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Grobner Basis for Bivariate Normal with Missing Data Model Estimation Problem

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Abstract: The goal of this study is to study maximum likelihood estimates for a bivariate distribution with missing data using an algebraic geometry tool, namely, Grobner basis techniques. In maximum likelihood estimation, the parameters of the model are estimated by maximizing the likelihood function which maps the parameters to the likelihood of observing the given data. By transforming this optimization problem into a polynomial optimization problem, it can be shown that the solutions of the likelihood equations can be computed using Grobner basis technique.

Key words: Bivariate normal distribution, Buchberger's algorithm, Grobner bases, algebraic geometry, maximum likelihood estimation, s-polynomial

INTRODUCTION

Originally, the method of Grobner bases was introduced by Buchberger (1965, 1970) for the algorithmic solution of some of the fundamental problems in commutative algebra (polynomial ideal theory, algebraic geometry). A Grobner basis technique was first introduced by Bruno Buchberger in his PhD dissertation research (1965) (Buchberger, 1970). They are named after Buchberger's advisor, Wolfgang Groebner. Grobner basis technique is applied to solve systems of polynomial equations in several variables. In this study, we will use this technique to obtain the maximum likelihood estimation of the parameters in a bivariate normal distribution.

MATERIALS AND METHODS

Some definitions and theorems: We will assume that the reader is familiar with the definitions of the following: ring and field.

Definition 2.1 (Cox *et al*., 1991): Let N denote the non-negative integers. Let α $(\alpha_1, ..., \alpha_n)$ be a power vector in N^n and let $x_1, ..., x_n$ be any n variables. Then a monomial x^α in $x_1, ..., x_n$ is defined as the product $x^\alpha = x_1^{\alpha 1} ... x_n^{\alpha n}$. Moreover, the total degree of the monomial X^α is defined as $|\alpha| = \alpha_1 + ...+\alpha_n$.

Definition 2.2 (Cox *et al*., 1991): Let k be any field and let $f = \sum_\alpha a_\alpha x^\alpha$ be a polynomial in k $[x_1, ..., x_n]$:

- We call a_α the coefficient of the monomial x^α
- If $a_\alpha \neq 0$, then we call $a_\alpha x^\alpha$ term of f

The total degree of f denoted deg (f) is the maximum $|\alpha|$ such that the coefficient is a_α nonzero.

Definition 2.3 (Cox *et al*., 1991): Given a field k and a positive integer n, we define the n-dimensional affine space over k to be the set $k^n = \{(a_1, ..., a_n): a_1, ..., a_n \in k\}$.

Definition 2.4 (Pistone *et al*., 2000): Let be a subset of k^s. The set of polynomials defined by:

$$\text{Ideal}(S) = \left\{ \begin{matrix} f \in k[x_1,...,x_s] : f(a_1,...,a_s) = \\ 0 \text{ for all } (a_1,...,a_s) \in S \end{matrix} \right\}$$

Is an ideal called ideal of S. The variety generated by a polynomial ideal $I \subseteq k [x_1, ..., x_s]$ is:

$$\text{Variety}(I) = \left\{ \begin{matrix} (a_1, ..., a_s) \in K^s : \\ f(a_1, ..., a_s) = 0 \text{ for all } f \in I \end{matrix} \right\}$$

A subset of K^s which is a variety of a polynomial ideal in k $[x_1, ..., x_s]$ is called a variety.

Definition 2.5 (Cox *et al*., 1991): A subset $I \in k [x_1, ..., x_n]$ is an ideal if it satisfies:

- $0 \in I$
- If f, g $\in I$ then $f+g \in I$
- If f $\in I$ and h $\in k[x_1 ,..., x_n]$ then h f $\in I$

Definition 2.6 (Cox *et al*., 2005): Let $f_1, ..., f_s$ be polynomials in k $[x_1 ,..., x_n]$. We let $<f_1, ..., f_s>$ denote the collection $< f_1, ..., f_s> = \left\{ \sum_{i=1}^{s} h_i f_i : h_1, ..., h_s \in k[x_1, ..., x_n] \right\}$ is an ideal.

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Definition 2.7 (Adams and Loustaunau, 1994): A greatest common divisor of polynomials f, g∈k [x] is a polynomial h such that:

- h divides f and g
- If p ∈k [x] divides f and g then p divides h
- LC (h) = 1 (that is h is monic)

Definition 2.8 (Cox *et al*., 1991): Let α and β in N^n.

