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Common Fixed Point Theorems for Weak (ψ, φ) -Contraction in B-Rectangular Metric Spaces

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Abstract: In this research, we extend the results of Roshan and eta. For four mappings in b-rectangular metric spaces.

Key words: Point theorems, weak (ψ, φ), b-rectangular, Roshan and eta, -contraction, literature

INTRODUCTION

Fixed point and common fixed point theorems for (weak) contractive mappings have been studied by many researchers (Chandok, 2011, 2012; Chandok *et al.*, 2013; Ciric, 1974). Many researchers have recently obtained fixed point, common fixed point, coupled fixed point results in partially ordered metric space (Abbas *et al.*, 2011; Aydi and Karapinar, 2012; Bhaskar and Lakshmikantham, 2006; Chandok, 2011; Chandok, 2012; Nieto and Lopez, 2005; Nieto and Rodriguez-Lopez. 2007; Ran and Reurings, 2004) and other spaces (Aydi *et al.*, 2012; Chandok *et al.*, 2013; Olatinwo and Postolache, 2012; George *et al.*, 2015).

Definition 1; Abbas *et al.* (2011): A pair (f, g) is named weakly increasing where, f, g: X→X and (X, ≤) is a partially ordered set, if f(x)≤g(x) and g(x)≤fg(x), ∀x∈X. A pair (f, g) is named partially weakly increasing if (x)≤f(x), ∀x∈X.

Definition 2; Abbas *et al.* (2011): A mapping f:X→X where (X, ≤) is a partially ordered set is named annihilator of a mapping g: X→X if f g(x)≤x, ∀x∈X. And a mapping f is named a dominating mapping if x≤f(x), ∀x∈X. If every two elements in Y are comparable where Y⊆X, then Y is named well ordered. A point x∈K is a common (coincidence) fixed point of self mappings of a metric space (X, ρ) if Tx = Sx = x (i.e., Tx = Sx = u) where, Ø≠K⊆X.

Definition 3; Abbas *et al.* (2011): A function T:X→X where (X, ρ) is a metric space are named compatible if:

$$\lim_{n\to\infty}\rho(STx_n, TSx_n)=0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z∈X.

Definition 4; Jungck and Rhoades (1998): A functions S and T are named weakly compatible if they commute at coincidence points of S and T (that is STx = TSx = u).

Definition 5: "Let, Ψ denote the family of all non decreasing and continuous function $\psi: R^+ \to R^+ \text{s.t.} \lim_{n\to\infty}\psi^n(t)=0 \forall t>0$ where ψ^n denotes the nth iterate of ψ. It is easy to show that for each ψ∈Ψ, the following is satisfied:

- ψ(t)<t ∀t>0
- ψ(0) = 0

Definition 6; Djoudi and Nisse (2003): A function φ: R^+→R^+ is named an altering distance function if the following properties are satisfied:

- φ(r) = 0⇔r = 0
- φ is non-decreasing and continuous

By Roshan *et al.* (2016), introduced b-rectangular or b-generalized metric space (b-gms) as follows:

Definition 7; Roshan *et al.* (2016): A function ρ: X×X→[0, ∞) on a non empty set X is a generalized metric or b-rectangular metric [b-gms] with parameter s≥1 if the triangle inequality in definition of a metric is replaced with the (b-rectangular inequality):

$$\rho(x,y) \le s\left[\rho(x,u)+\rho(u,v)+\rho(v,y)\right] (\text{b-rectangular inequality}) \quad (1)$$

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For all distinct points x, y, u, v∈X. The pair (X, ρ) is named a b-rectangular or a b-generalized metric space (b-gms).

MATERIALS AND METHODS

Lemma 1; Roshan *et al*. (2016): "Let (X, ρ) be a (b-gms) and let $\{x_n\}$ be a Cauchy sequence in X such that $x_m \neq x_n$ whenever $m \neq n$. Then $\{x_n\}$ can converge to at most one point."

By Roshan *et al*. (2016) introduced almost generalized weakly contractive mappings. The purpose of this research is to extend the results of Roshan for four mappings in partially b-rectangular metric space.

RESULTS AND DISCUSSION

Now, we will prove our result.

