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Odd Generalized Exponential Weibull Exponential Distribution

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Abstract: We introduce a new five-parameters continuous distribution called the Odd Generalized Exponential Weibull Exponential (OGE-W-E) distribution for modeling life time data. We derive an explicit expressions for the moments, quantile and median, the moment generating functions, Renyi entropy and order statistics. The five parameters of the suggested model are estimated by the maximum likelihood method. We illustrate its usefulness by means of an application to a real data set.

Key words: Odd generalized exponential family, moment, maximum likelihood, estimation, order statistics, five parameters

INTRODUCTION

In a statistical analysis, lifetime distributions such as the exponential distribution, the Weibull distribution, the normal distribution and the gamma distribution play an important role in many fields of the real life such as reliability, survival analysis, ecology, medicine and social sciences. There are continuous motivations to develop these lifetime distributions to become more flexible or more fitting for specific real data sets. So, in recent years many different families of distributions have been developed by generalizing the common families of continuous distributions such as Weibull distribution and exponential distribution by adding one or more one additional parameter(s) to baseline model. Among of these, exponentiated Weibull family (Mudholkar and Srivastava, 1993), Generalized Exponential (GE) distribution (Gupta and Kundu, 2007), modified Weibull distribution (Sarhan and Zaindin, 2009; Lai *et al*., 2003), beta-Weibull distribution (Famoye *et al*., 2005) a flexible Weibull extension (Bebbington *et al*., 2007), beta modified Weibull distribution (Silva *et al*., 2010; Nadarajah *et al*., 2011), beta generalized Weibull distribution (Singla *et al*., 2012) and a new modified Weibull distribution (Almalki and Yuan, 2013) among others. Gupta and Kundu (2007) proposed an important generalization of the exponential distribution called Generalized Exponential (GE). The cumulative distribution function (cdf) of GE is given by:

$$F(x;\alpha,\lambda)=\left[1-e^{-\lambda x}\right]^{\alpha},\ x>0,\ \alpha>0,\lambda>0 \quad (1)$$

Recently, a new family of continuous distributions called the Odd Generalized Exponential (OGE) family has been introduced by El-Damcese *et al*. (2015), Tahir *et al*. (2015). This family is flexible because of hazard rate shapes could be decreasing, increasing, bathtub and upside down bathtub. Many special OGE distributions have been introduced such as the Odd Generalized Exponential Weibull (OGE-W) distribution, Odd Generalized Exponential Normal (OGE-N) distribution by Tahir *et al*. (2015), Odd Generalized Exponential Generalized Linear Exponential (OGE-GLE) distribution by Luguterah and Nasiru (2017) and Odd Generalized Exponential Flexible Weibull Extension (OGE-FEW) distribution by Mustafa *et al*. (2018). The pdf and cdf of the Odd Generalized Exponential (OGE) family are defined as follows.

If G(x), x>0 is cumulative distribution function (cdf) of a random variable X then the corresponding probability density function (pdf) is g (x) and the survival function is $\bar{G}(x)=1-G(x)$ then we define the cdf of the OGE family by replacing in Eq. 1 by $\frac{G(x)}{\bar{G}(x)}$ leading to:

$$F(x;\alpha,\lambda)=\left[1-e^{-\lambda\frac{G(x)}{\bar{G}(x)}}\right]^{\alpha},\ x>0,\alpha>0,\lambda>0 \quad (2)$$

The pdf corresponding to Eq. 2 is given by:

$$f(x;\alpha,\lambda)=\frac{\alpha\lambda g(x)}{\bar{G}(x)^{2}}e^{-\lambda\frac{G(x)}{\bar{G}(x)}}\left[1-e^{-\lambda\frac{G(x)}{\bar{G}(x)}}\right]^{\alpha-1},x>0,\alpha>0,\lambda>0 \quad (3)$$

In this study we present and study a new continuous distribution called the Odd Generalized Exponential Weibull Exponential (OGE-W-E) distribution. We use Eq. 2 to define the cdf of this distribution by taking G (x) equals to the cdf of Weibull distribution and $\bar{G}(x)$ equals to the survival function of the exponential distribution. After we define the cdf of this distribution, we get the corresponding pdf of this new distribution.

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THE ODD GENERALIZED EXPONENTIAL WEIBULL EXPONENTIAL DISTRIBUTION

In this section, we studied the five parameters Odd Generalized Weibull Exponential (OGE-W-E) distribution. By using Eq. 2, we define the cdf of the OGE-W-E by taking G (x) equals to the cdf of Weibull distribution and $\bar{G}$ (x) equals to the survival function of the exponential distribution where the cdf of the Weibull distribution is given by:

$$G(x; a,b) = 1-e^{-ax^{b}},\ x>0,\ a>0,\ b>0 \quad (4)$$

And the survival function of the exponential distribution is given by:

$$\bar{G}(x;k) = 1-\left(1-e^{-kx}\right) = e^{-kx},\ x>0, k>0 \quad (5)$$

Then by substituting from Eq. 4 and 5 into Eq. 2, we obtain the cumulative distribution function (cdf) of the OGE-W-E distribution as follows:

$$F(x;\alpha,\lambda,a,b,k) = \left[1-e^{-\lambda\left(e^{kx}-e^{kx-ax^{b}}\right)}\right]^{\alpha},\ x>0,\ \alpha,\lambda,a,b,k>0 \quad (6)$$

The pdf corresponding to Eq. 6 is given by:

$$f(x;\alpha,\lambda,a,b,k) = \alpha\lambda\left[ke^{kx}-\left(k-abx^{b-1}\right)e^{kx-ax^{b}}\right]e^{-\lambda\left(e^{kx}-e^{kx-ax^{b}}\right)}\times \left[1-e^{-\lambda\left(e^{kx}-e^{kx-ax^{b}}\right)}\right]^{\alpha-1},\ x>0,\ \alpha,\lambda,a,b,k>0 \quad (7)$$