- Lexicographic order: $\alpha >_{lex} \beta$ if and only if the left-most nonzero entry in α-β is positive
- Graded lex order: $\alpha >_{grlex} \beta$ if and only if |α|>|β| or (|α| = |β| and $\alpha >_{lex} \beta$)
- Graded reverse lex order (tdeg): $\alpha >_{grevlex} \beta$ if and only if |α|>|β| or (|α| = |β| and the right-most nonzero entry in α-β is negative)

Definition 2.9 (Cox *et al*., 1991): Assume an arbitrary admissible ordering >is fixed. Given a nonzero polynomial f ∈k $[x_1, ..., x_n]$, we define:

- The multidegree of f as: multideg (f) = max (α ∈ N^n: $\alpha_a \neq 0$)
- The leading monomial of f as: LM (f) = $x^{\text{multideg (f)}}$
- The leading coefficient of f as: LC = (f) = $\alpha_{\text{multideg (f)}}$
- The leading term of f as: LT (f) = LCL (f). LM (f)

Theorem2.10:(Division Algorithm) (Cox *et al*., 1991): Fix a monomial order >on $\mathbb{Z}^n_{\geq 0}$ and let F = $(f_1, ..., f_s)$ be an ordered s-tuple of polynomials in k $[x_1, ..., x_n]$. Then every f ∈k $(x_1, ..., x_n)$ can he written as $f = a_1 f_1 + ... + a_s f_s + r$ where a_i, r ∈ k $[x_1, ..., x_s]$ and either r = 0 or r is a linear combination with coefficients in k of monomials, none of which is divisible by any LT (f_1), ..., LT (f_s). We will call r a remainder of f on division by F.

Definition 2.11 (Pistone *et al*., 2000): Let I ⊂ k $[x_1, ..., x_n]$ be an ideal other than{0}:

- We denote by LT (I) the set of leading terms of elements of I . Thus, LT (I) = {cx^α: there exists f ∈ I with LT (f) = cx^α}
- We denote by ⟨LT(I)⟩ the ideal generated by the elements LT (I)

Proposition 2.12 (Pistone *et al*., 2000): Let I ⊂ k $[x_1, ..., x_n]$ be an ideal:

- ⟨LT (I)⟩ is a monomial ideal
- There are $g_1, ..., g_s$ ∈I such that ⟨LT (I)⟩ = ⟨LT (g_1), ... , LT (g_s)⟩

Theorem 2.13: (Hilbert basis theorem) (Cox *et al*., 1991): Every ideal I ⊂ k $[x_1, ..., x_n]$ has a finite generating set. That is i = ⟨$g_1, ..., g_s$⟩ for some $g_1, ..., g_s$ ∈I.

Grobner basis: In this study we define the fundamental object of this study, namely, Grobner basis.

Definition 3.1 (Cox *et al*., 1991): Fix a monomial order. A finite subset G = {$g_1, ..., g_s$} of an ideal I is said to be a Groebner basis (or standard basis) if ⟨LT (g_1), ...,⟨LT (g_s)⟩ = ⟨LT (I)⟩. Equivalently, a set {$g_1, ... ,g_s$} ⊂ I is a Groebner basis of I if and only if the leading term of any element of I is divisible by one of the LT (g_i).

Corollary 3.2 (Adams and Loustaunau, 1994): Every non-zero ideal I ∈ k $[x_1, ..., x_n]$ has a Groebner basis.

Theorem 3.3 (Pistone *et al*., 2000): Let I be a non-zero ideal of k $[x_1, ..., x_n]$. The following statements are equivalent for a set of non-zero polynomials G = {$g_1, ..., g_s$} ⊂ I:

- G = {$g_1, ..., g_s$} is a Groebner basis for I
- f ∈I if and only if $\bar{f}^G = 0$ where $\bar{f}^G$ means the remainder on division of f by the ordered s-tuple G
- f ∈I if and only if $f = \sum_{i=1}^{s} h_i g_i$ with LT (f) = max (LT (hi), LT (gi))
- LT (G) = LT (I)

S-polynomials and Buchberger's algorithm: Before describing the Buchberger algorithm we define S-polynomials (S). In particular S-polynomials are used to test whether a set of polynomials is a Groebner basis.