Theorem 1: Let (X, ≤, ρ) be a complete ordered (b-gms) with parameter s>1. Let, I:X→X be an increasing function in relation to ≤, suppose that I, J, S and T are self–mappings on X, the pairs (T, I) and (S, J) are a partially weakly increasing where I and J are dominating mappings and the mappings I and J are weak annihilators of T and S, respectively with I(X)⊆T(X) and J(X)⊆S(X). Further, assume that for any two comparable elements u, v∈X and ψ∈Ψ holds:

$$\left.\begin{aligned}&\psi(sd(Iu, Jv)) \le \psi(M(u,v))-\varphi(M(u,v)+L\psi(N(u,v))\\ &\text{where}\\ &M(u,v)=\max\{\rho(Su,Tv),\rho(Su, Iu),\rho(Tv, Jv)\}\\ &\text{and}\\ &N(u,v)=\min\{\rho(Su, Iu),\rho(Su, Jv),\rho(Tv, Iu),\rho(Tv, Jv)\}\end{aligned}\right\} \quad (2)$$

If, for a non decreasing sequenc $\{u_n\}$ where $u_n \le v_n \forall n \ge 1$, $v_n \to t$ implies that $u_n \le t$ and either. "I and S are compatible, I or S is continuous and J, T are weakly compatible. J and T are compatible, J or T is continuous and I, S are weakly compatible, then I, J, S and T have a common fixed point in X. Moreover, the set of common fixed points of I, J, S and T is well ordered if and only if I, J, S and T have one and only one common fixed point in X".

Proof

Step I: We show that $\lim_{n\to\infty}\rho(v_n, v_{n+1}) = 0$. Let $u_0 \in X$ be an element, we construct the sequences $\{u_n\}$ and $\{v_n\}$ in X s.t.

$$v_{2n-1} = Iu_{2n-2} = Tu_{2n-1}, v_{2n} = Ju_{2n-1} = Su_{2n} \quad \forall n \ge 1 \quad (3)$$

By assumptions, we get:

$$\left.\begin{aligned}&u_{2n-2} \le Iu_{2n-2} = Tu_{2n-1} \le ITu_{2n-1} \le u_{2n-1},\\ &u_{2n-1} \le Ju_{2n-1} = Su_{2n} \le JSu_{2n} \le u_{2n}, \ \forall n \ge 1\end{aligned}\right\} \quad (4)$$

Then, ∀n≥1, we get $u_n \le u_{n+1}$. Without loss of generality, we assume that $v_{2n} \neq v_{2n+1} \forall n \ge 1$. Now, we will prove that ∀n∈N, we have:

$$\rho(v_{n+1}, v_{n+2}) < \rho(v_n, v_{n+1}) \quad (5)$$

By using the contrary that $\rho(v_{2n}, v_{2n+1}) \le \rho(v_{2n+1}, v_{2n+2})$ for some n∈N, since, v_{2n} and v_{2n+1} are comparable from (Eq. 2):

$$\begin{aligned}&\psi(s\rho(v_{2n+1}, v_{2n+2})) = \psi(s\rho(Iu_{2n}, Ju_{2n+1})) \le \psi(M(u_{2n}, u_{2n+1}))-\\ &\varphi(M(u_{2n}, u_{2n+1}))+L\psi(N(u_{2n}, u_{2n+1}))\end{aligned} \quad (6)$$

where:

$$\begin{aligned}M(u_{2n}, u_{2n+1}) &= \max\left\{\begin{matrix}\rho(Su_{2n}, Tu_{2n+1}),\rho(Su_{2n}, Iu_{2n}),\\ \rho(Tu_{2n+1}, Ju_{2n+1})\end{matrix}\right\}\\ &= \max\{\rho(v_{2n}, v_{2n+1}),\rho(v_{2n}, v_{2n+1}),\rho(v_{2n+1}, v_{2n+2})\}\\ M(u_{2n}, u_{2n+1}) &= \max\{\rho(v_{2n}, v_{2n+1}),\rho(v_{2n+1}, v_{2n+2})\}\end{aligned} \quad (7)$$

and:

$$\begin{aligned}N(u_{2n}, u_{2n+1}) &= \min\left\{\begin{matrix}\rho(Su_{2n}, Tu_{2n}),\rho(Su_{2n}, Ju_{2n+1}),\\ \rho(Tu_{2n+1}, Iu_{2n}),\rho(Tu_{2n+1}, Ju_{2n+1})\end{matrix}\right\}\\ &= \min\left\{\begin{matrix}\rho(v_{2n}, v_{2n+1}),\rho(v_{2n}, v_{2n+2}),\\ \rho(v_{2n+1}, v_{2n+1}),\rho(v_{2n+1}, v_{2n+2})\end{matrix}\right\}\\ N(u_{2n}, u_{2n+1}) &= \min\{\rho(v_{2n}, v_{2n+1}),\rho(v_{2n+1}, v_{2n+2}),0\} = 0\end{aligned} \quad (8)$$

