRP-based conjugate gradient direction.

The performances of these approaches are compared with reference to the number of iterations N_i, the number of function evaluations N_f and CPU time. In order to compare these algorithms, some well-known test problems by Ahookhosh *et al*. (2013) and Yan *et al*. (2010) are used where the dimensions are confined between 5000-50000 for the taken primary points.

The tests were run on a PC with CPU 2.70 GHz and 4 GB RAM. All of the codes were written in MATLAB R2014 a programming environment. The running of the codes checks if the provided data for problems in all algorithms converges to the equal points. All of the algorithms terminate whenever $\|F_k\|\leq 10^{-8}$ or $\|F(z_k)\|\leq 10^{-4}$ or the whole number of iterates surpasses 500000. In all of the algorithms, the parameters are stated as follows $\theta = 0.4$, $\beta = 0.9$, $\lambda = 0.1$, $\epsilon = 10^{-8}$. The numerical results of consecutively the algorithms are registered in Table 1 and 2. Table 1 contains N_i and N_f while Table 2 contains the numerical results of CPU time.

From Table 1, we can see that the new approach NBM is better than the other two methods SBM and DFPB1 that it has a number of iterations and number of evaluation functions less than in the other methods in most of the problems with most of initial points. As well as the results in Table 2, we can see that the CPU time spent by the new technique NBM is lower than in the other two methods in most of problem that indicated the efficiency and quality of our new method.

Table 1: Numerical results (Ni and Nf)

		NBM		MHS		DFPB	
p-value/Dim	SP	Ni	Nf	Ni	Nf	Ni	Nf
P_1							
50000	x_1	11	91	1421	13288	1421	13288
50000	x_2	19	200	1421	13288	1421	13288
50000	x_3	13	110	142	880	142	880
50000	x_4	16	162	142	880	142	880
50000	x_5	18	281	5404	53012	9	21
50000	x_6	9	96	17	65	17	65
50000	x_7	25	188	4439	40363	88	504
50000	x_8	25	198	2252	19238	88	504
P_2							
50000	x_1	11	91	1421	13288	1421	13288
50000	x_2	18	228	1371	12933	1421	13288
50000	x_3	13	110	142	880	142	880
50000	x_4	19	256	155	1086	142	880
50000	x_5	80	1120	3859	36651	9	21
50000	x_6	9	96	17	65	17	65
50000	x_7	56	666	7274	72430	88	504
50000	x_8	49	498	581	3673	88	504
P_3							
10000	x_1	16975	213353	418081	3099340	178397	721899
10000	x_2	67640	981540	415836	3005275	187559	759768
10000	x_3	14647	189321	390297	2977500	162500	656715
10000	x_4	55909	839601	402816	2960816	178197	720729
10000	x_5	47471	729546	371496	2767595	165523	668916
10000	x_6	54499	855259	368053	2766114	162292	655726
10000	x_7	22901	321922	163175	1263922	68435	276645
10000	x_8	22313	312113	153214	1138115	68436	276649
P_4							
10000	x_1	24	255	20751	230201	469	4001
10000	x_2	370	5289	7709	73104	1494	14896
10000	x_3	18	183	2231	20120	155	1029
10000	x_4	365	5294	12747	135225	244	1713
10000	x_5	198	2896	27023	302729	85	395
10000	x_6	144	2092	11484	121948	76	314
10000	x_7	45	568	17308	191471	113	620
10000	x_8	27	291	2215	18879	113	620
P_5							
5000	x_1	494	5834	154735	2156312	75926	1027957
5000	x_2	379	4285	147988	2085617	75401	1019862
5000	x_3	278	3260	153015	2148172	75883	1027291
5000	x_4	341	3944	149814	2109510	341	3944
5000	x_5	346	3797	158114	2198306	75839	1026614
5000	x_6	333	3718	156045	2174637	75843	1026679
5000	x_7	384	4216	146366	2075659	75872	1027109
5000	x_8	301	3459	152515	2142193	75850	1026796
P_6							
50000	x_1	11	82	3353	28949	1009	8383
50000	x_2	18	210	8993	88244	1914	17404
50000	x_3	19	245	3932	35528	262	1813
50000	x_4	23	300	142	880	554	4288
50000	x_5	17	210	4450	40138	386	2805
50000	x_6	15	179	17	65	376	2725
50000	x_7	19	250	1615	12611	333	2381
50000	x_8	21	284	2057	16704	333	2381
P_7							
50000	x_1	5	12	18560	45738	1421	13288
50000	x_2	22	280	1675	15191	1421	13288
50000	x_3	13	110	17325	34836	142	880
50000	x_4	19	256	158	1099	142	880
50000	x_5	12	138	993	1988	9	21
50000	x_6	4	16	16926	33854	17	65
50000	x_7	151	1728	17089	34332	88	504
50000	x_8	45	430	17089	34332	88	504