Definition 4.1 (Cox *et al*., 1991): Let f and g be two polynomials in R. The S-polynomial of f and g is the following combination:

$$S(f,g) = \frac{L}{LT(f)}.f - \frac{L}{LT(g)}.g$$

where, L is the least common multiple. $L = LCM(LT(f), LT(g))$.

Theorem 4.2 (Cox *et al*., 1991): Let I be a polynomial ideal. Then a basis G = {$g_1, ..., g_s$} is a Groebner basis for I if and only if for all pairs i ≠ j the remainder on division of S (g_i, g_j) by G is zero.

Theorem 4.3: (Buchberger's algorithm) (Cox *et al*., 1991): Let I = ⟨$f_1, ..., f_s$ ≠ {0} be a polynomial ideal. Then a Groebner basis for I can be constructed in a finite number of steps by the following algorithm.

Input: F = $(f_1, ..., f_s)$

Output: A Groebner basis G = (g_1, ..., g_t) for I with F $\subset$ G G: = F

Repeat:
G': = G
FOR each pair {p, q}, p ≠ q in G' DO

$$S := \overline{S(p,q)}^{G'}$$

If S ≠ 0 then G:= G U {S}

Until G = G'

Definition4.4 (Adams and Loustaunau 1994): A Groebner basis G = {g_1, ..., g_s} is called minimal if for all i, LC (g_i) = 1and for all, i ≠ j, LP (g_i) does not divide LP (g_i).

Definition 4.5 (Adams and Loustaunau 1994): A Groebner basis G = {g_1, ..., g_s} is called a reduced Groebner basis if for all I, LC (g_i) = 1 and g_i is reduced with respect to G - {g_i}. That is for all, no non-zero term in g_i is divisible by any LP (g_j) for any i ≠ j.

Definition 4.6 (Cox *et al*., 1991): A minimal Groebner basis for a polynomial ideal I is a Groebner basis G for I such that:

- LC (p) = 1 for all p ∈G
- For all p∈ G LT (p) ∉ ⟨LT (G-{p})⟩

RESULTS AND DISCUSSION

Computation: maximum likelihood estimation and Grobner basis

Maximum likelihood estimates for a bivariate distribution with missing data: The Maximum Likelihood Estimators (MLE) are obtained for the parameters of a bivariate normal distribution with equal variances when some of the observations are missing on one of the variables.

Maximum likelihood estimates: Let us consider the incomplete bivariate sample:

$$\begin{matrix} x_1, ..., x_n, x_{n+1}, ..., x_N \\ y_1, ..., y_n \end{matrix}$$

From a bivariate normal distribution with mean vector (μ_1, μ_2) and a covariance matrix with common variance σ^2 and correlation coefficient ρ. It may be noted that (x_i, y_i), i = 1, ..., n are paired observations. The likelihood function can be written as:

$$L(\mu_1,\mu_2,\sigma,\rho) = \left(2\pi\sigma^2\right)^{-(N+n)/2}\left(1-\rho^2\right)^{-n/2} \times\exp\left(-\frac{\sum_{i=1}^{N}\left(x_i-\mu_1\right)^2}{2\sigma^2}-\frac{1}{2\sigma^2\left(1-\rho^2\right)}\sum_{i=1}^{N}\left\{y_i-\mu_2-\rho\left(x_i-\mu_1\right)\right\}^2\right) \quad (1)$$

The log-likelihood function is:

$$\log L = -(N+n)\log\sigma-\frac{n}{2}\log\left(1-\rho^2\right)-\frac{\sum_{i=1}^{N}\left(x_i-\mu_1\right)^2}{2\sigma^2}-\frac{1}{2\sigma^2\left(1-\rho^2\right)}\sum_{i=1}^{n}\left\{y_i-\mu_2-\rho\left(x_i-\mu_1\right)\right\}^2 \quad (2)$$