From Eq. 6 and 8 and definitions of ψ and φ, we have:

$$\begin{aligned}&\psi(s\rho(v_{2n+1}, v_{2n+2})) < \psi(\max\{\rho(v_{2n}, v_{2n+1}),\rho((v_{2n+1}, v_{2n+2})\}\\ &\text{i.e., } \rho(v_{2n+1}, v_{2n+2}) < \frac{1}{s}\max\{\rho(v_{2n}, v_{2n+1}),\rho((v_{2n+1}, v_{2n+2})\}\end{aligned} \quad (9)$$

If max $\{\rho(v_{2n}, v_{2n+1}), \rho((v_{2n+1}, v_{2n+2})\} = \rho((v_{2n+1}, v_{2n+2})$ then, we have:

$$\rho((v_{2n+1}, v_{2n+2}) < \frac{\rho((v_{2n+1}, v_{2n+2})}{s}$$

This is contradiction. Hence, max $\{\rho(v_{2n}, v_{2n+1}), \rho((v_{2n+1}, v_{2n+2})\} = \rho(v_{2n}, v_{2n+1})$, therefore, Eq. 9 is that:

$$\rho(v_{2n+1}, v_{2n+2}) < \frac{1}{s}\rho(v_{2n}, v_{2n+1}) \quad (10)$$

Is the sequence $\rho(v_n, v_{n+1})$ is decreasing. Then, $\exists\ \varepsilon\geq 0$ s.t. $\lim_{n\to\infty}\rho(v_n, v_{n+1}) = \varepsilon$. Suppose that, $\varepsilon>0$ taking $n\to\infty$ in Eq. 10, we have $\varepsilon<\varepsilon/s$ which is contradiction (since, $s>1$). Hence, $\varepsilon = 0$ that is:

$$\lim_{n\to\infty}\rho(v_n, v_{n+1}) = 0 \tag{11}$$

Step II: We will prove $\{v_n\}$ is a b-gms Caushy sequence in X. Assume that $\{v_n\}$ is not a Caushy sequence in X. Then, $\exists\ \varepsilon>0$, we can find two subsequences $\{v_{m_k}\}$ and $\{v_{n_k}\}$ of $\{v_n\}$ and of such that n_k is the smallest index:

$$n_k>m_k>k \text{ and } \rho(v_{m_k}, v_{n_k})\geq\varepsilon \tag{12}$$

This means that:

$$\rho(v_{m_k}, v_{m_k-2})<\varepsilon \tag{13}$$

From Eq. 13 and taking limsup as $k\to\infty$, we have:

$$\limsup_{k\to\infty}\rho(v_{m_k}, v_{n_k-2})\leq\varepsilon \tag{14}$$

On the other hand, we have:

$$\rho(v_{m_k}, v_{n_k})\leq s\rho(v_{m_k}, v_{m_k+1})+s\rho(v_{m_k+1}, v_{n_k-1})+s\rho(v_{n_k-1}, v_{n_k})$$

Taking limsup as $k\to\infty$ and using Eq. 11 and 12, we have:

$$\frac{\varepsilon}{s}\leq\limsup_{k\to\infty}\rho(v_{m_k+1}, v_{n_k-1}) \tag{15}$$

Using Eq. 1 again, we get the following:

$$\rho(v_{m_k}, v_{n_k})\leq s\rho(v_{m_k}, v_{n_k-2})+s\rho(v_{n_k-2}, v_{n_k-1})+s\rho(v_{n_k-1}, v_{n_k})$$

Using Eq. 11 and 12 and taking limsup as $k\to\infty$, we have:

$$\frac{\varepsilon}{s}\leq\limsup_{k\to\infty}\rho(v_{m_k}, v_{n_k-2}) \tag{16}$$