Table 2: Numerical results (CPU time)

p-value/Dim	SP	NBM	MHS	FPB
P_1				
50000	x_1	0.15625	21.21875	20.62500
50000	x_2	0.26562	22.26562	20.90625
50000	x_3	0.140625	1.15625	1.06250
50000	x_4	0.15625	1.14062	1.00000
50000	x_5	0.32812	55.01562	0.01562
50000	x_6	0.14062	0.18750	0.10937
50000	x_7	0.26562	43.78125	0.60937
50000	x_8	0.25	20.68750	0.51562
P_2				
50000	x_1	0.12500	21.50000	20.79687
50000	x_2	0.28125	21.09375	20.93750
50000	x_3	0.10937	1.12500	1.00000
50000	x_4	0.31250	1.32812	1.04687
50000	x_5	1.10937	39.98437	0.10937
50000	x_6	0.14062	0.10937	0.12500
50000	x_7	0.62500	80.98437	0.53125
50000	x_8	0.54687	4.65625	0.68750
P_3				
10000	x_1	0.42221	6.36439	1.47290
10000	x_2	1.95898	6.28664	1.57473
10000	x_3	0.37418	6.14095	1.33835
10000	x_4	1.66826	6.05081	1.48701
10000	x_5	1.47228	5.68717	1.36782
10000	x_6	1.68854	5.71673	1.35753
10000	x_7	0.63462	2.60884	0.57164
10000	x_8	0.61637	2.35051	0.57096
p-value/Dim	SP	NBM	MHS	**DFPB**
P_4				
10000	x_1	0.09375	70.75000	1.20312
10000	x_2	1.34375	22.57812	4.18750
10000	x_3	0.03125	6.34375	0.31250
10000	x_4	1.375	41.62500	0.50000
10000	x_5	0.8125	92.67187	0.12500
10000	x_6	0.53125	37.10937	0.09375
10000	x_7	0.125	58.79687	0.15625
10000	x_8	0.0625	5.85937	0.18750
P_5				
5000	x_1	0.98437	3.34671	1.55750
5000	x_2	0.6875	3.17734	1.54468
5000	x_3	0.51562	3.32609	1.56765
5000	x_4	0.53125	3.32687	0.00593
5000	x_5	0.59375	3.40265	1.56156
5000	x_6	0.60937	3.35593	1.56953
5000	x_7	0.71875	3.16562	1.56218
5000	x_8	0.57812	3.2864	1.57656
P_6				
50000	x_1	0.21875	0.58468	16.76562
50000	x_2	0.4375	1.73859	32.28125
50000	x_3	0.53125	0.71421	3.25000
50000	x_4	0.60937	0.01062	8.23437
50000	x_5	0.48437	0.79390	5.18750
50000	x_6	0.35937	0.00109	5.43750
50000	x_7	0.5625	0.25359	4.46875
50000	x_8	0.67187	0.34156	4.34375
P_7				
50000	x_1	0.01562	92.42187	20.92187
50000	x_2	0.34375	22.23437	21.37500
50000	x_3	0.09375	74.35937	0.92187
50000	x_4	0.20312	1.15625	1.09375
50000	x_5	0.09375	4.14062	0.03125
50000	x_6	0.03125	73.79687	0.12500
50000	x_7	1.65625	74.93750	0.59375
50000	x_8	0.40625	73.34375	0.60937

CONCLUSION

The current research suggests a new projection technique for solving a system of large-scale nonlinear monotone equations. The projection-based algorithms belongs to the class of derivative-free function-value based approaches and it does not use any feature function and derivatives. Likewise, this method allows a simple globalization. The global convergence of the suggested algorithm is proved under standard assumptions. The numerical experiments indicated that the suggested algorithm is very efficient.

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