Take the partial derivatives of log L with respect to μ_1, μ_2, σ_2 and ρ equate them to zero and rearrange the equations, we get:

$$\sum_{i=1}^{N}x_i-N\mu_1-\rho^2\sum_{i=1}^{N}x_i+N\mu_1\rho^2-\rho\sum_{i=1}^{n}y_i+n\mu_2\rho+\rho^2\sum_{i=1}^{n}x_i-n\mu_1\rho^2=0 \quad (3)$$

$$\sum_{i=1}^{n}y_i-n\mu_2-\rho\sum_{i=1}^{n}x_i+n\mu_1\rho=0 \quad (4)$$

$$-(N+n)\sigma^2+(N+n)\sigma^2\rho^2+\sum_{i=1}^{N}x_i^2-2\mu_1\sum_{i=1}^{N}x_i+N\mu_1^2-\rho^2\sum_{i=1}^{N}x_i^2+2\mu_1\rho^2\sum_{i=1}^{N}x_i-N\mu_1^2\rho^2+\sum_{i=1}^{n}y_i^2-2\mu_2\sum_{i=1}^{n}y_i+n\mu_2^2-2\rho\sum_{i=1}^{n}x_iy_i+2\mu_2\rho\sum_{i=1}^{n}x_i+2\mu_1\rho\sum_{i=1}^{n}y_i-2n\rho\mu_1\mu_2+\rho^2\sum_{i=1}^{n}x_i^2-2\mu_1\rho^2\sum_{i=1}^{n}x_i+n\rho^2\mu_1^2=0 \quad (5)$$

$$n\rho\sigma^2-n\sigma^2\rho^3+\sum_{i=1}^{n}x_iy_i-\mu_1\sum_{i=1}^{n}y_i-\mu_2\sum_{i=1}^{n}x_i+n\mu_1\mu_2-\rho\sum_{i=1}^{n}y_i^2+2\mu_2\rho\sum_{i=1}^{n}y_i-n\mu_2^2\rho+\rho^2\sum_{i=1}^{n}x_iy_i-\mu_1\rho^2\sum_{i=1}^{n}y_i-\mu_2\rho_2\sum_{i=1}^{n}x_i+n\mu_1\mu_2\rho^2-\rho\sum_{i=1}^{n}x_i^2+2\mu_1\rho\sum_{i=1}^{n}x_i-n\mu_1^2\rho=0 \quad (6)$$

Using the data that are given in Table 1:

$$N=5, n=3, \sum_{i=1}^{N}x_i=15, \sum_{i=1}^{n}x_i=6, \sum_{i=1}^{n}y_i=9, \sum_{i=1}^{N}x_i^2=55, \sum_{i=1}^{n}x_i^2=14, \sum_{i=1}^{n}y_i^2=29 \text{ and } \sum_{i=1}^{n}x_iy_i=19$$

Table 1: Data

i	x_i	y_i	x_i^2	y_i^2	x_iy_i
1	3	3	9	4	9
2	1	2	1	16	2
3	2	4	4	9	8
4	5	-	25	-	-
5	4	-	16	-	-
Sum	15	9	55	29	19

The Eq. 2-5 become:

$$15\text{-}5\mu_1\text{-}9\rho^2+2\mu_1\rho^2\text{-}9\rho+3\mu_2\rho=0 \quad (7)$$

$$3\text{-}\mu_2\text{-}2\rho+\mu_1\rho=0 \quad (8)$$

$$\begin{aligned}&\text{-}8\sigma^2+8\sigma^2\rho^2+84\text{-}30\mu_1+5\mu_1^2\text{-}41\rho^2+18\mu_1\rho^2\text{-}2\mu_1^2\rho^2\text{-}\\&18\mu_2+3\mu_2^2\text{-}38\rho+112\mu_2\rho+18\mu_1\rho\text{-}6\rho\mu_1\mu_2=0\end{aligned} \quad (9)$$

$$\begin{aligned}&3\rho\sigma^2\text{-}3\sigma^2\rho^3+19\text{-}9\mu_1\text{-}6\mu_2+3\mu_1\mu_2\text{-}43\rho+18\mu_2\rho\text{-}3\mu_2^2\rho+\\&19\rho^2\text{-}9\mu_1\rho^2\text{-}6\mu_2\rho^2+3\mu_1\mu_2\rho^2+12\mu_1\rho\text{-}3\mu_1^2\rho=0\end{aligned} \quad (10)$$