From Eq. 2, we get:

$$\begin{aligned}&\psi\left(s\rho(v_{2m_k+1}, v_{2n_k-1})\right)=\psi\left(s\rho(Iu_{2m_k}, Ju_{2n_k-2})\right)\leq\\&\leq\psi\left(M(u_{2m_k}, u_{2n_k-2})\right)-\varphi\left(M(u_{2m_k}, u_{2n_k-2})\right)+\\&L\psi\left(N(u_{2m_k}, u_{2n_k-2})\right)\end{aligned} \tag{17}$$

Where:

$$M(u_{2m_k}, u_{2n_k-2}) = \max\left\{\begin{matrix}\rho(Su_{2m_k}, Tu_{2n_k-2}),\rho(Su_{2m_k}, Iu_{2m_k}),\\ \rho(Tu_{2n_k-2}, Ju_{2n_k-2})\end{matrix}\right\}$$

and:

$$N(u_{2m_k}, u_{2n_k-2}) = \min\left\{\begin{matrix}\rho(Su_{2m_k}, Iu_{2m_k}),\rho(Su_{2m_k}, Ju_{2n_k-2}),\\ \rho(Tu_{2n_k-2}, Iu_{2m_k}),\rho(Tu_{2n_k-2}, Ju_{2n_k-2})\end{matrix}\right\}$$

From Eq. 3, we have:

$$M(u_{2m_k}, u_{2n_k-2}) = \max\left\{\begin{matrix}\rho(v_{2m_k}, v_{2n_k-2}),\rho(v_{2m_k}, v_{2m_k+1}),\\ \rho(v_{2n_k-2}, v_{2n_k-1})\end{matrix}\right\} \tag{18}$$

and:

$$N(u_{2m_k}, u_{2n_k-2}) = \min\left\{\begin{matrix}\rho(v_{2m_k}, v_{2m_k+1}),\ \rho(v_{2m_k}, v_{2n_k-1}),\\ \rho(v_{2n_k-2}, v_{2m_k+1}),\ \rho(v_{2n_k-2}, v_{2n_k-1})\end{matrix}\right\} \tag{19}$$

Taking $\limsup_{k\to\infty}$ in Eq. 18 and 19 and using Eq. 11 and 14, we have:

$$\limsup_{k\to\infty} M(u_{2m_k}, u_{2n_k-2}) = \max\left\{\limsup_{k\to\infty}\rho(v_{2m_k}, v_{2n_k-2}), 0, 0\right\}\leq\varepsilon$$

So, we have:

$$\limsup_{k\to\infty} M(u_{2m_k}, u_{2n_k-2})\leq\varepsilon \tag{20}$$

and:

$$\limsup_{k\to\infty} N(u_{2m_k}, u_{2n_k-2}) = 0 \tag{21}$$

Similarly by taking $\liminf_{k\to\infty}$ in Eq. 18 and using (Eq. 11 and 16), we have:

$$\frac{\varepsilon}{s}\leq\liminf_{k\to\infty} M(u_{2m_k}, u_{2n_k-2}) \tag{22}$$

Now, taking $\limsup_{x\to\infty}$ in Eq. 17 and from Eq. 15, 20 and 21, we have:

$$\begin{aligned}&\psi\left(s.\frac{\varepsilon}{s}\right)\leq\psi\left(s\limsup_{k\to\infty}\rho(v_{2m_k+1}, v_{2n_k-1})\right)\leq\\&\psi\left(\limsup_{k\to\infty} M(u_{2m_k}, u_{2n_k-2})\right)-\liminf_{k\to\infty}\varphi\left(M(u_{2m_k}, u_{2n_k-2})\right)\\&\leq\psi(\varepsilon)-\liminf_{k\to\infty}\varphi\left(M(u_{2m_k}, u_{2n_k-2})\right)\end{aligned}$$

Which implies that:

$$\varphi\left(\liminf_{k\to\infty}\left(M\left(u_{2m_k}, u_{2n_k-2}\right)\right)\right) = 0$$

Hence, $\liminf_{k\to\infty}\left(M\left(u_{2m_k}, u_{2n_k-2}\right)\right) = 0$, a contradiction with (Eq. 22). Thus, $\{v_n\}$ is a b-g m s. Caushy sequenc in X. Since, X is complete, $\exists z \in X$ s.t. $v_{n_k} \to z$ $\forall k\in N$, therefore, we have:

$$\left.\begin{aligned} \lim_{n\to\infty} v_{2n+1} = \lim_{n\to\infty} Tu_{2n+1} = \lim_{n\to\infty} Iu_{2n} = z \\ \lim_{n\to\infty} v_{2n+2} = \lim_{n\to\infty} Su_{2n+2} = \lim_{n\to\infty} Ju_{2n+1} = z \end{aligned}\right\} \quad (23)$$