Where:
α and b : Shape parameters
a, k and λ : Scale parameters

The survival function $\bar{F}$ (x), hazard rate function h (x), reversed hazard function r (x) and cumulative hazard rate function H (x) of X~ OGE-W-E (α, λ, a, b, k) are given by:

$$\bar{F}(x;\alpha,\lambda,a,b,k) = 1-\left[1-e^{-\lambda\left(e^{kx}-e^{kx-ax^{b}}\right)}\right]^{\alpha},\ x>0,\ \alpha,\lambda,a,b,k>0 \quad (8)$$

$$h(x;\alpha,\lambda,a,b,k) = \frac{\alpha\lambda\left[ke^{kx}-\left(k-abx^{b-1}\right)e^{kx-ax^{b}}\right]e^{-\lambda\left(e^{kx}-e^{kx-ax^{b}}\right)}}{1-\left[1-e^{-\lambda\left(e^{kx}-e^{kx-ax^{b}}\right)}\right]^{\alpha}}\times \left[1-e^{-\lambda\left(e^{kx}-e^{kx-ax^{b}}\right)}\right]^{\alpha-1},\ x>0,\ \alpha,\lambda,a,b,k>0 \quad (9)$$

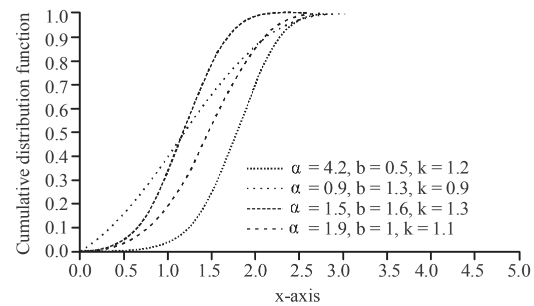

Fig. 1: The cdf of the OGE-W-E for different values of parameters; a = 0.6, λ = 0.4

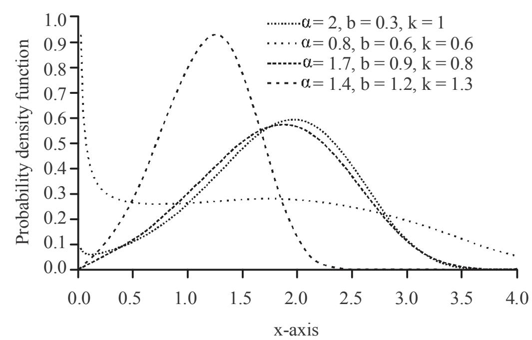

Fig. 2: The pdf of the OGE-W-E; a = 0.2, λ = 0.9

$$r(x;\alpha,\lambda,a,b,k) = \frac{\alpha\lambda\left[ke^{kx}-\left(k-abx^{b-1}\right)e^{kx-ax^{b}}\right]e^{-\lambda\left(e^{kx}-e^{kx-ax^{b}}\right)}}{\left[1-e^{-\lambda\left(e^{kx}-e^{kx-ax^{b}}\right)}\right]},\ \alpha,\lambda,a,b,k>0 \quad (10)$$

$$H(x;\alpha,\lambda,a,b,k) = -\ln\left[1-\left(1-e^{-\lambda\left(e^{kx}-e^{kx-ax^{b}}\right)}\right)^{\alpha}\right],\ x>0 \quad (11)$$

Respectively, in Fig. 1-5, we display some plots of the cdf, pdf, survival function $\bar{F}$ (x), hazard rate function h (x) and cumulative hazard rate function H (x) of the OGE-W-E (α, λ, a, b, k) distribution for some different values of parameters.

SOME STATISTICAL PROPERTIES

In this section, we investigate some statistical properties of the OGE-W-E distribution including quantile function and simulation median, the moments, the mean, moments about the mean, skewness and kurtosis.

Quantile and median: The quantile x_q of the OGE-W-E distribution is given by:

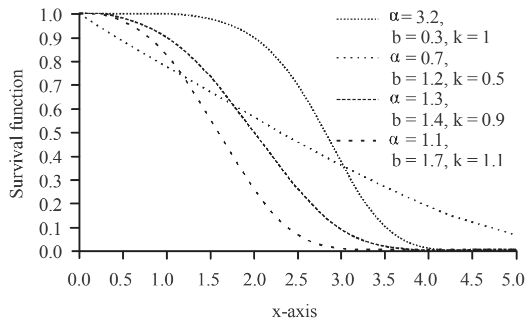

Fig. 3: The survival function of the OGE-W-E for different values of parameters; a = 0.5, λ = 0.2

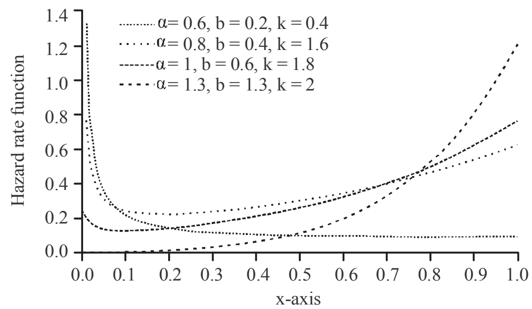

Fig. 4: The hazard rate function of the OGE-W-E for different values of parameters; a = 0.3, λ = 1.8