Now we will apply Grobner basis method to solve this system of polynomials, let:

$$f_1=15\text{-}5\mu_1\text{-}9\rho^2+2\mu_1\rho^2\text{-}9\rho+3\mu_2\rho \quad (11)$$

$$f_2=3\text{-}\mu_2\text{-}2\rho+\mu_1\rho \quad (12)$$

$$\begin{aligned}f_3=&\text{-}8\sigma^2+8\sigma^2\rho^2+84\text{-}30\mu_1+5\mu_1^2\text{-}41\rho^2+18\mu_1\rho^2\text{-}\\&2\mu_1^2\rho^2\,18\mu_2+3\mu_2^2\text{-}38\rho+12\mu_2\rho+18\mu_1\rho\text{-}6\rho\mu_1\mu_2\end{aligned} \quad (13)$$

$$\begin{aligned}f_4=&3\rho\sigma^2\text{-}3\sigma^2\rho^3+19\text{-}9\mu_1\text{-}6\mu_2+3\mu_1\mu_2\text{-}43\rho+18\mu_2\\&\rho\text{-}3\mu_2^2\rho+19\rho^2\text{-}9\mu_1\rho^2\text{-}6\mu_2\rho^2+3\mu_1\mu_2\rho^2+12\mu_1\rho\text{-}3\mu_1^2\rho\end{aligned} \quad (14)$$

To solve these equations we consider the ideal $F=\langle f_1, f_2, f_3, f_4\rangle$ and choose the lexicographic order where $\mu_1>_{lex}\mu_2>_{lex}\sigma>_{lex}\rho$. We will finding S-polynomial of f_1 and f_2. Since:

$$LT(f_1)=2\mu_1\rho^2 \text{ and } LT(f_2)=\mu_1\rho$$

Then:

$$\begin{aligned}&L=LCM(LT(f_1),LT(f_2))=\\&LCM(2\mu_1\rho^2,\mu_1\rho)=2\mu_1\rho^2\end{aligned}$$

$$S(f_1,f_2)=\frac{L}{LT(f_1)}.f_1\text{-}\frac{L}{LT(f_2)}.f_2$$

$$\begin{aligned}&S(f_1,f_2)=\frac{2\mu_1\rho^2}{2\mu_1\rho^2}\begin{pmatrix}15\text{-}15\mu_1\text{-}9\rho^2+2\mu_1\rho^2\text{-}\\9\rho+3\mu_2\rho\end{pmatrix}\text{-}\frac{2\mu_1\rho^2}{\mu_1\rho}\\&(3\text{-}\mu_2\text{-}2\rho+\mu_1\rho)=15\text{-}5\mu_1\text{-}5\rho^2\text{-}15\rho+5\mu_2\rho\end{aligned}$$

And:

$$\overline{S(f_1,f_2)}^F=15\text{-}15\mu_1\text{-}5\rho^2\text{-}15\rho+5\mu_2\rho\neq 0$$

Therefore, we must add:

$$f_5=15\text{-}15\mu_1\text{-}5\mu_1\text{-}5\rho^2\text{-}15\rho+5\mu_2\rho \quad (15)$$

To the generating set. The ideal becomes:

$$F=\langle f_1,f_2,f_3,f_4,f_5\rangle$$

And:

$$\overline{S(f_1,f_2)}^F=0$$

Similarly:

$$\begin{aligned}S(f_1,f_3)=&3\mu_1\mu_2\rho+8\sigma^2\text{-}8\sigma^2\rho^2\text{-}84+15\mu_1+\\&41\rho^2\text{-}9\mu_1\rho^2+18\mu_2\text{-}3\mu_2^2+38\rho\text{-}12\mu_2\rho\text{-}9\mu_1\rho\end{aligned}$$

$$\overline{S(f_1,f_3)}^F=\text{-}12\text{-}8\sigma^2\rho^2+2\rho+8\rho^2+8\sigma^2\neq 0$$

Therefore, we must add:

$$f_6=\text{-}12\text{-}8\sigma^2\rho^2+2\rho+8\rho^2+8\sigma^2 \quad (16)$$

To the generating set. The ideal becomes:

$$F=\langle f_1,f_2,f_3,f_4,f_5,f_6\rangle \text{and} \overline{S(f_1,f_2)}^F=0$$