Suppose that S is continuous. Since, I and S are compatible, we get:

$$\lim_{n\to\infty} ISu_{2n+2} = \lim_{n\to\infty} SIu_{2n+2} = Sz \quad (24)$$

Also $v_{2n+1} \le Ju_{2n+1} = Su_{2n+2}$. Now, using Eq. 23 and 24, we get:

$$\begin{aligned} &\psi\left(sd\left(ISu_{2n}, Ju_{2n+1}\right)\right) \le \psi\left(M\left(Su_{2n}, u_{2n+1}\right)\right) - \\ &\varphi\left(M\left(Su_{2n}, u_{2n+1}\right) + L\psi\left(N\left(Su_{2n}, u_{2n+1}\right)\right)\right) \end{aligned} \quad (25)$$

Where:

$$M\left(Su_{2n}, u_{2n+1}\right) = \max\left\{\begin{aligned} &\rho\left(S^2u_{2n}, Tu_{2n+1}\right), \rho\left(S^2u_{2n}, ISu_{2n}\right), \\ &\rho\left(Tu_{2n+1}, Ju_{2n+1}\right) \end{aligned}\right\} \quad (26)$$

and:

$$N\left(Su_{2n}, u_{2n+1}\right) = \min\left\{\begin{aligned} &\rho\left(S^2u_{2n}, ISu_{2n}\right), \rho\left(S^2u_{2n}, Ju_{2n+1}\right), \\ &\rho\left(Tu_{2n+1}, ISu_{2n}\right), \rho\left(Tu_{2n+1}, Ju_{2n+1}\right) \end{aligned}\right\} \quad (27)$$

Letting $n\to\infty$ in Eq. 25 and using Eq. 26 and 27, we get:

$$\psi\left(s\rho\left(Sz, z\right)\right) < \psi\left(\max\left\{\rho\left(Sz, z\right), \rho\left(Sz, Sz\right), \rho\left(z, z\right)\right\}\right)$$

We get $\rho(Sz, z) < \rho(Sz, z)$, this is contradiction, since, $(s>1)$ which implies that $Sz = z$. Hence, z is a fixed point of S:

$$\begin{aligned} &\psi\left(s\rho\left(Iz, Ju_{2n+1}\right)\right) \le \psi\left(M\left(z, u_{2n+1}\right)\right) - \\ &\varphi\left(M\left(z, u_{2n+1}\right) + L\psi\left(N\left(z, u_{2n+1}\right)\right)\right) \end{aligned} \quad (28)$$

Where:

$$M\left(z, u_{2n+1}\right) = \max\left\{\begin{aligned} &\rho\left(Sz, Tu_{2n+1}\right), \rho\left(Sz, Iz\right), \\ &\rho\left(Tu_{2n+1}, Ju_{2n+1}\right) \end{aligned}\right\} \quad (29)$$

and:

$$N\left(z, u_{2n+1}\right) = \min\left\{\begin{aligned} &\rho\left(Sz, Iz\right), \rho\left(Sz, Ju_{2n+1}\right), \\ &\rho\left(Tu_{2n+1}, Iz\right), \rho\left(Tu_{2n+1}, Ju_{2n+1}\right) \end{aligned}\right\} \quad (30)$$

Letting $n\to\infty$ in Eq. 28 and using Eq. 29 and 30, we have:

$$\begin{aligned} &\psi\left(s\rho\left(Iz, z\right)\right) < \psi\left(\max\left\{\rho\left(z, z\right), \rho\left(z, Iz\right), \rho\left(z, z\right)\right\}\right) - \\ &\varphi\left(\max\left\{0, \rho\left(z, Iz\right), 0\right\}\right) + \psi\left(\min\left\{\begin{aligned} &\rho\left(z, Iz\right), \rho\left(z, z\right), \\ &\rho\left(z, Iz\right), \rho\left(z, z\right) \end{aligned}\right\}\right) < \psi\left(\rho\left(z, Iz\right)\right) \end{aligned}$$