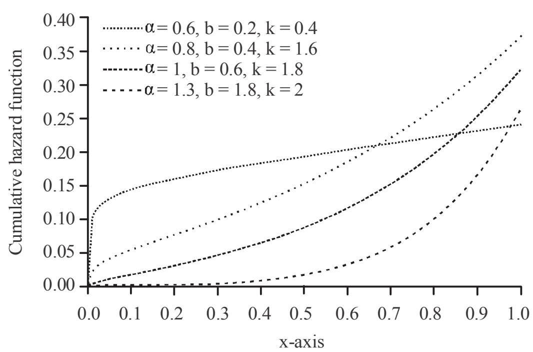

Fig. 5: The cumulative hazard function of the OGE-W-E for different values of parameters; a = 0.3, λ = 1.8

$$F(x_q) = P(x_q \le q) = q,\ 0<q<1 \tag{12}$$

From Eq. 6, we obtain:

$$\left[1-e^{-\lambda\left(e^{kx_q}-e^{kx_q-ax_q^b}\right)}\right]^{\alpha} = q \tag{13}$$

We can obtain x_q by solving the following equation numerically:

$$\ln\left(e^{kx_q}\right)+\ln\left(1-e^{-ax_q^b}\right)+\frac{1}{\lambda}\ln\left(1-q^{\frac{1}{\alpha}}\right) = 0 \tag{14}$$

And we can obtain the median of OGE-W-E by setting q = 0.5 in Eq. 14 and solve this equation numerically.

The moments: The rth moment for the OGE-W-E distribution is given by the following theorem.

Theorem 1: The rth moment for a random variable X~OGE-W-E (α, λ, a, b, k) is give by:

$$\mu_r' = \frac{\alpha\lambda}{b}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{1}\sum_{p=0}^{n}\sum_{q=0}^{\infty}\frac{\binom{j}{m}\binom{\alpha-1}{i}\binom{1}{n}\binom{n}{p}\Gamma\left(\frac{r+pb-p+q+1}{b}\right)}{j!q!\left(a(m+n)\right)^{\left(\frac{r+pb-p+q+1}{b}\right)}}\times (-1)^{i+j+m+n+p}(j+1)^q(i+1)^j\lambda^j k^{1-p+q}a^p b^p \tag{15}$$

Proof: The rth moment of X is obtained by:

$$\mu_r' = E\left(X^r\right) = \int_0^{\infty} x^r f(x;\alpha,\lambda,a,b,k) \tag{16}$$

Substituting from Eq. 7 into Eq. 16, we get:

$$\mu_r' = \alpha\lambda\int_0^{\infty} x^r\left[ke^{kx}-\left(k-abx^{b-1}\right)e^{kx-ax^b}\right]e^{-\lambda\left(e^{kx}-e^{kx-ax^b}\right)}\times\left[1-e^{-\lambda\left(e^{kx}-e^{kx-ax^b}\right)}\right]^{\alpha-1}dx \tag{17}$$

Using the binomial series expansion of $\left[1-e^{-\lambda\left(e^{kx}-e^{kx-ax^b}\right)}\right]^{\alpha-1}$ yields:

$$\left[1-e^{-\lambda\left(e^{kx}-e^{kx-ax^b}\right)}\right]^{\alpha-1} = \sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}e^{-\lambda i\left(e^{kx}-e^{kx-ax^b}\right)} \tag{18}$$

Substituting from Eq. 18 into Eq. 17, we obtain:

$$\mu_r' = \alpha\lambda\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\int_0^{\infty}x^r\left[ke^{kx}-\left(k-abx^{b-1}\right)e^{kx-ax^b}\right]e^{-\lambda(i+1)\left(e^{kx}-e^{kx-ax^b}\right)}dx \tag{19}$$

Using series expansion of $e^{-\lambda(i+1)\left(e^{kx}-e^{kx-ax^b}\right)}$ yields:

$$e^{-\lambda(i+1)\left(e^{kx}-e^{kx-ax^b}\right)} = \sum_{j=0}^{\infty}\frac{(-1)^j\lambda^j(i+1)^j\left(e^{kx}-e^{kx-ax^b}\right)^j}{j!} \tag{20}$$

Using the binomial series expansion of $\left(e^{kx}-e^{kx-ax^b}\right)^j$ yields:

$$\left(e^{kx}-e^{kx-ax^b}\right)^j = \sum_{j=0}^{\infty}(-1)^m\binom{j}{m}e^{kjx-amx^b} \tag{21}$$

Substituting from Eq. 21 into Eq. 20, we obtain:

$$e^{-\lambda(i+1)\left(e^{kx}-e^{kx-ax^b}\right)}=\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}\binom{j}{m}\lambda^j(i+1)^j}{j!}e^{kjx-amx^b} \quad (22)$$

Substituting from Eq. 22 into Eq. 19, we obtain:

$$\mu'_r=\alpha\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j+m}\lambda^j(i+1)^j\binom{j}{m}\binom{\alpha-1}{i}}{j!}\times \int_0^{\infty}x^r\left[ke^{kx}-\left(k-abx^{b-1}\right)e^{kx-ax^b}\right]e^{kjx-amx^b}dx \quad (23)$$

Using the binomial series expansion of $\left[ke^{kx}-\left(k-abx^{b-1}\right)e^{kx-ax^b}\right]$ yields:

$$\left[ke^{kx}-\left(k-abx^{b-1}\right)e^{kx-ax^b}\right]=\sum_{n=0}^{1}\binom{1}{n}k^{1-n}(-1)^n\left(k-abx^{b-1}\right)^n e^{kx-anx^b} \quad (24)$$

Using the binomial series expansion of $\left(k-abx^{b-1}\right)^n$ yields:

$$\left(k-abx^{b-1}\right)^n=\sum_{p=0}^{n}\binom{n}{p}k^{n-p}(-1)^p a^p b^p x^{pb-p} \quad (25)$$