Similarly:

$$\begin{aligned}S(f_1,f_4)=&\text{-}45\sigma^2\rho+27\sigma^2\rho^3+9\sigma^2\mu_1\text{-}9\sigma^2\rho^2\mu_2+\\&27\sigma^2\rho^2\text{-}38\mu_1^2+12\,\mu_1\mu_2\text{-}6\mu_1^2\mu_2+86\,\mu_1\rho\text{-}24\mu_1^2\rho+\\&6\mu_1^3\rho\text{-}36\mu_1\mu_2\rho+18\mu_1^2\rho^2+6\mu_1\mu_2^2\rho+12\mu_1\mu_2\rho^2\text{-}\\&6\mu_1^2\mu_2\rho^2\text{-}38\mu_1\rho^2\end{aligned}$$

And:

$$\begin{aligned}\overline{S(f_1,f_4)}^F=&\text{-}237+353\mu_2\text{-}184\mu_1\text{-}135\mu_2^2+\\&60\mu_1\mu_2+45\mu_1^2\text{-}15\rho^3+15\mu_2^3\text{-}15\mu_1^2\mu_2\neq 0\end{aligned}$$

Therefore, we must add:

$$\begin{aligned}f_7=&\text{-}237+353\mu_2\text{-}184\mu_1\text{-}135\mu_2^2+\\&60\mu_1\mu_2+45\mu_1^2\text{-}15\rho^3+15\mu_2^3\text{-}15\mu_1^2\mu_2\end{aligned} \quad (17)$$

To the generating set. The idealo becomes:

$$F=\langle f_1,f_2,f_3,f_4,f_5,f_6,f_7\rangle \text{ and } \overline{S(f_1,f_4)}^F=0$$

Similarly:

$$\begin{aligned}S(f_1,f_5)=&75\mu_2\text{-}45\mu_2\rho^2\text{-}45\mu_2\rho^2\text{-}25\mu_1\mu_2+15\mu_2^2\rho\text{-}\\&45\mu_2\rho\text{-}30\mu_1\rho+10\mu_1\rho^3+10\mu_1^2\rho+30\mu_1\rho^2\left(\overline{S(f_1,f_5)}^F=0\right.\end{aligned}$$

And similarly:

$$\begin{aligned}S(f_1,f_6)=&\text{-}60\sigma^2+36\sigma^2\rho^2+12\sigma^2\mu_1\text{-}12\sigma^2\rho\mu_2+\\&36\sigma^2\rho+12\mu_1\text{-}12\mu_1\rho\text{-}8\mu_1\rho^3\end{aligned}$$

And:

$$\overline{S(f_1,f_6)}^F = -6+2\rho+4\mu_1+2\mu_2 \neq 0$$

Therefore, we must add:

$$f_8 = -6+2\rho+4\mu_1+2\mu_2 \quad (18)$$

To the generating set. The ideal becomes:

$$F = (f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8) \text{ and } \overline{S(f_1,f_6)}^F = 0$$

Similarly:

$$S(f_1\text{-}f_7) = -225\mu_1\mu_2+15\mu_1\mu_2\rho^2+75\mu_1^2\mu_2- 45\mu_2^2\mu_1\rho+135\mu_1\mu_2\rho+474\rho^2-14\rho^3-706\mu_2\rho^2+ 368\mu_1\rho^2+270\mu_2^2\rho^2-90\mu_1^2\rho^2+30\rho^5-30\mu_2^3\rho^2$$

And:

$$\overline{S(f_1,f_7)}^F = -54-18\rho+18\mu_2 \neq 0$$

Therefore, we must add:

$$f_9 = -54-18\rho+18\mu_2 \quad (19)$$

To the generating set. The ideal becomes:

$$F = \langle f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8,f_9\rangle \text{ and } \overline{S(f_1,f7)}^F = 0$$

Similarly:

$$S(f_3,f_4) = -36\sigma^2\rho^2\mu_2+90\sigma^2\rho\mu_1-12\mu_1^2\mu_2\rho^2+24\mu_1^3\rho+ 24\sigma^4\rho+18\sigma^2\rho^2\mu_1\mu_2-24\sigma^4\rho^3+114\sigma^2\rho^2-18\mu_1^3- 9\mu_2^2\rho\sigma^2-86\mu_1^2\rho+6\mu_1^3\mu_2-6\mu_1^4\rho-18\mu_1^3\rho^2-252\rho\sigma^2+ 38\mu_1^2+38\mu_1^2\rho^2+123\sigma^2\rho^3+36\mu_1^2\mu_2\rho-54\sigma^2\rho^2\mu_1+ 6\mu_1^3\mu_2\rho^2-6\mu_1^2\mu_2^2\rho-9\sigma^2\rho\mu_1^2-54\sigma^2\rho^3\mu_1+54\sigma^2\rho\mu_2$$