We get, $s\rho(Iz, z) < \rho(z, Iz)$. We have $Iz = z$. Since, $I(X) \subseteq T(X)$, $\exists w\in X$ s.t. $Sz = z = Iz = Tw$, assume that $Jw \ne Tw$. Since, $z \le Iz = Tw \le ITw \le w$ implies $z \le w$ from (Eq. 2), we obtain:

$$\begin{aligned} &\psi\left(s\rho\left(Tw, Jw\right)\right) = \psi\left(s\rho\left(Iz, Jw\right)\right) \le \psi\left(M\left(z, w\right)\right) - \\ &\varphi\left(M\left(z, w\right) + L\psi\left(N(z, w\right)\right) \end{aligned} \quad (31)$$

where:

$$\begin{aligned} M\left(z, uw\right) &= \max\left\{\rho\left(Sz, Tw\right), \rho\left(Sz, Iz\right), \rho\left(Tw, Jw\right)\right\} \\ &= \max\left\{0, 0, \rho\left(Tw, Jw\right)\right\} \\ &= \rho\left(Tw, Jw\right) \end{aligned} \quad (32)$$

and:

$$\begin{aligned} N\left(z, w\right) &= \min\left\{\rho\left(Sz, Iz\right), \rho\left(Sz, Jw\right), \rho\left(Tw, Iz\right), \rho\left(Tw, Jw\right)\right\} \\ &= \min\left\{0, \rho\left(Sz, Jw\right), 0, \rho\left(Tw, Jw\right)\right\} \\ &= 0 \end{aligned} \quad (33)$$

Using Eq. 32 and 33 in Eq. 37, we get:

$$s\rho\left(Tw, Jw\right) < \rho\left(Tw, Jw\right)$$

which is contradiction. Therefor, $Tw = Jw$. Since, J and T are weakly compatible, hence, $Jz = JIz = JTw = TJw = TIz = Tz$. Thus, z is a coincidence point of T and J. Now, $u_{2n} \le Iu_{2n}$ and $Iu_{2n} \to z$ implies $u_{2n} \le z$. Thus, from Eq. 2, we get:

$$\psi\left(s\rho\left(Iu_{2n}, Jz\right)\right) \le \psi\left(M\left(u_{2n}, z\right)\right) - \varphi\left(M\left(u_{2n}, z\right) + L\psi\left(N\left(u_{2n}, z\right)\right)\right) \quad (34)$$

where:

$$M\left(u_{2n}, z\right) = \max\left\{\rho\left(Su_{2n}, Tz\right), \rho\left(Su_{2n}, Iu_{2n}\right), \rho\left(Tz, Jz\right)\right\} \quad (35)$$

and:

$$N(u_{2n}, z) = \min\left\{\begin{matrix}\rho(Su_{2n}, Iu_{2n}), \rho(Su_{2n}, Jz), \\ \rho(Tz, Iu_{2n}), \rho(Tz, Jz)\end{matrix}\right\} \quad (36)$$

Using Eq. 35 and 36 in Eq. 34 and taking limit as $n \to \infty$, we have:

$$s\rho(z, Jz) < \max\{\rho(z, Jz), \rho(z, z), \rho(Tz, Jz)\} =$$
$$\max\{\rho(z, Jz), 0, 0\} s\rho(z, Jz) < \rho(z, Jz)$$

This is contradiction, hence, Jz = z, therefore, we have Iz = Jz = Sz = Tz = z. In the same way, the result satisfies when (ii) be available. Now, assume that the set of common fixed points of T, S, I and J is well ordered. We shall show that common fixed point of T, S, I and J is single. Suppose that and $Tz_1 = Sz_1 = Iz_1 = Jz_1 = z_1$ and $Tz_2 = Sz_2 = Iz_2 = Jz_2 = z_2$ where $z_1 \neq z_2$. Then, from Eq. 2, we have:

$$\psi(s\rho(z_1, z_2)) = \psi(s\rho(Iz_1, Jz_2)) \leq \psi(M(z_1, z_2)) - \varphi(M(z_1, z_2)) + L\psi(N(z_1, z_2)) \quad (37)$$

where:

$$M(z_1, z_2) = \max\{\rho(Sz_1, Tz_2), \rho(Sz_1, Iz_1), \rho(Tz_2, Jz_2)\} = \max\{\rho(z_1, z_2), 0, 0\} = \rho(z_1, z_2) \quad (38)$$

and:

$$N(z_1, z_2) = \min\{\rho(Sz_1, Iz_1), \rho(Sz_1, Jz_2), \rho(Tz_2, Iz_1), \rho(Tz_2, Jz_2)\} = \min\{0, \rho(z_1, z_2), \rho(z_2, z_1), 0\} = 0 \quad (39)$$