Substituting from Eq. 25 into Eq. 24, we obtain:

$$\left[ke^{kx}-\left(k-abx^{b-1}\right)e^{kx-ax^b}\right]=\sum_{n=0}^{1}\sum_{p=0}^{n}\binom{1}{n}\binom{n}{p}(-1)^{n+p}k^{1-p}a^p b^p x^{pb-p}e^{kx-anx^b} \quad (26)$$

Substituting from Eq. 26 into Eq. 23, we obtain:

$$\mu'_r=\alpha\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{n}\sum_{q=0}^{1}\frac{\binom{j}{m}\binom{\alpha-1}{i}\binom{1}{n}\binom{n}{p}(-1)^{i+j+m+n+p}(i+1)^j\lambda^j k^{1-p}a^p b^p}{j!}\times \int_0^{\infty}x^{r+pb-p}e^{k(j+1)x}e^{-a(m+n)x^b}dx \quad (27)$$

Using series expansion of $e^{k(j+1)x}$, yields:

$$e^{k(j+1)x}=\sum_{q=0}^{\infty}\frac{k^q x^q (j+1)^q}{q!} \quad (28)$$

Substituting from Eq. 28 into Eq. 27, we obtain:

$$\mu'_r=\alpha\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{n}\sum_{n=0}^{1}\sum_{q=0}^{\infty}\frac{\binom{j}{m}\binom{\alpha-1}{i}\binom{1}{n}\binom{n}{p}(j+1)^q(-1)^{i+j+m+n+p}(i+1)^j\lambda^j k^{1-p+q}a^p b^p}{j!\ q!}\times \int_0^{\infty}x^{r+pb-p+q}e^{-a(m+n)x^b}dx \quad (29)$$

To find:

$$\int_0^{\infty}x^{r+pb-p+q}e^{-a(m+n)x^b}dx$$

Let $u = a\ (m+n)\ x^b$ when $x = 0 \Rightarrow u = 0$ and when $x = \infty \Rightarrow \lim_{x\to\infty} u = \infty \Rightarrow x = u^{1/b}a^{-1/b}\ (m+n)^{-1/b} \Rightarrow dx = 1/b\ u^{1/b-1}a^{-1/b}\ (m+n)^{-1/b}\ du$ then:

$$\int_0^{\infty}x^{r+pb-p+q}e^{-a(m+n)x^b}dx=\int_0^{\infty}\frac{1}{b}u^{\frac{r+pb-p+q+1}{b}} a^{-\left(\frac{r+pb-p+q+1}{b}\right)}(m+n)^{-\left(\frac{r+pb-p+q+1}{b}\right)}e^{-u}du \quad (30)$$

Using the definition of gamma function $(\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}dt$ we get:

$$\int_0^{\infty}e^{-u}du^{\frac{r+pb-p+q+1}{b}-1}du=\Gamma\left(\frac{r+pb-p+q+1}{b}\right)$$

So, we get:

$$\int_0^{\infty}x^{r+pb-p+q}e^{-a(m+n)x^b}dx=\frac{1}{b}a^{-\left(\frac{r+pb-p+q+1}{b}\right)}(m+n)^{-\left(\frac{r+pb-p+q+1}{b}\right)}\Gamma\left(\frac{r+pb-p+q+1}{b}\right) \quad (31)$$

Finally by substituting from Eq. 31 into Eq. 29 we get the rth moment of OGE-W-E distribution in the form:

$$\mu'_r=\frac{\alpha\lambda}{b}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{1}\sum_{p=0}^{n}\sum_{q=0}^{\infty}\frac{\binom{j}{m}\binom{\alpha-1}{i}\binom{1}{n}\binom{n}{p}\Gamma\left(\frac{r+pb-p+q+1}{b}\right)}{j!q!\left(a(m+n)\right)^{\left(\frac{r+pb-p+q+1}{b}\right)}}\times (-1)^{i+j+m+n+p}(j+1)^q(i+1)^j\lambda^j k^{1-p+q}a^p b^p \quad (32)$$

The mean: The mean for the OGE-W-E distribution is given by the following corollary.

Corollary 1: The mean (μ) for a random variable X~OGE-W-E (α, λ, a, b, k) is given by:

$$\mu=\frac{\alpha\lambda}{b}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{1}\sum_{p=0}^{n}\sum_{q=0}^{\infty}\frac{\binom{j}{m}\binom{\alpha-1}{i}\binom{1}{n}\binom{n}{p}\Gamma\left(\frac{pb-p+q+2}{b}\right)}{j!q!\left(a(m+n)\right)^{\left(\frac{pb-p+q+2}{b}\right)}}\times (-1)^{i+j+m+n+p}(j+1)^q(i+1)^j\lambda^j k^{1-p+q}a^p b^p \quad (33)$$

Proof: The mean (μ) of X~OGE-W-E (α, λ, a, b, k) is obtained by putting r = 1 in Eq. 32.

The moment about the mean: The rth moment about the mean of OGE-W-E distribution is given by the following theorem.