And:

$$\overline{S(f_3,f_4)}^F = 11-\frac{65}{2}\rho-10\sigma^2+18\rho^2+20\sigma^2\rho \neq 0$$

Therefore, we must add:

$$f_{10} = 11-\frac{65}{2}\rho-10\sigma^2+18\rho^2+20\sigma^2\rho \quad (20)$$

To the generating set. The ideal becomes:

$$F = \langle f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}\rangle \text{ and } \overline{S(f_3,f_4)}^F = 0$$

Similarly:

$$(f_3,f_5) = 40\sigma^2\mu_2+420\mu_2+40\sigma^2\mu_2\rho^2-205\mu_2\rho^2+90\rho^2\mu_1\mu_2- 150\mu_1\mu_2+25\mu_1^2\mu_2-90\mu_2^2+15\mu_2^3\ 190\ \mu_2\rho+90\mu_1\mu_2\rho+ 60\mu_2^2\rho-30\mu_1\mu_2^2\rho+30\mu_1^2-10\mu_1^2\sigma^3-10\mu_1^3\rho-30\mu_1^2\rho^2$$

And:

$$\overline{S(f_3,f_5)}^F = \frac{40}{3}-\frac{20}{3}\rho-40\rho^3+\frac{10}{3}\rho^2 \neq 0$$

Therefore, we must add:

$$f_{11} = \frac{40}{3}-\frac{20}{3}\rho-40\rho^3+\frac{10}{4}\rho^2 \quad (21)$$

To the generating set. The ideal becomes:

$$F = \langle f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8,f_9,f_{10},f_{11}\rangle \text{and} \overline{S(f_3,f_5)}^F = 0$$

Similarly:

$$(f_3,f_{10})-80\sigma^4+840\sigma^2+80\sigma^2\rho^2-410\sigma^2\rho^2+180\sigma^2\rho^2\mu_1- 300\sigma^2\mu_1+50\mu_1^2\sigma^2-180\sigma^2\mu_2+30\sigma^2\mu_2^2-380\sigma^2\rho+180\sigma^2\mu_1\rho+ 120\sigma^2\mu_2\rho-60\sigma^2\mu^2\mu_1\rho+11\mu_1^2\rho-\frac{65}{2}\mu_1^2\rho^2-10\mu_1^2\sigma^2\rho+6\mu_1^2\sigma^3$$

And:

$$\overline{S(f_3,f_{10})}^F = -\frac{405}{2}+\frac{27}{4}\rho+135\sigma^2-81\rho^3 \neq 0$$

Therefore, we must add:

$$f_{12} = -\frac{405}{2}+\frac{27}{4}\rho 135\sigma^2-81\rho^3 \quad (22)$$

To the generating set. The ideal becomes:

$$F = \left\langle \begin{matrix} f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8, \\ f_9,f_{10},f_{11},f_{12} \end{matrix} \right\rangle \text{and} \overline{S(f_3,f_{10})}^F = 0$$

Similarly, since, the reminders of the S-polynomials for all pairs polynomials f_8, f_9, f_{11} and f_{12} are zero, therefore, the Groebner basis of F is:

$$G = \langle f_8,f_9,f_{11},f_{12}\rangle = <-6+2\rho+4\mu_1+2\mu_2, -54-18\rho+ 18\mu_2, \frac{40}{3}-\frac{20}{3}\rho-40\rho^3+\frac{10}{3}\rho^2, -\frac{405}{2}+\frac{27}{4}\rho+135\sigma^2-81\rho^3>$$

So that, the reduced (minimum) Groebner basis of ideal F is:

$$G = <\mu_2-3-\rho, \mu_1-3, 20\sigma^2-30-12\rho^2+\rho, 12\rho^3-4-\rho^2-2\rho>$$