Using Eq. 38 and 39 in Eq. 37, we get $s\rho(z_1, z_2) < \rho(z_1, z_2)$, a contradiction, hence, $z_1 = z_2$. On the contrary, if T, S, I and J only one common fixed point to have then the set of common fixed point of I, J, S and T being singleton is well ordered. This completes the proof. If I = J, then we get the following result:

Corollary 1: Let $(X, \leq, \rho)$ be a complete ordered b-rectangular metric space with parameter s>1. Assume that I, S and T are self–mappings on X, the pairs (T, I) and (S, I) are partially weakly increasing with $I(X) \subseteq T(X)$ and $I(X) \subseteq S(X)$ and the dominating mapping I is a weak annihilator of T and S. Further, suppose that exists the function $\psi \in \Psi$ such that for any two comparable elements x, $y \in X$:

$$\psi(s\rho(Iu, Jv)) \leq \psi(M(u, v)) - \varphi(M(u, v) + L\psi(N(u, v))$$

where:

$$M(u, v) = \max\{\rho(Su, Tv), \rho(Su, Iu), \rho(Tv, Iv)\}$$

and:

$$N(u, v) = \min\{\rho(Su, Iu), \rho(Su, Iv), \rho(Tv, Iu), \rho(Tv, Iv)\}$$

holds. If, for a non decreasing sequence $\{v_n\}$ with $u_n \leq v_n$ $\forall n \geq 1$, $y_n \to v$ implies $u_n \leq v$ and either. "I, S are compatible, I or S is continuous and I, T are weakly compatible or I, T are compatible, I or T is continuous and I, S are weakly compatible. Then I, S and T have a common fixed point in X." If S = T, then, we get the following result:

Corollary 2: Let $(X, \leq, \rho)$ be a complete ordered b-rectangular metric space with parameter s>1. Assume that are self-mappings on X, the pairs (T, I) and (T, J) are partially weakly increasing with $I(X) \subseteq T(X)$ and $J(X) \subseteq T(X)$ and the dominating mappings I are J a weak annihilator of T. Further, suppose that there exists function $\psi \in \Psi$ such that, for any two comparable elements x, $y \in X$:

$$\psi(s\rho(Iu, Jv)) \leq \psi(M(u, v)) - \varphi(M(u, v) + L\psi(N(u, v))$$

where:

$$M(u, v) = \max\{\rho(Tu, Tv), \rho(Tu, Iu), \rho(Tv, Jv)\}$$

and:

$$N(u, v) = \min\{\rho(Tu, Iu), \rho(Tu, Jv), \rho(Tv, Iu), \rho(Tv, Jv)\}$$

holds. If, for any a non decreasing sequence $\{u_n\}$ with $u_n \leq u_n$ $\forall n \geq 1$, $v_n \to v$ implies that $u_n \leq v$ and either, "I, T are compatible, I or T is continuous, J and T are weakly compatible or", J, T are compatible, J or T is continuous and I, T are weakly compatible. Then I, J and T have a common fixed point in X."

Corollary 3: Let $(X, \leq, \rho)$ be a complete ordered b-rectangular metric space with parameter s>1. Assume that T and I are self-mappings on X, the pair (T, I) is partially weakly increasing with $I(X) \subseteq T(X)$ and the dominating mapping I is a weak annihilator of T. Further, suppose that exists the function $\psi \in \Psi$ such that for any two comparable elements x, $y \in X$:

$$\psi(s\rho(Iu, Jv)) \leq \psi(M(u, v)) - \varphi(M(u, v) + L\psi(N(u, v))$$

where:

$$M(u, v) = \max\{\rho(Tu, Tv), \rho(Tu, Iu), \rho(Tv, Iv)\}$$

and:

$$N(u, v) = \min\{\rho(Tu, Iu), \rho(Tu, Iv), \rho(Tv, Iu), \rho(Tv, Iv)\}$$

holds. If, for a non decreasing sequence $e\{u_n\}$ with $u_n \leq v_n$ $\forall n \geq 1$, $v_n \to v$ indicates that $x_n \leq v$, further I, T are compatible, I or T is continuous, I and T are weakly compatible. Then I and T have a common fixed point in X.

CONCLUSION

Our results improves some fixed point theorems for (ψ, φ) -contraction in the literature.

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