Theorem 2: The rth moment about the mean for a random variable X~OGE-W-E (α, λ, a, b, k) is give by:

$$\mu_r=E[X-\mu]^r=\frac{\alpha\lambda}{b}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{1}\sum_{p=0}^{n}\sum_{q=0}^{\infty}\sum_{s=0}^{r}\frac{\binom{j}{m}\binom{\alpha-1}{i}\binom{1}{n}\binom{n}{p}\binom{r}{s}\Gamma\left(\frac{s+pb-p+q+1}{b}\right)}{j!q!\left(a(m+n)\right)^{\left(\frac{s+pb-p+q+1}{b}\right)}}\times (-1)^{i+j+m+n+p+r-s}(j+1)^q(i+1)^j\lambda^j k^{1-p+q}a^p b^p\mu^{r-s} \quad (34)$$

Proof: The rth moment about the mean (μ_r) of the random variable X with pdf f (x) is given by:

$$\mu_r = E\left[(X-\mu)^r\right] = \int_0^\infty (x-\mu)^r f(x;\alpha,\lambda,a,b,k)dx \quad (35)$$

where μ is the mean of OGE-W-E distribution and f (x; α, λ, a, b, k) is the pdf by using the binomial series expansion of $(x-\mu)^r$ yields:

$$(x-\mu)^r = \sum_{s=0}^{\infty}\binom{r}{s}(-1)^{r-s}\mu^{r-s}x^s \quad (36)$$

Substituting from Eq. 36 into Eq. 35 yields:

$$\begin{aligned}\mu_r = E\left[(X-\mu)^r\right] &= \sum_{s=0}^{\infty}\binom{r}{s}(-1)^{r-s}\mu^{r-s} \\ \int_0^\infty x^s f(x;\alpha,\lambda,a,b,k) &= \sum_{s=0}^{\infty}\binom{r}{s}(-1)^{r-s}\mu^{r-s}\mu'_s\end{aligned} \quad (37)$$

Here μ'_s represents the sth moment of OGE-W-E by substituting from Eq. 32 (by replacing r by s) into Eq. 37 we get the rth moment about the mean of OGE-W-E distribution as follows:

$$\mu_r = E[X-\mu]^r = \frac{\alpha\lambda}{b}$$
$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{1}\sum_{p=0}^{n}\sum_{q=0}^{\infty}\sum_{s=0}^{r}\frac{\binom{j}{m}\binom{\alpha-1}{i}\binom{1}{n}\binom{n}{p}\binom{r}{s}\Gamma\left(\frac{s+pb-p+q+1}{b}\right)}{j!q!\left(a(m+n)\right)^{\left(\frac{s+pb-p+q+1}{b}\right)}}\times$$
$$(-1)^{i+j+m+n+p+r-s}(j+1)^q(i+1)^j\lambda^j k^{1-p+q}a^p b^p \mu^{r-s} \quad (38)$$

The skewness and kurtosis: In this subsection we drive the skewness and kurtosis of OGE-W-E distribution based on the moment about the mean as in the following theorem.

Theorem 3: The skewness and kurtosis for a random variable X~OGE-W-E (α, λ, a, b, k) are given in 1 and 2 as follows. The Coefficient of Skewness (CS) of OGE-W-E distribution is given by:

$$CS = \frac{A}{B} \quad (39)$$

Where:

$$A = \frac{\alpha\lambda}{b}$$
$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{1}\sum_{p=0}^{n}\sum_{q=0}^{\infty}\sum_{s=0}^{3}\frac{\binom{j}{m}\binom{\alpha-1}{i}\binom{1}{n}\binom{n}{p}\binom{3}{s}\Gamma\left(\frac{s+pb-p+q+1}{b}\right)}{j!q!\left(a(m+n)\right)^{\left(\frac{s+pb-p+q+1}{b}\right)}}\times$$
$$(-1)^{i+j+m+n+p+3-s}(j+1)^q(i+1)^j\lambda^j k^{1-p+q}a^p b^p \mu^{3-s}$$

And:

$$B = [\frac{\alpha\lambda}{b}$$
$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{1}\sum_{p=0}^{n}\sum_{q=0}^{\infty}\sum_{s=0}^{2}\frac{\binom{j}{m}\binom{\alpha-1}{i}\binom{1}{n}\binom{n}{p}\binom{2}{s}\Gamma\left(\frac{s+pb-p+q+1}{b}\right)}{j!q!\left(a(m+n)\right)^{\left(\frac{s+pb-p+q+1}{b}\right)}}\times$$
$$(-1)^{i+j+m+n+p+2-s}(j+1)^q(i+1)^j\lambda^j k^{1-p+q}a^p b^p \mu^{2-s}]^{\frac{3}{2}}$$

The Coefficient of kurtosis (Ck) of OGE-W-E distribution is given by:

$$CK = \frac{C}{D} \quad (40)$$

Where:

$$C = \frac{\alpha\lambda}{b}$$
$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{1}\sum_{p=0}^{n}\sum_{q=0}^{\infty}\sum_{s=0}^{4}\frac{\binom{j}{m}\binom{\alpha-1}{i}\binom{1}{n}\binom{n}{p}\binom{4}{s}\Gamma\left(\frac{s+pb-p+q+1}{b}\right)}{j!q!\left(a(m+n)\right)^{\left(\frac{s+pb-p+q+1}{b}\right)}}\times$$
$$(-1)^{i+j+m+n+p+4-s}(j+1)^q(i+1)^j\lambda^j k^{1-p+q}a^p b^p \mu^{4-s}$$

And

$$D = [\frac{\alpha\lambda}{b}$$
$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{1}\sum_{p=0}^{n}\sum_{q=0}^{\infty}\sum_{s=0}^{2}\frac{\binom{j}{m}\binom{\alpha-1}{i}\binom{1}{n}\binom{n}{p}\binom{2}{s}\Gamma\left(\frac{s+pb-p+q+1}{b}\right)}{j!q!\left(a(m+n)\right)^{\left(\frac{s+pb-p+q+1}{b}\right)}}\times$$
$$(-1)^{i+j+m+n+p+2-s}(j+1)^q(i+1)^j\lambda^j k^{1-p+q}a^p b^p \mu^{2-s}]^2$$

Proof 1: We start with the following equation of the coefficient of skewness by Oja (1981):

$$CS = \frac{E\left[(X-\mu)\right]^3}{\left[E\left[(X-\mu)\right]^2\right]^{\frac{3}{2}}} \quad (41)$$

$$\text{Let } A = E\left[(X-\mu)\right]^3 \quad (42)$$

And

$$\text{Let } B = \left[E\left[(X-\mu)\right]^2\right]^{\frac{3}{2}} \quad (43)$$

We can find CS by finding A and B as follows: we can obtain A by setting r = 3 in Eq. 34 and we can obtain B by substituting from Eq. 34 (by setting r = 2) into Eq. 43.