The variety of the reduced Groebner basis G is:

$$V(G)=\{3, 3.849, 1.3596, 0.8049\}$$

The maximum likelihood estmator of μ_1, μ_2, σ_2 and ρ are:

$$\hat{\mu}_1=3, \hat{\mu}_2=3.849, \hat{\sigma}^2=1.8485 \text{ and } \hat{\rho}=0.8049$$

Using maple, we get the following results:

> with (Grobner)

[Basis, FGLM, Hilbert Dimension, HilbertPolynomial, HilbertSeries, Homogenize, InitialForm, IneterReduce, Is Proper, Is Zero Dimensional, LeadingCoefficient, Leading Monomial, LeadingTerm, MatrixOrder, Maximall Independent Set, Monomial Order, Multiplication Matrix, Multivariate cyclic Vector, Normal Form, Normal Set, Rational Univariate Representation, Reduce, Remember Basis, Spolynomial, Solve, Suggest Variable Order, Test Order, Toric Ideal Basis, Trailing Term, Univariate Polynomial, Walk, Weighted Degree]

$$F:=\begin{bmatrix} 15\text{-}9\rho^2\text{-}5\mu_1+2\mu_1\rho^2+3\mu_2\rho\text{-}9\rho, 3\text{-}\mu^2\text{-}2\rho+ \\ \mu_1\rho, \text{-}8\sigma^2+84+8\sigma^2\rho^2\text{-}41\rho^2\text{-}2\mu_1^2\rho^2+ \\ 18\mu_1\rho^2\text{-}30\mu_1+5\mu_1{}^2\text{-}18\mu_2+3\mu_2^2\text{-}38\rho+ \\ 18\mu_1\rho+12\mu_2\rho\text{-}6\mu_1\mu_2\rho, 3\sigma^2\rho\text{-}3\sigma^2\rho^3+ \\ 19\text{-}9\mu_1\text{-}6\mu_2+3\mu_1\mu_2\text{-}43\rho+12\mu_1\rho\text{-}3\mu_1^2\rho+ \\ 18\mu_2\rho\text{-}9\mu_1\rho^2\text{-}3\mu_2{}^2\rho\text{-}6\ \mu_2\rho^2+ \\ 3\mu_1\mu_2\rho^2+19\rho^2 \end{bmatrix}$$

>

$$F:=\begin{bmatrix} 15\text{-}9\rho^2\text{-}5\mu_1+2\mu_1\rho^2+3\mu_2\rho\text{-}9\rho, 3\text{-}\mu_2\text{-}2\rho+ \\ \mu_1\rho, \text{-}8\sigma^2+84+8\sigma^2\rho^2\text{-}41\rho^2\text{-}2\mu_1^2\rho^2+ \\ 18\mu_1\rho^2\text{-}30\mu_1+5\mu_1{}^2\text{-}18\mu_2+3\mu_2^2\text{-}38\rho+ \\ 18\mu_1\rho+12\mu_2\rho\text{-}6\mu_1\mu_2\rho, 3\sigma^2\rho\text{-}3\sigma^2\rho^3+ \\ 19\text{-}9\mu_1\text{-}6\mu_2+3\mu_1\mu_2\text{-}43\rho+12\mu_1\rho\text{-}3\mu_1^2\rho+ \\ 18\mu_2\rho\text{-}9\mu_1\rho^2\text{-}3\mu_2^2\rho\text{-}6\mu_2\rho^2+ \\ 3\mu_1\mu_2\rho^2+19\rho^2 \end{bmatrix}$$

G:= basis (F, tdeg (μ_1, μ_2, σ, ρ))

$$G:=\begin{bmatrix} \mu_2\text{-}3\text{-}\rho, \mu_1\text{-}3, 20\sigma^2\text{-}30\text{-}12\rho^2+ \\ \rho, \text{-}4\text{-}2\rho+12\rho^3\text{-}\rho^2 \end{bmatrix}$$

> V (G) := fsolve ((3))

$$\begin{Bmatrix} \rho=0.8049038101, \sigma=1.359587031 \\ \mu_1=3.000000000, \mu_2=3.804903810 \end{Bmatrix}$$

CONCLUSION

In this study we have applied the Grobner basis technique to study the maximum likelihood estimation for bivariate normal model with missing data. For future research, this research can be extended to further application of this important technique in various scientific fields.

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