Proof 2: We start with the following equation of the coefficient of kurtosis by Oja (1981):

$$Ck = \frac{E\left[(X-\mu)\right]^4}{\left[E\left[(X-\mu)\right]^2\right]^2} \tag{44}$$

$$\text{Let } C = E\left[(X-\mu)\right]^4 \tag{45}$$

And:

$$\text{Let } D = \left[E\left[(X-\mu)\right]^2\right]^2 \tag{46}$$

We can find CK by finding C and D as follows: we can obtain C by setting r = 4 in Eq. 34 and we can obtain D by substituting from Eq. 34 (by setting r = 2) into Eq. 46.

The moment generating function: The moment generating function of OGE-W-E distribution is given by the following theorem.

Theorem 4: The moment generating function M_X (t) of OGE-W-E distribution is given by:

$$M_X(t) = \frac{\alpha\lambda}{b} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{1}\sum_{p=0}^{n}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} \frac{\binom{j}{m}\binom{\alpha-1}{i}\binom{1}{n}\binom{n}{p}\Gamma\left(\frac{r+pb-p+q+1}{b}\right)}{j!q!r!\left(a(m+n)\right)^{\left(\frac{r+pb-p+q+1}{b}\right)}} \times (-1)^{i+j+m+n+p}(j+1)^q(i+1)^j\lambda^j k^{1-p+q}a^p b^p t^r \tag{47}$$

Proof: The moment generating function M_X (t) of the random variable X is given by:

$$M_X(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f(x;\alpha,\lambda,a,b,k)\,dx \tag{48}$$

Using series expansion of e^{tx}, yields:

$$e^{tx} = \sum_0^{\infty}\frac{t^r x^r}{r!} \tag{49}$$

Substituting from Eq. 49 into Eq. 48, yields:

$$M_X(t) = \sum_0^{\infty}\frac{t^r}{r!}\int_0^{\infty} x^r f(x;\alpha,\lambda,a,b,k)\,dx = \sum_0^{\infty}\frac{t^r}{r!}\mu'_r \tag{50}$$

Finally by substituting from Eq. 32 into Eq. 50, we get:

$$M_X(t) = \frac{\alpha\lambda}{b} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{1}\sum_{p=0}^{n}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} \frac{\binom{j}{m}\binom{\alpha-1}{i}\binom{1}{n}\binom{n}{p}\Gamma\left(\frac{r+pb-p+q+1}{b}\right)}{j!q!r!\left(a(m+n)\right)^{\left(\frac{r+pb-p+q+1}{b}\right)}} \times (-1)^{i+j+m+n+p}(j+1)^q(i+1)^j\lambda^j k^{1-p+q}a^p b^p t^r \tag{51}$$

Order statistics: In this section, the PDF of the jth order statistic and the PDF of the smallest and largest order statistics of OGE-W-E distribution are derived. Let $X_1, X_2, ..., X_n$ be a random sample from an OGE-W-E distribution and $X_{1:n}, X_{2:n}, ..., X_{n:n}$ denote the order statistics obtain- ned from this sample then the pdf of $X_{j:n}$ is given by:

$$f_{j:n}(x;\Phi) = \frac{1}{B(j,n-j+1)}\left[F(x;\Phi)\right]^{j-1}\left[1-F(x;\Phi)\right]^{n-j} f(x;\Phi) \tag{52}$$

where, f (x; Φ) is the pdf of OGE-W-E distribution given by Eq. 7, F (x; Φ) is the cdf of OGE-W-E distribution given by Eq. 6, Φ = (α, λ, a, b, k) and B (. , .) is the beta function. Because of, 0<F (x; Φ) <1, we can use the binomial series expansion of $[1-F(x;\Phi)]^{n-j}$ as follows:

$$\left[1-F(x;\Phi)\right]^{n-j} = \sum_{i=0}^{n-j}\binom{n-j}{i}(-1)^i\left[F(x;\Phi)\right]^i \tag{53}$$

Substituting from Eq. 53 into Eq. 52, we get:

$$f_{j:n}(x;\Phi) = \sum_{i=0}^{n-j}\frac{(-1)^i n!}{i!(j-1)!(n-j-i)!}\left[F(x;\Phi)\right]^{j+i-1} f(x;\Phi) \tag{54}$$

Substituting from Eq. 6 and 7 into Eq. 54, we get the pdf for the jth order statistic as follows:

$$f_{j:n}(x;\Phi) = \alpha\lambda\sum_{i=0}^{n-j}\frac{(-1)^i n!}{i!(j-1)!(n-j-i)!}\left[1-e^{-\lambda\left(e^{kx}-e^{kx-ax^b}\right)}\right]^{\alpha(j+1)-1} \times \left[ke^{kx}-\left(k-abx^{b-1}\right)e^{kx-ax^b}\right]e^{-\lambda\left(e^{kx}-e^{kx-ax^b}\right)} \tag{55}$$

From Eq. 55, we can find the pdf of the smallest order statistics, say $f_{1:n}$ (x; Φ) and the largest order statistics, say $f_{1:n}$ (x; Φ) as follows:

$$f_{1:n}(x;\Phi) = \alpha\lambda\sum_{i=0}^{n-1}\frac{(-1)^i n!}{i!(n-j-i)!}\left[1-e^{-\lambda\left(e^{kx}-e^{kx-ax^b}\right)}\right]^{\alpha(j+1)-1} \times \left[ke^{kx}-\left(k-abx^{b-1}\right)e^{kx-ax^b}\right]e^{-\lambda\left(e^{kx}-e^{kx-ax^b}\right)} \tag{56}$$

$$f_{n:n}(x;\Phi) = n\alpha\lambda\left[1-e^{-\lambda\left(e^{kx}-e^{kx-ax^b}\right)}\right]^{\alpha n-1} \times \left[ke^{kx}-\left(k-abx^{b-1}\right)e^{kx-ax^b}\right]e^{-\lambda\left(e^{kx}-e^{kx-ax^b}\right)} \tag{57}$$

Renyi entropy (Renyi, 1961): An entropy is a measure of variation or uncertainty of a random variable X. The Renyi entropy of a random variable X with probability density function f (x) is defined by:

$$I_R(\delta) = \frac{1}{1-\delta}\log\left(\int_0^\infty f^\delta(x)dx\right) \text{ where } \delta>0 \text{ and } \delta \neq 1 \quad (58)$$

Proposition 1: If X a random variable has a OGE-W-E distribution then the Renyi entropy of X is given by:

$$I_R(\delta) = \frac{1}{1-\delta}\log\Big[\frac{\alpha^\delta\lambda^\delta}{b}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\binom{j}{m}\binom{\delta\alpha-\delta}{i}\Gamma\left(\frac{q+pb-p+1}{b}\right)}{j!q!\left(a(m+n)\right)^{\left(\frac{q+pb-p+1}{b}\right)}}\times (-1)^{i+j+m+n+p}(jk+\delta k)^q(i+\delta)^j\lambda^j k^{\delta-p}a^p b^p \quad (59)$$

Proof: Substituting from Eq. 7 into Eq. 58, we get:

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[\alpha^\delta\lambda^\delta\int_0^\infty\left[ke^{kx}-\left(k-abx^{b-1}\right)e^{kx-ax^b}\right]^\delta e^{-\lambda\delta\left(e^{kx}-e^{kx-ax^b}\right)}\times\left[1-e^{-\lambda\left(e^{kx}-e^{kx-ax^b}\right)}\right]^{\delta\alpha-\delta}dx\right] \quad (60)$$

From Eq. 59, we get the Renyi entropy of X given in Eq. 58 by applying the same steps for finding u'_r.

Parameters estimation of OGE-W-E distribution: In this section, we use the maximum likelihood estimation method to estimate the five parameters of OGE-W-E distribution.

Maximum likelihood estimation: If $x_1, x_2, ..., x_n$ denote a random sample from the OGE-W distribution then the likelihood function is given by:

$$L = \prod_{i=1}^{n} f(x_i;\alpha,\lambda,a,b,k) \quad (61)$$

Substituting from Eq. 7 into Eq. 61, we obtain:

$$L = \prod_{i=1}^{n}\alpha\lambda\left[ke^{kx_i}-\left(k-abx_i^{b-1}\right)e^{kx_i-ax_i^b}\right]e^{-\lambda\left(e^{kx_i}-e^{kx_i-ax_i^b}\right)}\times\left[1-e^{-\lambda\left(e^{kx_i}-e^{kx_i-ax_i^b}\right)}\right]^{\alpha-1} \quad (62)$$

The log-likelihood function is:

$$\ell = n\ln(\alpha)+n\ln(\lambda)+\sum_{i=1}^{n}\ln\left[ke^{kx_i}-\left(k-abx_i^{b-1}\right)e^{kx_i-ax_i^b}\right]-\lambda\sum_{i=1}^{n}\left(e^{kx_i}-e^{kx_i-ax_i^b}\right)+(\alpha-1)\sum_{i=1}^{n}\ln\left[1-e^{-\lambda\left(e^{kx_i}-e^{kx_i-ax_i^b}\right)}\right] \quad (63)$$

By take in the partial derivatives of ℓ with respect to the parameters α, λ, a, b and k setting the result to zero, we get the following equations:

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha}+\sum_{i=1}^{n}\ln\left[1-e^{-\lambda\left(e^{kx_i}-e^{kx_i-ax_i^b}\right)}\right] = 0 \quad (64)$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda}-\sum_{i=1}^{n}\left(e^{kx_i}-e^{kx_i-ax_i^b}\right)+(\alpha-1)\sum_{i=1}^{n}\frac{\left(e^{kx_i}-e^{kx_i-ax_i^b}\right)e^{-\lambda\left(e^{kx_i}-e^{kx_i-ax_i^b}\right)}}{\left[1-e^{-\lambda\left(e^{kx_i}-e^{kx_i-ax_i^b}\right)}\right]} = 0 \quad (65)$$

$$\frac{\partial\ell}{\partial a} = \sum_{i=1}^{n}\frac{\left[\left(k-abx_i^{b-1}\right)x_i^b+bx_i^{b-1}\right]e^{kx_i-ax_i^b}}{\left[ke^{kx_i}-\left(k-abx_i^{b-1}\right)e^{kx_i-ax_i^b}\right]}-\lambda\sum_{i=1}^{n}x_i^b e^{kx_i-ax_i^b}+\lambda(\alpha-1)\sum_{i=1}^{n}\frac{x_i^b e^{kx_i-ax_i^b}e^{-\lambda\left(e^{kx_i}-e^{kx_i-ax_i^b}\right)}}{\left[1-e^{-\lambda\left(e^{kx_i}-e^{kx_i-ax_i^b}\right)}\right]} = 0 \quad (66)$$

$$\frac{\partial\ell}{\partial b} = \sum_{i=1}^{n}\frac{\left[ax_i^b\ln(x_i)\left(k-abx_i^{b-1}\right)+abx_i^{b-1}\ln(x_i)+ax_i^{b-1}\right]e^{kx_i-ax_i^b}}{\left[ke^{kx_i}-\left(k-abx_i^{b-1}\right)e^{kx_i-ax_i^b}\right]}-\lambda\sum_{i=1}^{n}ax_i^b\ln(x_i)e^{kx_i-ax_i^b}+a\lambda(\alpha-1)\sum_{i=1}^{n}\frac{x_i^b\ln(x_i)e^{kx_i-ax_i^b}e^{-\lambda\left(e^{kx_i}-e^{kx_i-ax_i^b}\right)}}{\left[1-e^{-\lambda\left(e^{kx_i}-e^{kx_i-ax_i^b}\right)}\right]} = 0 \quad (67)$$

$$\frac{\partial\ell}{\partial k} = \frac{\left(ke^{kx_i}x_i+e^{kx_i}\right)-\left[\left(k-abx_i^{b-1}\right)x_ie^{kx_i-ax_i^b}+e^{kx_i-ax_i^b}\right]}{\left[ke^{kx_i}-\left(k-abx_i^{b-1}\right)e^{kx_i-ax_i^b}\right]}-\lambda\sum_{i=1}^{n}\left(e^{kx_i}x_i-e^{kx_i-ax_i^b}x_i\right)+\lambda(\alpha-1)\sum_{i=1}^{n}\frac{\left(e^{kx_i}x_i-e^{kx_i-ax_i^b}x_i\right)e^{-\lambda\left(e^{kx_i}-e^{kx_i-ax_i^b}\right)}}{\left[1-e^{-\lambda\left(e^{kx_i}-e^{kx_i-ax_i^b}\right)}\right]} = 0 \quad (68)$$

We can obtain the MLEs of the parameters α, λ, a, b and k by solving the Eq. 64-68, numerically for α, λ, a, b and k.

Application: In this section, we provide an application to real data to demonstrate the importance of the OGE-W-E distribution and we will compare OGE-W-E distribution with the following distributions:

- Modified Weibull Distribution (MWD) with cdf, $F(x; \alpha, \beta, \gamma) = 1-e^{-\alpha x-\beta x^\gamma}$, $x>0$
- Flexible Weibull (FW) with cdf, $F(x; \alpha, \gamma, \beta, \theta) = 1-e^{-e(\beta x^\gamma+\theta x^\alpha)}$, $x>0$, $\alpha, \gamma, \beta, \theta>0$
- Odd Generalized Exponential-Exponential (OGE-E) with cdf, $F(x; \alpha, \alpha, \lambda) = \left[1-e^{-\lambda\left(e^{\alpha x}-1\right)}\right]^\alpha$

In order to compare the OGE-W-E distribution with the above distributions, we use the measures of goodness-of-fit including the Akaike Information

Table 1: Parameters estimates and their standard errors (in parentheses) for data set

Models	Parameters estimates				
OGE-W-E (α, λ, a, b, k)	$\hat{\alpha}$ = 1.373 (0.254)	$\hat{\lambda}$ = 2.362 (0.778)	$\hat{a}$ = 0.056 (0.003)	$\hat{b}$ = 1.134 (0.091)	$\hat{k}$ = 0.14 (0.005)
MWD (α, β, γ)	$\hat{\alpha}$ = 0.008, (0.015)	$\hat{\beta}$ = 0.86 (0.734)	$\hat{\gamma}$ = 1.052 (1.514)	-	-
(OGE-E) (a, α, λ)	$\hat{a}$ = 0.022 (0.001)	$\hat{\alpha}$ = 0.819 (0.042)	$\hat{\lambda}$ = 2.816 (0.154)	-	-
(FW) (α, γ, β, θ)	$\hat{\alpha}$ = 0.61 (0.763)	$\hat{\gamma}$ = 0.66 (1.02)	$\hat{\beta}$ = 0.138 (0.059)	$\hat{\theta}$ = 0.007 (1.143)	-

Table 2: The values of the statistics ($\hat{\ell}$ AIC, HQIC, CAIC and BIC for the data set)

Models	$\hat{\ell}$	AIC	HQIC	CAIC	BIC
OGE-W-E (α, λ, a, b, k)	-410.0163	830.0326	835.8266	830.5244	844.2928
MWD (α, β, γ)	-414.0939	834.1877	837.6641	834.3813	842.7438
(OGE-E) (a, α, λ)	-425.1476	856.2951	859.7715	856.4887	864.8512
(FW) (α, γ, β, θ)	-554.6617	1117.3000	1122.0000	1117.6000	1128.7000

Criterion (AIC), Hannan-Quinn Information Criterion (HQIC), Consistent Akaike Information Criterion (CAIC) and Bayesian Information Criterion (BIC). In general, the distribution which gives the smallest values from the criteria, shows the better fit to the data.

The MLEs of the model parameters are given in Table 1 and the numerical values of the model selection statistics $\hat{\ell}$, AIC, HQIC, CAIC, BIC are listed in Table 2. We have a real data set corresponding toremission times (in months) of a random sample 0f 128 bladder cancer patients (Al-Zahrani and Sagor, 2014). The data set are: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.2, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39,10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, ,1.76, 3.25, 4.50, 6.25, 8.37, 11.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69. From Table 2 we can see that the OGE-W-E distribution represents the data set better than the other selected models.

CONCLUSION

In this study, a new five parameters distribution called the OGE-W-E has been proposed and studied. We study some statistical properties of this distribution. For estimating the distribution parameters, we use the maximum likelihood method. Finally, by mean of an application using real data set we demonstrate the usefulness of this distribution and we see that the OGE-W-E distribution provides better fit than the other competing models.

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